L^p-uniqueness of Schrödinger operators and capacitary positive improving property

Liming WU

email : wuliming@ucfma.univ-bpclermont.fr

Laboratoire de Mathématiques Appliquées, CNRS-UMR 6620
Université Blaise Pascal, 63177 AUBIERE, France

and

Department of Mathematics, Wuhan Universtity
430072 Hubei, China

January 2000

Abstract : We prove several L^p-uniqueness results for Schrödinger operators $-\mathcal{L}+V$ by means of the Feynman-Kac formula. Using the (m,p)-capacity theory for general Markov semigroups, we show that the associated Feynman-Kac semigroup is positive improving in the sense of (m,p)-capacity, improving the well known one in the sense of measure. Using that capacitary positive improving property and two new inequalities for generalized Ornstein-Uhlenbeck generators, we show the essential self-adjointness of the ground state diffusion generator $\mathcal{L}^\phi = \mathcal{L} + 2\Gamma(\phi,\cdot)/\phi$ associated with two dimensional Euclidean quantum fields.

Key words : L^p-uniqueness, essential self-adjointness, Schrödinger operators, Euclidean quantum fields, (m,p)-capacity, positive improving property.

AMS Mathematical Subject Classification (1991) : 47D07, 31C15; 81T08 ; 60J35.

1. Problems and Introduction

1.1. Problems

Following Nelson [19](1973), the free Euclidean quantum field is described by a centered Gaussian measure $\mathbb{P}^C$ on the space $\mathcal{S}'(\mathbb{R}^d)$ of temperate distributions on $\mathbb{R}^d$ with covariance

$$\mathbb{E}^{\mathbb{P}^C} X(f)X(g) = \langle f, Cg\rangle_{L^2(\mathbb{R}^d,dx)}, \quad \forall f,g \in \mathcal{S}(\mathbb{R}^d)$$

where $C = (-\Delta_d + m^2)^{-1}$, $d \geq 2$ is the *time-space* dimension (Δ_d being the Laplacian on $\mathbb{R}^d$), $m > 0$ is the mass of each particle in the system of indistinguishable identical particles, $X(f)$ is the usual dual bilinear relation between $X \in \mathcal{S}'(\mathbb{R}^d)$ and $f \in \mathcal{S}(\mathbb{R}^d)$.

It is well known that $X_t = X|_{\{t\}\times\mathbb{R}^{d-1}} \in \mathcal{S}'(\mathbb{R}^{d-1})$ is well defined $\mathbb{P}^C - a.s.$ and $(X_t)_{t\in\mathbb{R}}$ is a continuous Gaussian Markov process with values in $\mathcal{S}'(\mathbb{R}^{d-1})$ and with covariance

$$\mathbb{E}^{\mathbb{P}^C} X_{s+t}(f)X_s(g) = \frac{1}{2}\langle f, B^{-1}e^{-Bt}g\rangle_{L^2(\mathbb{R}^{d-1},dx)}, \tag{1.1}$$

where $B = \sqrt{-\Delta_{d-1} + m^2}$ (see [9], Prop.6.2.5). In particular the law μ of X_t under $\mathbb{P}^C$ is the centered Gaussian measure on $\mathcal{S}'(\mathbb{R}^{d-1})$ with covariance operator $(2B)^{-1}$, called *time-zero Minkowski field* ([22]).

The *free Schrödinger operator* will be the generator $\mathcal{L}$ of the Gaussian Markov process $(X_t)_{t\geq 0}$ on $L^2\left(\mathcal{S}'(\mathbb{R}^{d-1}), \mu\right)$ (it is in fact a generalized Ornstein-Uhlenbeck operator). Our space of test functions will be

$$\mathcal{F}C_0^\infty := \left\{F(S(f_1), \cdots, S(f_n)) \mid n \in \mathbb{N}, f_i \in C_0^\infty(\mathbb{R}^{d-1}), F \in C_0^\infty(\mathbb{R}^n)\right\}, \tag{1.2}$$

where $C_0^\infty(\mathbb{R}^n)$ is the space of all infinitely differentiable functions with compact support on $\mathbb{R}^n$. If F varies in $C_b^\infty(\mathbb{R}^n)$ in the expression above, we get a larger space $\mathcal{F}C_b^\infty$. We have the following explicit expression for $\mathcal{L}$ (a consequence of (5.8a,b) in [29] and the path continuity of (X_t)),

$$\mathcal{L}u = \sum_{i=1}^n \partial_i F(\cdots)S(-Bf_i) + \frac{1}{2}\sum_{i,j=1}^n \partial_{ij}F(\cdots)\langle f_i, f_j\rangle_{L^2(\mathbb{R}^{d-1},dx)} \tag{1.3}$$

for all $u := F(S(f_1), \cdots, S(f_n)) \in \mathcal{F}C_b^\infty$.

Now let $\mathcal{V} : \mathcal{S}'(\mathbb{R}^{d-1}) \to \mathbb{R}$ be a μ-measurable interaction potential (only the space variable is involved), satisfying

$$\mathcal{V} \in L^{p_0+}(\mu) := \bigcup_{q>p_0} L^q(\mu), \quad e^{-\mathcal{V}} \in \bigcap_{1\le q<+\infty} L^q(\mu). \tag{1.4}$$

The corresponding Euclidean quantum field with interaction $\mathcal{V}$ is given by the thermodynamic limit

$$\mathbb{P}^{\mathcal{V}} = \lim_{T\to\infty} \frac{\exp\left(-\int_{-T}^{T} \mathcal{V}(X_t)dt\right)}{\mathbb{E}^{\mathbb{P}^C} \exp\left(-\int_{-T}^{T} \mathcal{V}(X_t)dt\right)} \cdot \mathbb{P}^C. \tag{1.5}$$

See [9] (1985) for the weak convergence of finite dimensional marginal laws in (1.5), and [30] (Theorem 6.1) for the process level weak convergence. To present a more explicit description of $\mathbb{P}^{\mathcal{V}}$, let us recall the following basic result,

Theorem 1.1 (due to Hoegh-Krohn and Simon [10], Theorem 4.5, 1972) *Assume (1.4) for $p_0 = 2$. Then $(-\mathcal{L} + \mathcal{V}, \mathcal{F}C_b^\infty)$ is essentially self-adjoint (in short : e.s.a.), and the infimum $\lambda(\mathcal{V})$ of the spectrum of its closure $\mathcal{H}$ is an isolated eigenvalue whose eigenspace is generated by some unique $\mu - a.s.$ strictly positive ϕ with $\int \phi^2 d\mu = 1$ (called ground state).*

By [9] or [30], restricted to $\mathcal{F}_T^0 := \sigma(X_t; 0 \le t \le T)$,

$$\mathbb{P}^{\mathcal{V}}|_{\mathcal{F}_T^0} = \phi(X_0)\phi(X_T) \exp\left(\lambda(\mathcal{V})T - \int_0^T \mathcal{V}(X_t)dt\right) \cdot \mathbb{P}^C$$

i.e., the ground state diffusion (its transition semigroup is called *semigroup of transfer matrices* in [22]). Moreover under $\mathbb{P}^{\mathcal{V}}$, $(X_t)_{t\in\mathbb{R}}$ is a continuous symmetric Markov process with invariant measure $\mu_\phi := \phi^2\mu$, and its generator acting on $\mathcal{F}C_0^\infty$ is given by

$$\mathcal{L}^\phi F = \mathcal{L}F + 2\frac{\Gamma(\phi, F)}{\phi}, \ \forall F \in \mathcal{F}C_0^\infty, \tag{1.6}$$

where Γ is the *carré-du-champs* operator associated with $\mathcal{L}$, given at first by $\Gamma(F,G) := (1/2)\left(\mathcal{L}(FG) - (\mathcal{L}F)G - F\mathcal{L}G\right)$ for $F, G \in \mathcal{F}C_0^\infty$, and extended next for all F, G belonging to the domain of Dirichlet form associated with $\mathcal{L}$ (since $\mathcal{F}C_0^\infty$ is a core for $\mathcal{L}$ in $L^2(\mu)$, see [26]).

It has the following explicit expression for $u = F(S(f_1), \cdots, S(f_n)), v = G(S(g_1), \cdots, S(g_m)) \in \mathcal{F}C_0^\infty$ (a consequence of (1.3)):

$$\Gamma(u,v) = \frac{1}{2} \sum_{1\le i\le n, 1\le j\le m} \partial_i F(\cdots)\partial_j G(\cdots)\langle f_i, g_j\rangle_{L^2(\mathbb{R}^{d-1},dx)}.$$

Remark that the operator $\mathcal{L}^\phi$ is well defined for any ϕ belonging to the domain of Dirichlet form associated with $\mathcal{L}$.

To state the questions studied in this paper, let us recall the following notion of uniqueness used in [27] (1998).

Definition 1.2 *An operator A on a Banach space X is called an essential generator, if it is closable and its closure $\bar{A}$ is the generator of a C_0-semigroup (T_t) on X (i.e., a strongly continuous semigroup of bounded linear operators).*

By Arendt [2] and [27] (Lemma 2.6), if and only if A is an essential generator, (T_t) is the unique C_0-semigroup whose generator is an extension of A. (And for an upper bounded symmetric operator A on a Hilbert space, A is an essential generator if and only if it is essentially self-adjoint). That's why often in this paper the above property is called X*-uniqueness.*

Our first question consists to extend Theorem 1.1 :

Question 1. $(\mathcal{L} - \mathcal{V}, \mathcal{F}C_0^\infty)$ *is-it* $L^p(\mu)$*-unique ?*

$-\mathcal{L}^\phi$ will be served as the Hamiltonian on $L^2(\mu_\phi = \phi^2\mu)$ of the interaction Euclidean quantum field observed at the ground state ϕ. The e.s.a. of $\left(\mathcal{L}^\phi, \mathcal{F}C_0^\infty\right)$ is equivalent to the unique solvability of the corresponding Schrödinger equation or of the heat equation $\partial_t u = \mathcal{L}^\phi u$ in $L^2(\mathcal{S}'(\mathbb{R}^{d-1}), \mu_\phi)$. Now the reader see clearly the importance of the open

Question 2 . *Whether is* $\left(\mathcal{L}^\phi, \mathcal{F}C_0^\infty\right)$ *essentially self-adjoint (in short : e.s.a.) on* $L^2(\mathcal{S}'(\mathbb{R}^{d-1}), \mu_\phi)$ *?*

We were rather close (we believed it but in reality quite far) to solve the open question 2 above in Theorem 3.5 of [26] in the following sense :

if the ground state ϕ of $-\mathcal{L}+\mathcal{V}$ satisfies $\phi \in \bigcap_{1<p<+\infty} \mathbb{D}_{2,p}$ and for every $p \in (1,+\infty)$,

$$Cap_{2,p}(\widetilde{\phi} < \epsilon) \longrightarrow 0 \quad \text{as} \quad \epsilon \to 0, \tag{1.7}$$

where $\widetilde{\phi}$ is the $Cap_{2,p}$-quasi-continuous version of ϕ, then the question 2 has a positive answer.

Up to the knowledge of the author, Question 2 remains open.

1.2. Several known results

An enormous number of pioneering and important works are realized on the L^2-uniqueness (or equivalently e.s.a.) of Schrödinger operator $-\Delta + V$. We are content here only to cite Chung and Zhao [4], Kato [11], Reed and Simon [21], where the reader can find a large number of references. Below we focus on operators of type (1.6), whose studies are more recent.

1) About the L^2-uniqueness. In the finite dimensional context where $\mathcal{S}'(\mathbb{R}^{d-1})$ is replaced by $\mathbb{R}^n$, $\mathcal{L}$ by the usual Laplacian $\Delta/2$, and $\mathcal{F}C_0^\infty$ by $C_0^\infty(\mathbb{R}^n)$, the operator $\mathcal{L}^\phi$ given in (1.6) becomes

$$\mathcal{L}^\phi f = \frac{1}{2}\Delta f + \frac{\nabla\phi \cdot \nabla f}{\phi}, \ \forall f \in C_0^\infty(\mathbb{R}^n), \tag{1.8}$$

A lot of studies are realized about the e.s.a. of $(\mathcal{L}^\phi, C_0^\infty(\mathbb{R}^n))$ on $L^2(\mathbb{R}^n, \phi^2 dx)$, even for general ϕ belonging locally to H^1 (the usual Sobolev space), see Wielens [25] (1985), Liskevitch and Semenov [14](1992), Liskevitch [13] (1999), Eberle [5] (1997), the author [28] (1999) and the references therein. A result due to Wielens [25] (1985) says that $(\mathcal{L}^\phi, C_0^\infty(\mathbb{R}^n))$ is e.s.a. if ϕ is locally lipchizian and $\phi > 0$ **over** $\mathbb{R}^n$ (our condition (1.7) can be read as a (very partial) extension of his result to the infinite dimensional case).

In [14] (1992), Liskevitch and Semenov established the e.s.a. of $(\mathcal{L}^\phi, C_0^\infty(\mathbb{R}^n))$ under the global integrability condition $\beta := |\nabla\phi|/\phi \in L^4(\mathbb{R}^n, \phi^2 dx)$. More recently Liskevitch [13] (Th.1, 1999) gives a local version of that result.

For the e.s.a. in the infinite dimensional framework, see Albeverio, Kondratiev, Röckner [1] (1992), the author [26] (1994), Song [23] (1996), Eberle [5] (1997), and Liskevitch and Röckner [15] (1998) (where the applications to stochastic quantization are investigated).

2) About the L^1-uniqueness. This is studied by the author [27] (1998) for the Schrödinger operators $-\Delta + V$, and [28], [29] (1999) for generalized Schrödinger operators of type (1.8) and (1.6), where the explicit necessary and sufficient conditions in terms of V or ϕ are obtained. Stannat [24] (1997), Liskevitch [13] (1999) obtain same results for generalized Schrödinger operators of type (1.8) in some slightly different contexts (they were apparently unaware of my works [28], [29] (1999) which circulated during 1997, as well as I were so of their works in that period).

In [28], it is shown that $(\mathcal{L}^\phi, C_0^\infty(D)$ is $L^1(D, \phi^2 dx)-$unique, if and only if the corresponding diffusion is conservative, under the condition that ϕ is continuous and $\phi \in H^1_{loc}(D)$, where D is an open domain in $\mathbb{R}^n$. The last result is extended to a general framework including the infinite dimensional case in [29] (1999), under the condition that ϕ, ϕ^2 belong both to the domain of Dirichlet form. In particular we show that ([29], §5)

• for any eigenfunction ϕ of $-\mathcal{L} + \mathcal{V}$ given in §1.1, $\left(\mathcal{L}^\phi, \mathcal{F}C_0^\infty\right)$ defined by (1.6) is $L^1(\phi^2\mu)$-unique ; but

• for any eigenfunction ϕ of $-\mathcal{L} + \mathcal{V}$ given in §1.1 with an eigenvalue different from the lowest energy $\lambda(\mathcal{V})$ (called often *excited state*), $\left(\mathcal{L}^\phi, \mathcal{F}C_b^\infty\right)$ is not L^2-unique.

The last non-uniqueness result shows the subtleness of the question 2 .

The main purpose of this paper is to solve both the questions 1 and 2. It is organized as follows. In the next section we shall establish the L^p-uniqueness of a general Schrödinger operator $(\mathcal{L} - V, \mathcal{D})$ under quite explicit conditions, where $\mathcal{L}$ is the generator of some Markov semigroup. The main results of this section, Theorem 2.1 and 2.2, extend the famous Kato theorem from $\mathcal{L} = \Delta$ to general $\mathcal{L}$ and from $p = 2$ to any $1 \leq p < +\infty$. In particular Question 1 is completely solved in Corollary 2.6, where the Nelson hypercontractivity (or equivalently the Gross log-Sobolev inequality) and the Meyer inequality are used in an essential way.

The remained part of the paper are devoted to the studies of Question 2, which is much more delicate. Our first remark is that condition (1.7) is equivalent to

$$Cap_{2,p}(\widetilde{\phi} \leq 0) = 0, \tag{1.9}$$

for every $p \in (1, +\infty)$. For verifying (1.9), we are brought to show that $e^{-\lambda(\mathcal{V})t}\widetilde{\phi} = \widetilde{P_t^{\mathcal{V}}\phi}$ is $Cap_{2,p}$-quasi-everywhere positive. The objective of Section 3 is to establish that capacitary positive improving property for a general

Markov semigroup (P_t) and for the associated Feynman-Kac semigroup (P_t^V), instead of the well known one in the sense of measure μ. The key tool for realizing this aim is the theory of (m,p)-capacity, developed by Malliavin [17] (1984), Fukushima and Kaneko [8] (1985), Feyel and de la Pratelle [6], [7] (1991), and by Kazumi and Shigekawa [12] (1992) in a general framework, etc. See Malliavin [18] (1997) for an updated development and for the references.

Our second imperative consists to substitute the condition that $\phi \in \bigcap_{1<p<+\infty} \mathbb{D}_{2,p}$ by some weaker one suitable to the condition (1.4) for $p_0 = 2$ (as in the Hoegh-Krohn and Simon theorem). For that purpose we require two new inequalities proven in another recent paper [31] for generalized Ornstein-Uhlenbeck operators (both consequences of the deep Meyer-Bakry inequality). After recalling them, we give the solution of Question 2 in §4. Moreover as applications, three examples are treated in dimension $d = 2$: $P(\phi)$-model, Hoegh-Krohn's exponential model and the sine-Gordon trigonometric models.

2. L^p-uniqueness of Schrödinger operators

2.1. Feynman-Kac semigroup

Throughout this paper, E is a Polish space equipped with some nonnegative σ-finite measure μ on its Borel $\sigma-$field $\mathcal{B}$, charging all open subsets of E. Given a Markov semigroup (P_t) on $L^\infty(E,\mu)$ (i.e., P_t is nonnegative and $P_t 1 = 1$) such that

$$\int_E P_t f d\mu = \int_E f d\mu, \ \forall f \in \mathcal{B}^+, \tag{2.1}$$

(where $\mathcal{B}^+$ denotes the set of all nonnegative $\mathcal{B}$-measurable functions), (P_t) can be regarded as a semigroup of contractions on $L^p(E,\mu)$ for every $p \in [1,+\infty]$. We assume that (P_t) is strongly continuous on $L^p(E,\mu)$ for all $p \in [1,+\infty)$. Its generator in $L^p(E,\mu)$ will be denoted by $(\mathcal{L}, \mathbb{D}_p(\mathcal{L}))$. Given $\lambda, m > 0$, consider the operator

$$V_m(\lambda) := \frac{1}{\Gamma(m/2)} \int_0^\infty t^{m/2-1} e^{-\lambda t} P_t dt = (\lambda - \mathcal{L})^{-m/2}. \tag{2.2}$$

Then $R_\lambda := V_2(\lambda)$ is the resolvent. Define the Sobolev space $(\mathbb{D}_{m,p}, \|\cdot\|_{m,p})$ by

$$\mathbb{D}_{m,p} := V_m(1)\left(L^p(\mu)\right); \ \|V_m(1) f\|_{m,p} := \|f\|_p, \tag{2.3}$$

for any $p \in [1, +\infty]$. Then $\mathbb{D}_{2,p} = \mathbb{D}_p(\mathcal{L}) = R_\lambda(L^p)$ for any $\lambda > 0$.

For some initial probability measure $\nu_0 \sim \mu$ with $d\nu_0/d\mu \in L^\infty(\mu)$ fixed, one can construct a probability measure $\mathbb{P}_{\nu_0}$ on the product space $(\Omega := E^{\mathbb{R}^+}, \mathcal{F}^0_\infty := \mathcal{B}^{\mathbb{R}^+})$ such that the coordinates process (X_t) is a conservative Markov process with initial measure ν_0 and with transition semigroup (P_t). Let $\mathcal{F}$ be the completion of $\mathcal{F}^0_\infty$ by $\mathbb{P}_{\nu_0}$. There is always a $\mathcal{B}(\mathbb{R}^+) \times \mathcal{F}$-measurable $\mathbb{P}_{\nu_0}$-version of (X_t). So without loss of generality we shall assume from now on that (X_t) is itself $\mathcal{B}(\mathbb{R}^+) \times \mathcal{F}$-measurable. Now for any measure $\nu \ll \mu$ and any nonnegative $\mathcal{F}$-measurable function F on Ω, set

$$\mathbb{P}_\nu(A) := \int_A \frac{d\nu}{d\nu_0}(X_0) d\mathbb{P}_{\nu_0}, \ \forall A \in \mathcal{F}$$
$$\mathbb{E}^x F := \mathbb{E}^{\mathbb{P}_{\nu_0}}(F | X_0 = x), \ \nu_0 \sim \mu - a.e.$$

Then $\mathbb{E}^x F = \mathbb{E}^{\mathbb{P}_\nu}(F|X_0 = x), \ \nu - a.e.$

Now given a measurable potential $V : E \to \mathbb{R}$ such that

$$\int_0^t |V|(X_s) ds < +\infty, \ \forall t > 0, \ \mathbb{P}_{\nu_0} - a.e. \tag{2.4}$$

define the Feynman-Kac semigroup

$$P_t^V f(x) := \mathbb{E}^x f(X_t) \exp\left(-\int_0^t V(X_s) ds\right) \tag{2.5}$$

and the adjoint Feynman-Kac semigroup :

$$\widehat{P}_t^V f(x) := \mathbb{E}^\mu \left[f(X_0) \exp\left(-\int_0^t V(X_s) ds\right) | X_t = x \right] \tag{2.6}$$

for any $f \in \mathcal{B}^+$. Then for any $f, g \in \mathcal{B}^+$,

$$\langle P_t^V f, g \rangle_\mu = \langle f, \widehat{P}_t^V g \rangle_\mu := \int_E f \widehat{P}_t^V g d\mu. \tag{2.7}$$

(All terms above are well defined with values in $[0, +\infty]$, μ-a.e.).

When V is bounded, it is well known that $(\mathcal{L} - V, \mathbb{D}_{2,p})$ is the generator of (P_t^V) on $L^p(\mu)$ for each $p \in [1, +\infty)$ (this correspondence is the so called *Feynman-Kac formula*). But when V is unbounded, the study of the Schrödinger operator $-\mathcal{L}+V$ becomes much more difficult, and its L^2-uniqueness is a classical subject, especially for $\mathcal{L}=\Delta/2$, the generator of the Brownian motion, see Kato [11] and Reed-Simon [21].

2.2. A general result for L^p-uniqueness

We begin with the nonnegative potential case $V \geq 0$.

Theorem 2.1 *Let $1 \leq p < +\infty$. Given $0 \leq V \in L^p(\mu)$ and $\mathcal{D} \subset \mathbb{D}_{2,p} \bigcap L^\infty$. If $(\mathcal{L}, \mathcal{D})$ is strongly L^p-unique in the following sense : for any $f \in \mathbb{D}_{2,p} \cap \mathbb{D}_{2,\infty}$, there exists a sequence $(f_n) \subset \mathcal{D}$ so that :*

$$\|f_n - f\|_{2,p} \to 0 \text{ and } \sup_n \|f_n\|_\infty < +\infty, \tag{2.8}$$

then $(\mathcal{L}-V, \mathcal{D})$ is closable in $L^p(\mu)$ and its closure is exactly the generator of the Feynman-Kac semigroup (P_t^V), which is a C_0-semigroup on $L^p(\mu)$. In particular $(\mathcal{L}-V, \mathcal{D})$ is L^p-unique.

Remarks (2.i) : It is a natural extension of the famous Kato theorem : if $0 \leq V \in L^2_{loc}(\mathbb{R}^n)$, $(-\Delta/2 + V; C_0^\infty(\mathbb{R}^n))$ is e.s.a. (it is easy to ckeck that $C_0^\infty(\mathbb{R}^n)$ satisfies (2.8), see the proof of Corollary 2.6(a) below). In [27], it is proven that if $0 \leq V \in L^1_{loc}(\mathbb{R}^n)$, $(-\Delta/2 + V; C_0^\infty(\mathbb{R}^n))$ is L^1-unique.

Now for $p \neq 1, 2$, the above result gives the L^p-uniqueness of $(-\Delta/2 + V; C_0^\infty(\mathbb{R}^n))$ once if $0 \leq V \in L^p(\mathbb{R}^n)$. But using the classical approach (Kato's inequality), we can prove that L^p-uniqueness even for $0 \leq V \in L^p_{loc}(\mathbb{R}^n)$ (this will be carried out in a forthcoming paper [32]).

Theorem 2.2 *Let $1 \leq p \leq q < +\infty$ and $r \in (p, +\infty]$ so that $1/q + 1/r = 1/p$. Given a potential V and a space of test-functions $\mathcal{D}$ satisfying the following two assumptions :*

(A) $\mathcal{D} \subset \mathbb{D}_{2,p} \bigcap L^r(\mu)$ and for any $f \in \mathbb{D}_{2,p} \cap \mathbb{D}_{2,\infty}$, there exists $(f_n) \subset \mathcal{D}$ such that

$$\|f_n - f\|_{2,p} \to 0 \text{ and } \sup_n \|f_n\|_r < +\infty, \tag{2.9}$$

(B) $V \in L^q(\mu)$ and for $a = p, r$,

$$\sup_{0 \leq t \leq 1} \|P_t^V\|_a := \sup_{0 \leq t \leq 1} \sup \left\{ \|P_t^V f\|_a ; f \in \mathcal{B}^+, \|f\|_a \leq 1 \right\} < +\infty, \tag{2.10}$$

then $(\mathcal{L}-V,\mathcal{D})$ *is closable in* $L^p(\mu)$ *and its closure is exactly the generator of the Feynman-Kac semigroup* (P_t^V)*, which is a* C_0*-semigroup on* $L^p(\mu)$*. In particular* $(\mathcal{L}-V,\mathcal{D})$ *is* L^p*-unique.*

Remarks (2.ii) : This result is an extension of Theorem 2.5 in [26]. It gives not only the L^p-uniqueness of the Schrödinger operator, but also the corresponding Feynman-Kac formula. Note also that it contains the e.s.a. result in Kazumi and Shigekawa [12] (Theorem 3.6) as particular case (they used the Kato inequality). The key idea behind this result is : the Schrödinger operator $\mathcal{L}-V$, being unbounded, is a difficult analytical object; but the Feynman-Kac semigroup, being of an explicit probabilistic expression, is bounded and more easier to handle.

The extension from $p=2$ to $p\neq 2$ given in Theorem 2.1 and 2.2 may be quite delicate and have interesting consequences in partial differential equations (see [5], [27] (§6), [28] (Prop.1.1) and [32] etc). Let us explain it by a simple example. Consider the Laplacian Δ on $\mathcal{D}=C_0^\infty(\mathbb{R}^n\backslash\{o\})$ (the space of all infinitely differentiable functions with compact support contained in $\mathbb{R}^n\backslash\{o\}$). It is well known that $(\Delta, C_0^\infty(\mathbb{R}^n\backslash\{o\}))$ is e.s.a. in $L^2(\mathbb{R}^n,dx)$ iff $n\geq 4$ (see [21]), but it is $L^p(\mathbb{R}^n,dx)$-unique iff $p\leq n/2$ (the critical index in the Sobolev space theory, see [5]).

Proof of Theorem 2.1: It follows from Theorem 2.2, because if one takes $q=p$, $r=+\infty$, condition (2.9) becomes (2.8), and condition (2.10) for $a\in[1,\infty]$ is automatically verified for $V\geq 0$. □

The proof of the lemma below is direct, so omitted (it is not used in the proof Theorem 2.2 below, but it is useful for understanding condition (2.10) and for later purpose).

Lemma 2.3 *Let* $p\in[1,+\infty]$*. If* $V:E\to\mathbb{R}$ *satisfies (2.4) and*

$$\sup_{0\leq t\leq 1}\|P_t^V\|_p<+\infty,$$

then (P_t^V) *(resp.* $(\widehat{P}_t^V)$*) is a* C_0*-semigroup on* $L^p(\mu)$ *once if* $p<+\infty$ *(resp. on* $L^{p'}$ *once if* $p'<+\infty$*, where* $1/p+1/p'=1$*).*

Proof of Theorem 2.2 : We shall follow the proof of Theorem 2.5 in [26]. Assume at first $V\geq 0$. Set $V^n:=V\wedge n$. Note that

(i) $\mathcal{D} \subset \mathbb{D}_{2,p}$ = the domain of the generator $\mathcal{L} - V^n$ of $(P_t^{V^n})$;
(ii) $(\mathcal{L} - V^n)f \to (\mathcal{L} - V)f$ in L^p, for all $f \in \mathcal{D}$;
(iii) $\sup_{n\geq 0} \sup_{0\leq t\leq 1} \|P_t^{V^n}\|_p < +\infty$ (≤ 1 in reality).

By a variant version of the Kato-Trotter theorem in Pazy [20] (Th.4.5, p 88), if we can show

$$(1 - \mathcal{L} + V)(\mathcal{D}) \text{ is dense in } L^p(\mu), \tag{2.11}$$

then $(\mathcal{L} - V, \mathcal{D})$ is an essential generator of some C_0-semigroup (P_t^∞) on L^p, and $(P_t^{V^n})$ converges strongly to (P_t^∞) on L^p. On the other hand, by Fatou's lemma, for any $0 \leq f \in L^p$, $P_t^{V^n} f \to P_t^V f$, $\mu - a.e.$. Therefore $P_t^V = P_t^\infty$ and all conclusions of Theorem 2.2 follow. It remains thus to establish the key (2.11).

Let p' (resp. q', r') be the conjugated number of p, i.e., $1/p + 1/p' = 1$ (resp. q, r). For (2.11), it is enough to show that for any $h \in L^{p'}$

$$(\langle h, (1 - \mathcal{L} + V)f\rangle_\mu = 0,\ \forall f \in \mathcal{D}) \implies h = 0. \tag{2.12}$$

We shall do it in two steps.

Step 1. By conditions (A), (B) and Hölder's inequality, $hV \in L^{r'}$. Now for any $f \in \mathbb{D}_{2,p} \bigcap \mathbb{D}_{2,\infty} \subset \mathbb{D}_{2,r}$, let $(f_n) \subset \mathcal{D}$ be the sequence specified by condition (2.9). Since $1 \leq r' < +\infty$ and L^r is the dual space of $L^{r'}$, we can choose a subsequence (f_{n_k}) converging to f w.r.t. the weak* topology $\sigma(L^r, L^{r'})$. Consequently

$$\begin{aligned}\langle h, (1 - \mathcal{L})f\rangle_\mu &= \lim_{k\to\infty} \langle h, (1 - \mathcal{L})f_{n_k}\rangle_\mu \quad \text{(by (2.9))}\\ &= -\lim_{k\to\infty} \langle hV, f_{n_k}\rangle_\mu = -\langle hV, f\rangle_\mu.\end{aligned}$$

Noting that $\langle hV, f\rangle_\mu = \langle \widehat{R}_1(hV), (1 - \mathcal{L})f\rangle_\mu$, we get therefore

$$\langle h, (1 - \mathcal{L})f\rangle_\mu = -\langle \widehat{R}_1(hV), (1 - \mathcal{L})f\rangle_\mu,\ \forall f \in \mathbb{D}_{2,p} \bigcap \mathbb{D}_{2,\infty}. \tag{2.13}$$

Now for any $A \in \mathcal{B}$ with $\mu(A) < +\infty$, $f := R_1 1_A \in \mathbb{D}_{2,p} \bigcap \mathbb{D}_{2,\infty}$ and $(1 - \mathcal{L})f = 1_A$. From (2.13) we deduce

$$\int_A h d\mu = -\int_A \widehat{R}_1(hV) d\mu,$$

where it follows $h = -\widehat{R}_1(hV)$, $\mu - a.e.$. Consequently h belongs to the domain $\mathbb{D}_{r'}(\widehat{\mathcal{L}})$ of the generator $\widehat{\mathcal{L}}$ of $(\widehat{P}_t)$ on $L^{r'}$ and

$$(1 - \widehat{\mathcal{L}})h + Vh = 0 \text{ in } L^{r'}. \tag{2.14}$$

Step 2. For any $\lambda > 0$ and $a \in [1, +\infty)$, consider the resolvent

$$\widehat{R}_\lambda^V g := \int_0^\infty e^{-\lambda t} \widehat{P}_t^V g dt, \ \forall g \in L^a(\mu) \tag{2.15}$$

(the integral above, defined as Stietjes integral, is convergent in operator norm $\|\cdot\|_a$ for all $a \in [1, +\infty)$, because $V \geq 0$). We show now

$$h = \widehat{R}_1^V \left[(1 - \widehat{\mathcal{L}})h + Vh \right] = 0 \tag{2.16}$$

which is the desired result (2.12) . The second equality in (2.16) is trivial by (2.14) . To show the first, we have by the triangular inequality and $h = \widehat{R}_1^{V^n} \left((1 - \widehat{\mathcal{L}})h + V^n h \right)$ (recalling $V^n = V \wedge n$),

$$\begin{aligned}
&\left\| \widehat{R}_1^V \left((1 - \widehat{\mathcal{L}})h + Vh \right) - h \right\|_{r'} \\
&\leq \left\| \left(\widehat{R}_1^V - \widehat{R}_1^{V^n} \right) \left((1 - \widehat{\mathcal{L}})h + Vh \right) \right\|_{r'} + \left\| \widehat{R}_1^{V^n} ((V - V^n)h) \right\|_{r'} \\
&\leq \left\| \left(\widehat{R}_1^V - \widehat{R}_1^{V^n} \right) \left((1 - \widehat{\mathcal{L}})h + Vh \right) \right\|_{r'} + \|(V - V^n)h\|_{r'},
\end{aligned}$$

where the last inequality follows from $\|\widehat{R}_1^{V^n}\|_{r'} \leq 1$ (by the assumption $V \geq 0$). By dominated convergence, $\widehat{R}_1^{V^n} g \to \widehat{R}_1^V g$ in $L^{r'}$ for any $0 \leq g \in L^{r'}$. Thus the last two terms above tend to zero as $n \to \infty$. So (2.16) follows.

We turn now to the general case where $V \in L^q$. The proof is essentially the same and the reader see how condition (2.10) intervenes.

Let $V_n = V \vee (-n)$. Applying the nonnegative case's theorem to $V_n + n$, we get that $(\mathcal{L} - V_n, \mathcal{D})$ is an essential generator of $(P_t^{V_n})$. Note by condition (2.10),

$$\sup_{0 \leq t \leq 1} \|P_t^{V_n}\|_a \leq \sup_{0 \leq t \leq 1} \|P_t^V\|_a < +\infty, \ for \ a = p, r. \tag{2.17}$$

For applying [20] (Th. 4.5, p88) to obtain the desired result, it remains to establish (2.11) or equivalently (2.12) with 1 replaced by

$$\lambda_0 := 1 + \log^+ \left(\sup_{0 \leq t \leq 1} \left(\|P_t^V\|_p \vee \|P_t^V\|_r \right) \right).$$

For such λ_0, the integral (2.15) giving $\widehat{R}_{\lambda_0}^V$ is convergent in the operator norm $\|\cdot\|_{r'}$.

The proof of (2.12) in Step 1 above still works, which yields (2.14) with 1 replaced by λ_0. It remains to prove the first equality in (2.16) with 1 substituted by λ_0. With the same triangular inequality estimation, we are brought to show

- $\widehat{R}^{V_n}_{\lambda_0} g \to \widehat{R}^{V}_{\lambda_0} g$ in $L^{r'}$ for any $0 \le g \in L^{r'}$;
- $\sup_{n\ge 0} \left\| \widehat{R}^{V_n}_{\lambda_0} \right\|_{r'} < +\infty$.

The first property above follows from (2.10), our choice of λ_0 and the a.e. nondecreasing convergence $\widehat{R}^{V_n}_{\lambda_0} g \uparrow \widehat{R}^{V}_{\lambda_0} g$, by monotone convergence. The second follows from (2.17). The proof of Theorem 2.2 is completed. □

2.3. Several corollaries

We present now several corollaries. By following Kato [11] and Chung and Zhao [4] in the classical situation where $\mathcal{L} = \Delta$, we introduce similarly the weak Kato class $\mathcal{K}_w$:

$$V \in \mathcal{K}_w \quad \text{if} \quad esssup_{x\in E} \mathbb{E}^x \exp\left(\int_0^t |V|(X_s) ds \right) < +\infty, \ \forall t > 0. \tag{2.18}$$

(for the usual Kato class, one requires that the last quantity tends to 1 as $t \to 0+$). The following corollary extends the well known Kato theorem which is for $p = 2$, $\mathcal{L} = \Delta$ and $\mathcal{D} = C_0^\infty(\mathbb{R}^n)$:

Corollary 2.4 *Assume that (P_t) is symmetric w.r.t. μ. Under condition (2.8) on $\mathcal{D}$, $V \in L^p$ and $V^- \in \mathcal{K}_w$, then $(\mathcal{L} - V, \mathcal{D})$ is L^p-unique.*

Proof : Condition (2.10) is satisfied for $a = r = \infty$ and by the symmetry, it is also valid for $a = 1$. By interpolation, (P_t^V) satisfies (2.10) for all $a \in [1, +\infty]$. Now this corollary follows from Theorem 2.2 with $q = p$ and $r = +\infty$. □

The following corollary gives another important situation where the condition (2.10) can be verified easily :

Corollary 2.5 *Assume that μ is a probability and that (P_t) is a symmetric Markov semigroup satisfying the defected logarithmic Sobolev inequality*

$$\mathbb{E}^\mu f^2 \log f^2 \le \frac{4}{\lambda} \left(\mathcal{E}(f,f) + m_0 \right), \ \forall f \in \mathbb{D}(\mathcal{E}), \ \|f\|_2 = 1 \tag{2.19}$$

where $(\mathcal{E}(f,f) := \langle\sqrt{-\mathcal{L}}f, \sqrt{-\mathcal{L}}f\rangle_\mu, \mathbb{D}(\mathcal{E}) := \mathbb{D}(\sqrt{-\mathcal{L}}))$ *is the Dirichlet form associated with* (P_t) *acting on* L^2.

Given $1 < p < q, r < +\infty$ *verifying* $1/q + 1/r = 1/p$, *if the following conditions are satisfied by* $(V, \mathcal{D})$ *:*

(i) $V \in L^q$ *and* $e^{-V} \in \bigcap_{1<a<+\infty} L^a(\mu)$ *;*

(ii) $\mathcal{D}$ *satisfies (2.9) ;*

then $(\mathcal{L} - V, \mathcal{D})$ *is* L^p*-unique.*

Remarks (2.iii) : By the Gross Theorem, the defected log-Sobolev inequality (2.19) is equivalent to

- there are $q > p > 1$ and $t > 0$ so that $\|P_t\|_{p,q} < +\infty$.

See Bakry [3]. Corollary 2.5 extends several classical results such as Theorem 4.2 in Hoegh-Krohn and Simon [10], and Th.X.58 and X.59 in Reed-Simon [21] , from $p = 2$ to $p \neq 2$.

Proof of Corollary 2.5. By Theorem 2.2, it is enough to show that for every $r \in (1, \infty)$ fixed,

$$\sup_{0\le t\le 1} \|P_t^V\|_r < +\infty. \tag{2.20}$$

To this end, as $P_t^V f \le P_t^{-V^-} f$ for all $f \ge 0$ and $e^{bV^-} \le \max\{1, e^{-bV}\} \in L^1$ for all $b > 1$, we can assume without loss of generality that $V = -V^- \le 0$.

Now let $\mathbb{D}_{2,\infty}^+ := \{f \in \mathbb{D}_{2,\infty} | \exists \epsilon > 0 : \ f \ge \epsilon\}$, and $V_n = -(V^- \wedge n)$. For $p \in (1, +\infty)$, set

$$\mathcal{E}_p(f) := \langle f^{p-1}, -\mathcal{L}f\rangle_\mu, \ \mathcal{E}_p^V(f) := \langle f^{p-1}, (-\mathcal{L} + V)f\rangle_\mu$$

for $f \in \mathbb{D}_{2,\infty}^+$. The defected log-Sobolev inequality (2.19) implies

$$\int f^p \log f^p d\mu \le c(p)\left(\mathcal{E}_p(f) + m(p)\right)$$

for $f \in \mathbb{D}_{2,\infty}^+$ with $\|f\|_p = 1$, where

$$c(p) = \frac{p^2}{\lambda(p-1)}, \ m(p) := m_0 \frac{4(p-1)}{p^2}$$

(the defected log-Sobolev inequality in L^p, see Bakry [3]). We now translate this inequality into that for $\mathcal{E}_p^{V_n}(f)$.

By Donsker-Varadhan's variational formula for entropy, for any $b > 0$

$$b\int_E V^- f^p d\mu - \int f^p \log f^p d\mu \le \log \int_E e^{bV^-} d\mu := M(b) < +\infty,$$

for all $f \ge 0$ with $\|f\|_p = 1$. Then by the defected log-Sobolev inequality in L^p above, we get

$$b\int_E V^- f^p d\mu \le c(p)\left(\mathcal{E}_p(f) + m(p)\right) + M(b),\ \forall f \in \mathbb{D}^+_{2,\infty},\ \|f\|_p = 1.$$

Taking $b = b(p) := 2c(p)$, we obtain

$$\mathcal{E}_p^V(f) \ge \frac{1}{2}\mathcal{E}_p(f) - \frac{m(p)}{2} - \frac{M(2c(p))}{2c(p)}.$$

Substituting it into the the defected log-Sobolev inequality in L^p above, we get

$$\int f^p \log f^p d\mu \le 2c(p)\left(\mathcal{E}_p^V(f) + m(V;p)\right)$$

for all $f \in \mathbb{D}^+_{2,\infty}$ with $\|f\|_p = 1$, where

$$m(V;p) := \frac{m(p)}{2} + \frac{M(2c(p))}{2c(p)}.$$

The same is true for $\mathcal{E}_p^{V_n}$ too (because $\mathcal{E}_p^{V_n}(f) \ge \mathcal{E}_p^V(f)$). As $\mathbb{D}^+_{2,\infty}$ is stable by $(P_t^{V_n})$, the Gross theorem for equivalence between the L^p-log-Sobolev inequalities and the hyperboundedness (see e.g. [3], Th.3.2 and its proof, p39-41) is applicable for $(P_t^{V_n})$, and it gives us : for any $p > 1, t \ge 0$

$$\|P_t^{V_n}\|_{L^p \to L^{q(t,p)}} \le \exp(\widehat{m}(V;t,p))$$

where

$$q(t,p) := 1 + (p-1)e^{\lambda t/2},\ \widehat{m}(V;t,p) := \int_0^t m(V;q(s,p))ds. \tag{2.21}$$

Letting $n \to \infty$, we obtain by monotone convergence

$$\|P_t^V\|_{L^p \to L^{q(t,p)}} \le \exp(\widehat{m}(V;t,p)) \tag{2.22}$$

which is much stronger than (2.20). $\square$

We now turn to the physical objects introduced in §1.1 and give an affirmative answer to Question 1. The following result extends Theorem 1.1 from $p = 2$ to all $p \in [1, +\infty)$. The Meyer inequality plays a prominent role.

Corollary 2.6 *Let $1 \le p < +\infty$ and $\mathcal{L}$ be the generator of the free quantum field w.r.t. μ on $\mathcal{S}'(R^{d-1})$, given in §1.1.*

a) If $0 \le \mathcal{V} \in L^p$, then $(\mathcal{L} - \mathcal{V}, \mathcal{F}C_0^\infty)$ is L^p-unique ;

b) Under (1.4) for $p_0 = p$, $(\mathcal{L} - \mathcal{V}, \mathcal{F}C_0^\infty)$ is L^p-unique.

Proof : a) By Theorem 2.1, we have only to verify condition (2.8). It is known that $\mathcal{F}C_b^\infty$ is dense in $\mathbb{D}_{2,p}$ for every $1 < p < \infty$ (see [26], Prop.4.1). Since any element $\mathcal{F}C_b^\infty$ can be approximated by those of $\mathcal{F}C_0^\infty$ in $\mathbb{D}_{2,p}$ (easy from (1.3)), then $\mathcal{F}C_0^\infty$ is dense in $\mathbb{D}_{2,p}$ for every $1 < p < \infty$, then for $p = 1$ too.

Now for some $q \in (p, +\infty)$ and for any $f \in \mathbb{D}_{2,p} \bigcap \mathbb{D}_{2,\infty}$, choose $A > a > 0$ so that $\|f\|_\infty < a$, and $(f_n) \in \mathcal{F}C_0^\infty$ so that

$$\|f_n - f\|_{2,2q} \longrightarrow 0.$$

Let $h_A : \mathbb{R} \to \mathbb{R} \in C_0^\infty$ (with compact support) so that $h_A(x) = x$ for any $|x| \le a$ and $|h_A| \le A$ over $\mathbb{R}$. Obviously $\|h_A(f_n) - f\|_p \to 0$ and $\|h_A(f_n)\|_\infty \le A$. For condition (2.8), it remains to show that $h_A(f_n) \in \mathbb{D}_{2,p}$ and

$$\mathcal{L}h_A(f_n) \longrightarrow \mathcal{L}h_A(f) = \mathcal{L}f, \ \textit{in}\ L^p(\mu). \tag{2.23}$$

To this purpose let $\mathbb{D}_e(\mathcal{L})$ be the extended domain of $\mathcal{L}$ in the probabilistic sense given for instance in [31]. We have by the path continuity of the Markov process (X_t) and the Ito formula that $h_A(f_n) \in \mathbb{D}_e(\mathcal{L})$ and

$$\mathcal{L}h_A(f_n) = h'_A(f_n) \cdot \mathcal{L}f_n + h''_A(f_n)\Gamma(f_n, f_n). \tag{2.24}$$

The first term at the RHS converges to $\mathcal{L}f$ in L^p, because $\{h'_A(f_n)\}_n$, being bounded, tends in measure μ to $h'_A(f) = 1$. For the last term, $\{h''_A(f_n)\}_n$ is bounded and tends to $h''_A(f) = 0$ in measure μ. By Meyer's inequality (see [26], Prop.4.1), for some constants C_q, C'_q,

$$\|\sqrt{\Gamma(f_n, f_n)}\|_{2q} \le C_q\|\sqrt{-\mathcal{L}}f_n\|_{2q} \le C'_q\|(1 - \mathcal{L})f_n\|_{2q}. \tag{2.25}$$

Thus $\{\Gamma(f_n, f_n)^p; n \ge 0\}$ is uniformly integrable. Consequently $h_A(f_n) \in \mathbb{D}_{2,p}$ by Lemma 2.3 in [31], and the last term at the RHS of (2.24) tends to 0 in $L^p(\mu)$, too. Hence (2.23) follows.

b) Since the L^p-uniqueness is stronger than L^1-uniqueness in the actual situation (because μ is a probability), we have only to prove it for $1 < p < \infty$. Since $\mathcal{F}C_0^\infty$ is dense in $\mathbb{D}_{2,r}$ for every $1 < r < \infty$ (as recalled above), this part follows directly from Corollary 2.5 by the well known Nelson hypercontractivity for the free quantum field $((X_t), (P_t))$. □

3. Positive improving property in the sense of capacity

We assume from now on that μ is an invariant *probability* measure of (P_t) and the Markov semigroup (P_t) is μ-essentially irreducible, i.e., for any $\lambda > 0$ and for any $A \in \mathcal{B}$ with $\mu(A) > 0$,

$$R_\lambda 1_A := \int_0^\infty e^{-\lambda t} P_t 1_A dt > 0, \ \mu - a.s.$$

i.e., the resolvent R_λ is positive improving (this is equivalent to the ergodicity of the stationary process $\mathbb{P}_\mu$). When (P_t) is symmetric w.r.t. μ, it is well known that the essential irreducibility implies that $P_t, t > 0$ are positive improving. The aim of this section is to strengthen that property by means of the capacity.

We shall place in the set-up of Kazumi and Shigekawa [12]. In the following we fix $m > 0$ and $1 < p < +\infty$. The (m,p)-capacity $Cap_{m,p}$ of an open subset $G \subset E$ is defined as

$$Cap_{m,p}(G) := \inf \left\{ \|V_m f\|_{m,p}^p : V_m f \geq 1, \ \mu - a.s. \ on \ G \right\} \tag{3.1}$$

where $V_m = V_m(1)$ is given in (2.2), and for an arbitrary $A \subset E$,

$$Cap_{m,p}(A) := \inf \left\{ Cap_{m,p}(G) | G \text{ open and } G \supset A \right\}. \tag{3.2}$$

We assume throughout this section the basic assumptions below (in [12]):

(A.1) $\mathbb{D}_{m,p} \cap C_b(E)$ *is dense in* $\mathbb{D}_{m,p}$ *;*

(A.2) there exists an algebra $\mathcal{C} \subset \mathbb{D}_{m,p} \cap C_b(E)$*, separating points of* E *;*

(A.3) the capacity $Cap_{m,p}$ *is tight, i.e., for any* $\epsilon > 0$ *there exists a compact* $K \subset E$ *such that*

$$Cap_{m,p}(E \backslash K) < \epsilon.$$

Now introduce the dual Sobolev space $(\widehat{\mathbb{D}}_{-m,q}, \|\cdot\|\hat{}_{-m,q})$ with negative index as the completion of $L^q(\mu)$ w.r.t. the norm

$$\|g\|\hat{}_{-m,q} := \|\widehat{V}_m g\|_q, \ \widehat{V}_m := \frac{1}{\Gamma(m/2)} \int_0^\infty t^{m/2-1} e^{-t} \widehat{P}_t dt.$$

Theorem 3.1 (see Fukushima and Kaneko [8], Kazumi and Shigekawa [12]) *Let $m > 0$, $p \in (1, +\infty)$ and $1/p + 1/p' = 1$.*

(a) ([8], property (a) in p45) *Every $f \in \mathbb{D}_{m,p}$ admits a $Cap_{m,p}$-quasi-continuous version $\tilde{f}$. For $f, g \in \mathbb{D}_{m,p}$, if $f \geq g$, $\mu - a.s.$, then $\tilde{f} \geq \tilde{g}$, $Cap_{m,p}$-quasi-everywhere (i.e., except a set of $Cap_{m,p}$-zero).*

(b) ([12], Prop.2.6) *$(\widehat{\mathbb{D}}_{-m,p'}, \|\cdot\|\hat{}_{-m,p'})$ is the dual space of $(\mathbb{D}_{m,p}, \|\cdot\|_{m,p})$, and their dual relation $\langle\cdot,\cdot\rangle$ is an extension of the usual one $\langle g, f\rangle_\mu$.*

(c) ([12], Prop.3.3) *If $\varphi \in \widehat{\mathbb{D}}_{-m,p'}$ is nonnegative, i.e., $\langle \varphi, u\rangle \geq 0$ for any $0 \leq u \in \mathbb{D}_{m,p}$, then there is a unique nonnegative measure ν on E charging no set of $Cap_{m,p}$-zero, so that*

$$\langle \varphi, u\rangle = \int_E \tilde{u} d\nu$$

where $\tilde{u}$ is the $Cap_{m,p}$-quasi continuous version of u. The set of all such nonnegative functionals or measures will be denoted by $\widehat{\mathbb{D}}^+_{-m,p'}$.

(d) ([12], Theorem 4.7) *For any Borel subset A of E, $Cap_{m,p}(A) = 0$ if and only if*

$$\nu(A) = 0, \ \forall \nu = \varphi \in \widehat{\mathbb{D}}^+_{-m,p'}.$$

See Malliavin [18] for the theory over an abstract Wiener space.

We now apply those results to get the desired $Cap_{m,p}$-positive improving property. For applying the criterion (d) above, we isolate at first a key condition (ii) below :

Lemma 3.2 *Let $1 < q < +\infty$. Assume that*

(i) $P : L^q(\mu) \to L^q(\mu)$ is μ-positive improving, i.e., for any $f \in L^q_+(\mu)$ (the space of all elements nonnegative and not identically zero μ-a.s. of $L^q(\mu)$), $Pf > 0$, $\mu - a.s.$

(ii) $Q : L^q(\mu) \to \mathbb{D}_{m,p}$ *is a linear bounded and nonnegative operator such that its range* $Ran(Q) := Q(L^q(\mu))$ *is dense in* $\mathbb{D}_{m,p}$.

Then QP *is* $Cap_{m,p}$*-positive improving, i.e., for any* $f \in L^q_+(\mu)$,

$$Cap_{m,p}\left(\widetilde{QPf} \leq 0\right) = 0. \tag{3.3}$$

Proof : Though $[\widetilde{QPf} \leq 0]$ is not Borel measurable, but it is so up to a set of $Cap_{m,p}$-zero. Hence the criterion of Theorem 3.1.(d) is applicable, and it gives us : (3.3) holds if and only if for any nonzero nonnegative measure $\nu = \varphi \in \widehat{\mathbb{D}}^+_{-m,p'}$,

$$\int_E \widetilde{QPf} d\nu = \langle \varphi, QPf \rangle > 0. \tag{3.4}$$

Let $Q^* : \widehat{\mathbb{D}}_{-m,p'} \to L^{q'}(\mu)$ be the dual of $Q : L^q \to \mathbb{D}_{m,p}$. It is easy to see that $Q^*\varphi \geq 0, \mu - a.s.$ by the non-negativeness of Q. Since $Ran(Q)$ is assumed to be dense in $\mathbb{D}_{m,p}$, the kernel of its dual is trivial, i.e., $Q^*\varphi = 0 \implies \varphi = 0$ for any $\varphi \in \widehat{\mathbb{D}}_{-m,p'}$. Therefore $Q^*\varphi \in L^{q'}_+(\mu)$. Consequently by the positive improving property of P, we have

$$\langle \varphi, QPf \rangle = \langle Q^*\varphi, Pf \rangle_\mu > 0$$

the desired result . □

Proposition 3.3 *Let* $m \in (0, 2]$ *and* $1 < p < +\infty$. *For any* $f \in L^p_+(\mu)$,

$$Cap_{m,p}(\widetilde{R_1 f} \leq 0) = 0. \tag{3.5}$$

Proof : By resolvent equation, $R_1 f = R_2 f + R_1 R_2 f \geq R_1 R_2 f, \mu - a.s.$. By Theorem 3.1.(a),

$$\widetilde{R_1 f} \geq \widetilde{R_1 R_2 f},\ Cap_{m,p} - \text{quasi everywhere (in short : q.e.)}.$$

Read $P = R_2$, $Q = R_1$ and $p = q$ in Lemma 3.2. Condition (i) is satisfied by $P = R_2$, by the μ-essential irreducibility assumption. Since $Q = R_1 : L^p \to \mathbb{D}_{2,p}$ is an isomorphism and $\mathbb{D}_{2,p}$ is a dense subset of $\mathbb{D}_{m,p}$ (since $m \in (0, 2]$), condition (ii) in Lemma 3.2 is also satisfied. □

Proposition 3.4 *Let $m > 0$ and $1 < p < +\infty$. Assume moreover that (P_t) is symmetric w.r.t. μ. Then for any $f \in L^p_+(\mu)$ and $t > 0$, $P_t f \in \mathbb{D}_{m,p}$ and*

$$Cap_{m,p}(\widetilde{P_t f} \leq 0) = 0 \tag{3.6}$$

Proof : Since (P_t) is a holomorphic semigroup on $L^p(\mu)$ (see [21], Th.X.55), $P_t : L^p(\mu) \to \mathbb{D}_{m,p}$ is bounded for any $m, t > 0$. For any $t > 0$, writing $P_t = P_{t/2}P_{t/2}$, $P = P_{t/2}$ satisfies condition (i) of Lemma 3.2 (well known). For applying Lemma 3.2, it remains to check its condition (ii), i.e., to show that $P_{t/2}(L^p)$ is dense in $\mathbb{D}_{m,p}$. Let $k \in \mathbb{N}$ such that $2k \geq m$. Since $\mathbb{D}_{2k,p}$ is the domain of $\mathcal{L}^k$ equipped with the graph topology and it is dense in $\mathbb{D}_{m,p}$ (see [12], Prop.2.5. But this is easy), we have only to prove that $P_{t/2}(L^p)$ is a core for $\mathcal{L}^k$. As $P_{t/2}(L^p)$ is stable by $(P_s)_{s\geq 0}$, the last fact follows from the lemma below. □

The following general lemma is used in the proof above and will be useful in Theorem 3.7. It is known for $k = 1$, see [2] (our proof below is simpler and gives a stronger result).

Lemma 3.5 *Let (T_t) be a C_0-semigroup on some Banach space X with generator $\mathcal{L}$. Assume that $\mathcal{D}$ is a linear subspace of $\mathbb{D}(\mathcal{L}^k)$ $(k \geq 1)$, dense in X. If $\mathcal{D}$ is stable for (T_t) (i.e. $T_t(\mathcal{D}) \subset \mathcal{D}$ for all $t \geq 0$), then $\mathcal{D}$ is a core for $\mathcal{L}^k$.*

Proof : Fix some $\lambda > \sup_{0\leq t\leq 1} \log \|T_t\|_X$. λ belongs to the resolvent domain of $\mathcal{L}$. We have only to show that $(\lambda - \mathcal{L})^k(\mathcal{D})$ is dense in X. For this purpose, it is enough to show that for any $y \in X'$ (the dual space of X) satisfying

$$\langle y, (\lambda - \mathcal{L})^k x\rangle = 0, \ \forall x \in \mathcal{D},$$

then $y = 0$ (by the Hahn-Banach theorem). To that end, let (T_t^*) be the adjoint semigroup of (T_t), acting on X'. For any $x \in \mathcal{D} \subset \mathbb{D}(\mathcal{L}^k)$ and $t \geq 0$,

$$\frac{d^k}{dt^k}\langle e^{-\lambda t}T_t^* y, x\rangle = \frac{d^k}{dt^k}\langle y, e^{-\lambda t}T_t x\rangle = \langle y, (-\lambda + \mathcal{L})^k e^{-\lambda t}T_t x\rangle = 0$$

because $e^{-\lambda t}T_t x \in \mathcal{D}$ by our condition. Consequently $h(t) := \langle e^{-\lambda t}T_t^* y, x\rangle$ is a polynomial of degree less than k. But $h(t) \to 0$ as t goes to infinity by our choice of λ. Therefore $h(t) = \langle e^{-\lambda t}T_t^* y, x\rangle = 0$ for all $t \geq 0$. Since $\mathcal{D}$ is dense in X, the previous equality implies $e^{-\lambda t}T_t^* y = 0$ for all $t \geq 0$, i.e., $y = 0$. □

Remarks : In contrast, Lemma 3.5 is no longer true if the stability of $\mathcal{D}$ for (T_t) is substituted by that of $\mathcal{D}$ for the generator $\mathcal{L}$. That can be seen from the following counter-example : (T_t) is the semigroup of the Brownian motion on $\mathbb{R}^n$ and $\mathcal{D} = C_0^\infty(\mathbb{R}^n\backslash\{o\})$. $\mathcal{D}$ is well stable by $\mathcal{L} = \Delta/2$, but it is not a core for $\Delta/2$ in $L^p(\mathbb{R}^n, dx)$ once if $p > n/2$.

The following lemma should be known. Its proof is completely parallel to that of Th.X.55 in [21], then omitted.

Lemma 3.6 *Let (π_t) be a symmetric C_0-semigroup of nonnegative operators on $L^2(E,\mu)$, such that for some $p \neq 2$, $\sup_{t\in[0,1]} \|\pi_t\|_p < +\infty$. Then for any q strictly between p and its conjugated number p', (π_t) is an holomorphic semigroup of some angle $\theta(q) \in (0, \pi/2)$ on L^q. In particular, $\mathcal{L}\pi_t : L^q \to L^q$ is bounded, where $\mathcal{L}$ is the generator of (π_t).*

We now turn to the positive improving property of the Feynman-Kac semigroup (P_t^V).

Theorem 3.7 *Assume (A.1), (A.2) and (A.3) for some $m \in (0, 2]$ and some $1 < p < +\infty$. If*

(i) $V \in L^q(\mu)$ for some $q \in (p, +\infty)$;

(ii) $M := \log \sup_{t\in[0,1]} \left(\|P_t^V\|_p \vee \|P_t^V\|_r\right) < +\infty$, where $1/q + 1/r = 1/p$;

then for any $f \in L_+^r(\mu)$ and for any $\lambda > M$,

$$R_\lambda^V f := \int_0^\infty e^{-\lambda t} P_t^V f dt \in \mathbb{D}_{2,p} \quad \textit{and} \quad Cap_{m,p}(\widetilde{R_\lambda^V f} \leq 0) = 0. \tag{3.7}$$

In the symmetric case, if moreover $\sup_{t\in[0,1]} \|P_t^V\|_a < +\infty$ for some a outside the closed interval between r and r', then for any $t > 0$ and for any $f \in L_+^r(\mu)$, $P_t^V f \in \mathbb{D}_{2,p}$ and

$$Cap_{m,p}(\widetilde{P_t^V f} \leq 0) = 0. \tag{3.8}$$

Proof : 1) Under condition (ii), (P_t^V) is a C_0-semigroup on $L^a(\mu)$ for $a = p, r$, by Lemma 2.3. Let $\mathbb{D}_p(\mathcal{L}^V)$ be the domain of the generator $\mathcal{L}^V$ of (P_t^V) acting on $L^p(\mu)$, equipped with the graph norm $\|f\|_p + \|\mathcal{L}^V f\|_p$. We claim that $L^r \bigcap \mathbb{D}_p(\mathcal{L}^V)$ is continuously embedded in $\mathbb{D}_{2,p}$, i.e., there is some constant $C > 0$ so that

$$\|f\|_{2,p} \le C\left(\|f\|_r + \|\mathcal{L}^V f\|_p\right),\ \forall f \in L^r \bigcap \mathbb{D}_p(\mathcal{L}^V). \tag{3.9}$$

In fact, for all $f \in L^r$, we have by the explicit expression (2.5) and Newton-Leibniz formula,

$$\begin{aligned} P_t^V f(x) &= \mathbb{E}^x f(X_t)\left(1 - \int_0^t V(X_s)\exp\left(-\int_0^s V(X_u)du\right)ds\right) \\ &= P_t f(x) - \int_0^t P_s^V(VP_{t-s}f)ds. \end{aligned} \tag{3.10}$$

(formula of Duhamel type). By Hölder inequality, the equality $1/r + 1/q = 1/p$ and by the contractivity of (P_t), we have for all $t \in [0,1]$,

$$\begin{aligned} \left\|\int_0^t P_s^V(VP_{t-s}f)ds\right\|_p &\le t \cdot \sup_{1\ge t\ge s\ge 0} \|P_s^V(VP_{t-s}f)\|_p \\ &\le t \cdot \sup_{0\le s\le 1} \|P_s^V\|_p \cdot (\|V\|_q\|f\|_r). \end{aligned}$$

Therefore for all $f \in L^r \bigcap \mathbb{D}_p(\mathcal{L}^V)$, $f \in \mathbb{D}_{2,p}$ (by (3.10) and by the strong continuity of (P_t^V) on L^p as well as that of (P_t) on L^r) and

$$\begin{aligned} \sup_{0<t\le 1} \|(P_t f - f)/t\|_p &\le \sup_{0<t\le 1} \|(P_t^V f - f)/t\|_p + \sup_{0\le s\le 1} \|P_s^V\|_p \cdot (\|V\|_q\|f\|_r) \\ &\le \sup_{0\le s\le 1} \|P_s^V\|_p \left(\|\mathcal{L}^V f\|_p + \|V\|_q\|f\|_r\right) \end{aligned}$$

where the desired inequality (3.9) follows.

2) In this step we prove that $\mathbb{D}_r(\mathcal{L}^V)$ is densely embedded into $\mathbb{D}_{2,p}$ by means of (3.9) and of its counterpart below : provided that $1/b + 1/q = 1/a$ where $p \le a \le q$, the Banach space $L^b \bigcap \mathbb{D}_{2,a}$ equipped with norm $\|f\|_b + \|\mathcal{L}f\|_a$ is continuously embedded into $\mathbb{D}_a(\mathcal{L}^V)$. (It can be shown by the same argument as in Step 1) but with the roles of $\mathcal{L}$ and $\mathcal{L}^V$ exchanged.)

For the desired claim, since $\mathbb{D}_{2,\infty}$ is dense in $\mathbb{D}_{2,p}$, it is enough to show that every $f \in \mathbb{D}_{2,\infty}$ can be approximated by elements of $\mathbb{D}_r(\mathcal{L}^V)$ in the norm of $\mathbb{D}_{2,p}$. To this end, fix $f \in \mathbb{D}_{2,\infty}$. Then $f \in \mathbb{D}_p(\mathcal{L}^V)$ by the fact above. For any $\lambda > M$, $f_\lambda := \lambda R_\lambda^V f \in \mathbb{D}_r(\mathcal{L}^V)$ and then $f_\lambda \in \mathbb{D}_{2,p}$ by (3.9). As λ goes to infinity,

$$f_\lambda = \lambda R_\lambda^V f \to f \text{ in } \mathbb{D}_p(\mathcal{L}^V) \text{ and in } L^r.$$

In further by (3.9), as $\lambda \to +\infty$,

$$\|\mathcal{L}(f_\lambda - f)\|_p \le C\left(\|\mathcal{L}^V(f_\lambda - f)\|_p + \|f_\lambda - f)\|_r\right) \to 0$$

too, the desired approximation.

3) Notice that $R_\lambda^V f \in \mathbb{D}_r(\mathcal{L}^V) \subset \mathbb{D}_{2,p}$ (by Step 1)) for any $\lambda > \lambda_0 > M$. Since $R_\lambda^V f \ge (\lambda - \lambda_0) R_{\lambda_0}^V R_\lambda^V f$ (by resolvent equation), for (3.7), we have only to show that

$$Cap_{m,p}\left(\widetilde{R_{\lambda_0}^V R_\lambda^V f} \le 0\right) = 0.$$

We shall apply Lemma 3.2 for $P = R_\lambda^V$ and $Q = R_{\lambda_0}^V$. Obviously $P = R_\lambda^V : L^r \to L^r$ is bounded and μ-positive improving. Since $R_{\lambda_0}^V : L^r \to \mathbb{D}_r(\mathcal{L}^V)$ is an homeomorphism, $R_{\lambda_0}^V : L^r \to \mathbb{D}_{2,p}$ is bounded by (3.9). Then $Q = R_{\lambda_0}^V : L^r \to \mathbb{D}_{m,p}$ is bounded too (since $0 < m \le 2$). It remains to show that the range $R_{\lambda_0}^V(L^r) = \mathbb{D}_r(\mathcal{L}^V)$ is dense in $\mathbb{D}_{m,p}$. But in Step 2), we have shown that $\mathbb{D}_r(\mathcal{L}^V)$ is even densely embedded in $\mathbb{D}_{2,p}$, then in $\mathbb{D}_{m,p}$.

4) For proving (3.8) in the symmetric case, first notice that $P_{t/2}^V : L^r \to \mathbb{D}_r(\mathcal{L}^V)$ is continuous for any $t > 0$, by Lemma 3.6.

For any $t > 0$, write $P_t^V = P_{t/2}^V P_{t/2}^V$. $P = P_{t/2}^V$ is bounded on L^r and μ-positive improving (well known), i.e., it satisfies condition (i) of Lemma 3.2. By the fact shown above and (3.9), $Q = P_{t/2}^V : L^r \to \mathbb{D}_{2,p}$ is continuous. By Lemma 3.5, $Q(L^r) = P_{t/2}^V(L^r)$ is a dense subset of $\mathbb{D}_r(\mathcal{L}^V)$. Thus by Step 2), $Q(L^r) = P_{t/2}^V(L^r)$ is also dense in $\mathbb{D}_{2,p}$, then in $\mathbb{D}_{m,p}$. In other words $Q = P_{t/2}^V$ satisfies condition (ii) of Lemma 3.2. Hence the desired result follows by Lemma 3.2. □

Remarks (3.i) : In Lemma 3.2, we have seen that for the $Cap_{2,p}$-positive improving property, it is required that the range of Q is dense in $\mathbb{D}_{2,p}$, which means exactly that $Ran(Q)$ is a uniqueness domain for determining $\mathcal{L}$ in L^p. In other words the $Cap_{2,p}$-positive improving property is intimately linked with the uniqueness question.

Remarks (3.ii) : The (m,p)-capacity is sensible with m as well as with p. For instance, when $E = \mathbb{R}^n$, $\mathcal{L} = \Delta$, Theorem 3.1 holds for any $m > 0$

and $1 < p < +\infty$ (well known) and all results in this section still holds (though $\mu = dx$ is not a probability). It is well known that (see Ziemer [33], p73-75) $Cap_{m,p}(\{o\}) = 0$ iff $mp > n$. Therefore as soon as $mp > n$, any $Cap_{m,p}$-quasi continuous function is continuous, and any $Cap_{m,p}$-quasi everywhere positive function is positive everywhere. In case that $m = 2$ and $0 \le V \in L^q(\mathbb{R}^n)$, Theorem 3.7 yields that $\widetilde{P_t^V f}$ is continuous and strictly positive everywhere over $\mathbb{R}^n$ for every $f \in L_+^\infty(\mathbb{R}^n, dx)$ as soon as $q > n/2$. But it is well known that "$q > n/2$" is a sharp condition for the continuity and positivity of $P_t^V f$ in this special situation (see e.g. [4]). Now the reader see clearly that Theorem 3.7 is just an extension of that result to the general infinite dimensional setting.

It would be very interesting to prove that $P_t^V f$ is itself (m,p)-quasi-continuous once if (P_t) is "good" (Feller for example). That is the extension of the following well known result in the Dirichlet space $\mathbb{D}_{1,2}$: if P_t is the semigroup of transition of a symmetric Hunt-Markov process, then $P_t^V f$ is $(1,2)$-quasi-continuous for $V \in \mathcal{B}^+ \cap L^1(\mu)$ and $f \in L^2(\mu)$ (see [16]). Since the Hunt realization of (P_t) is up to $Cap_{1,2}$-unique, we can quess that the Hunt regularity is certainly not enough for this aim.

We end this section by a remark on condition (1.7), promised in the Introduction :

Proposition 3.8 *If $\phi \in \mathbb{D}_{m,p}$ and $Cap_{m,p}\left(\tilde{\phi} \le 0\right) = 0$, then as ϵ decreases to zero,*

$$Cap_{m,p}\left(\tilde{\phi} < \epsilon\right) \longrightarrow 0.$$

Proof : Without loss of generality we can assume that $\epsilon = 1/n \to 0$. If in contrary $\inf_{n\ge 1} Cap_{m,p}\left(\tilde{\phi} < 1/n\right) \ge \delta > 0$, then for each $n \ge 1$, we can find compact $K_n \subset A_n := [\tilde{\phi} < 1/n]$ so that

$$Cap_{m,p}(A_n \backslash K_n) < \frac{\delta}{2^{n+1}}$$

(by [12], (2.12)). Since

$$A_n \subset (\cap_{l=1}^n K_l) \bigcup (\cup_{l=1}^n (A_n \cap K_l^c)) \subset (\cap_{l=1}^n K_l) \bigcup (\cup_{l=1}^n (A_l \cap K_l^c))$$

we have

$$\delta \leq Cap_{m,p}(A_n) \leq Cap_{m,p}\left(\cap_{l=1}^{n} K_l\right) + \sum_{l=1}^{n} Cap_{m,p}(A_l \cap K_l^c)$$
$$\leq Cap_{m,p}\left(\cap_{l=1}^{n} K_l\right) + \sum_{l=1}^{n} \frac{\delta}{2^{l+1}}$$

where it follows

$$Cap_{m,p}\left(\cap_{l=1}^{n} K_l\right) \geq \delta/2, \ \forall n \geq 1.$$

But by the property of the (m,p)-capacity for decreasing sequence of compacts,

$$Cap_{m,p}\left(\cap_{l\geq 1} K_l\right) = \lim_{n\to\infty} Cap_{m,p}\left(\cap_{l=1}^{n} K_l\right) \geq \delta/2$$

which is in contradiction with the fact that

$$Cap_{m,p}\left(\cap_{l\geq 1} K_l\right) \leq Cap_{m,p}(\tilde{\phi} \leq 0) = 0.$$

The proof is finished. □

4. L^2-uniqueness of the ground state diffusion

The main purpose of this section is to solve Question 2.

Theorem 4.1 *In the Euclidean quantum field setting of §1.1, assume that the interaction potential $\mathcal{V}$ satisfies (1.4) for $p_0 = 2$. Then for the generator $\mathcal{L}^\phi$ given by (1.6), where ϕ is the ground state specified in Theorem 1.1, $(\mathcal{L}^\phi, \mathcal{F}C_0^\infty)$ is essentially self-adjoint on $L^2(\mathcal{S}'(\mathbb{R}^{d-1}), \mu_\phi = \phi^2\mu)$.*

Besides Theorem 3.7, we require also the following crucial

Lemma 4.2 (see [31]) *For the generalized Ornstein-Uhlenbeck generator $\mathcal{L}$ given by (1.3) and the associated square-field operator Γ, we have*

(a) If $h \in C^2(\mathbb{R})$ satisfies

$$|h'(t)| \leq M, \ |t \cdot h''(t)| \leq M, \ \forall t \in \mathbb{R}, \tag{4.1}$$

then for any $p \in [2, +\infty)$ and for all $u \in \mathbb{D}_{2,p}$ with $u \geq 0$, $\mu - a.s.$, $h(u) \in \mathbb{D}_{2,p}$ and there exists some universal constant $C(p)$ depending only on p such that

$$\|\mathcal{L}h(u)\|_p \leq MC(p)\|\mathcal{L}u\|_p \tag{4.2}$$

(b) Let $1 \le p \le q < r \le +\infty$ satisfy $1/q + 1/r = 1/p$ and $q \ge 2$. Then there exists some universal constant $C(p,q)$ such that

$$\|\Gamma(u,u)\|_p \le C(p,q)\|\mathcal{L}u\|_q \cdot \|u\|_r, \ \forall u \in L^r \bigcap \mathbb{D}_{2,q}. \tag{4.3}$$

In particular for all $u, v \in L^r \bigcap \mathbb{D}_{2,q}$,

$$\|\mathcal{L}(uv)\|_p \le \|u\|_r\|\mathcal{L}v\|_q + \|v\|_r\|\mathcal{L}u\|_q + 2C(p,q)\sqrt{\|u\|_r\|v\|_r\|\mathcal{L}u\|_q\|\mathcal{L}v\|_q}.$$

It is a particular case of Corollary 5.2 in [31] (where the extra condition "$h(0) = 0$" for part (a) can be removed automatically, because one may consider $h(x) - h(0)$ instead of h in the probability space case). In fact, let H be the usual Sobolev-Lax space on $\mathbb{R}^{d-1}$ with index $(-1/4, 2)$ equipped with inner product $\langle f, g\rangle_H := \langle (2B)^{-1/2}f, (2B)^{-1/2}g\rangle_{L^2(\mathbb{R}^{d-1},dx)}$ where $B = \sqrt{-\Delta_{d-1} + m^2}$. We can always find another separable Hilbert space $\tilde{H}$ such that $H \subset \tilde{H} \subset \mathcal{S}'(\mathbb{R}^{d-1})$ (continuous and dense embedding) and $(\tilde{H}, H, \mu)$ constitutes a triplet of abstract Wiener space. By (1.1), it is easy to see that the transition semigroup (P_t) of (X_t) is exactly the second quantization of $(T_t = e^{-Bt})$ acting on H. So Corollary 5.2 in [31] is applicable.

We are now ready to give the

Proof of Theorem 4.1: Its proof is divided into three steps : the first consists to draw informations about ϕ furnished by condition (1.4); in the second step we translate the question into the denseness of $\phi\mathcal{F}C_0^\infty$ in $L^r \bigcap \mathbb{D}_{2,2+}$, by means of Theorem 2.2; and that last property is verified in Step 3.

1) By Corollary 2.5 (and its proof) and Lemma 2.3, under the second condition in (1.4), $(P_t^{\mathcal{V}})$ is a C_0-semigroup on all L^r, $1 < r < +\infty$. Condition (ii) in Theorem 3.7 is verified for all $1 < p, r < +\infty$ and its condition (i) is satisfied for some $q > 2$. Moreover by the hyperboundedness of (P_t^V) shown in (2.22) and by $P_t^{\mathcal{V}}\phi = e^{-\lambda(\mathcal{V})t}\phi$, we have $\phi \in \bigcap_{r>1} L^r$. Then by Theorem 3.7, the ground state ϕ belongs to $\mathbb{D}_{2,a}$ for some $a \in (2, q)$.

To get the strict positivity of $\tilde{\phi}$ in the sense of (1.7) by means of Theorem 3.7, we should be sure that condition (A.1), (A.2) and (A.3) in §3 are satisfied for $m = 2$ and for any $p \in (1, +\infty)$. Conditions (A.1), (A.2) are contained in [26] (Prop.4.1). The key tightness condition (A.3) is due to Feyel and de la Pratelle [7].

Since $P_t^V\phi = e^{-\lambda(\mathcal{V})t}\phi$, $Cap_{2,a}(\tilde{\phi} \le 0) = 0$ by Theorem 3.7. Finally by Proposition 3.8, $Cap_{2,a}(\tilde{\phi} < \epsilon) \to 0$ as ϵ decreases to zero.

2) Note that $u \mapsto \phi u$ is an isomorphism from $L^2(\phi^2\mu = \mu_\phi)$ to $L^2(\mu)$ and for any $u \in \mathcal{F}C_0^\infty$, $u\phi \in \bigcup_{p>2} \mathbb{D}_{2,p}$ by Lemma 4.2.(b). We have the following ground state representation :

$$(\mathcal{L} - \mathcal{V} + \lambda(\mathcal{V}))(\phi u) = \phi \mathcal{L}^\phi u$$

for any $u \in \mathcal{F}C_0^\infty$. Consequently $(\mathcal{L}^\phi, \mathcal{F}C_0^\infty)$ is e.s.a. in $L^2(\mu_\phi)$ if and only if $(\mathcal{L} - \mathcal{V}, \phi\mathcal{F}C_0^\infty)$ is $L^2(\mu)$-unique. By Theorem 2.2, we have only to show that for any $r > 2$ sufficiently large and for any $u \in \mathbb{D}_{2,\infty}$, there exists a sequence $v_n \in \mathcal{F}C_0^\infty$ so that

$$\|\mathcal{L}(\phi v_n - u)\|_2 \to 0, \ \|\phi v_n - u\|_r \to 0. \tag{4.4}$$

In the other words, the closure of $\phi\mathcal{F}C_0^\infty$ w.r.t. the norm $\|\cdot\|_r + \|\cdot\|_{2,2}$ contains $\mathbb{D}_{2,\infty}$.

3) For the simplicity of notation we adopt the following convention : $2+$ denotes any number strictly larger than 2, but sufficiently close to 2 ; $\infty-$ denotes any number large enough. Moreover they may change from one place to other. For the clarity we divide the verification of (4.4) into three points.

3.1) Note that the closure of $\mathcal{D}$ w.r.t. the norm $\|\cdot\|_{\infty-} + \|\cdot\|_{2,2+}$ contains $\mathbb{D}_{2,\infty}$ by the proof of Corollary 2.6. Moreover $\mathbb{D}_{2,\infty}$ is dense in $L^{\infty-} \bigcap \mathbb{D}_{2,2+}$. This is quite easy. Indeed for any $f \in L^{\infty-} \bigcap \mathbb{D}_{2,2+}$ and for any $\epsilon > 0$,

$$\|\lambda R_\lambda(f) - f\|_{\infty-} + \|\mathcal{L}(\lambda R_\lambda(f) - f)\|_{2+} < \epsilon$$

when $\lambda > 0$ is sufficiently large. Fix such a λ and next choose $g \in L^\infty$ so that $\|f - g\|_{\infty-} < \epsilon$ and $\|f - g\|_{2+} < \epsilon/C$ where C is the norm of $\lambda R_\lambda : L^{2+} \to \mathbb{D}_{2,2+}$ (which is bounded). Therefore $f_\epsilon := \lambda R_\lambda g \in \mathbb{D}_{2,\infty}$ satisfies

$$\|f_\epsilon - f\|_{\infty-} + \|\mathcal{L}(f_\epsilon - f)\|_{2+} < 3\epsilon$$

the desired claim.

Thus for (4.4), by Lemma 4.2.(b), it is enough to show that any $u \in \mathbb{D}_{2,\infty}$ can be approximated by elements of $\phi(L^{\infty-} \bigcap \mathbb{D}_{2,2+})$ in $L^{\infty-} \cap \mathbb{D}_{2,2+}$.

3.2) This point is crucial : since $Cap_{2,2+}(\tilde{\phi} < \epsilon) \to 0$, then there exists some nonnegative function $g_\epsilon \in \mathbb{D}_{2,2+}$ so that (see [8]),

$$\tilde{g}_\epsilon \geq 1, \ Cap_{2,2+} - q.e. \text{ over } \ [\tilde{\phi} < \epsilon] \text{ and } \ \|g_\epsilon\|_{2,2+} = Cap_{2,2+}(\tilde{\phi} < \epsilon) \to 0$$

as $\epsilon \to 0$. Let $h \in C_b^2(\mathbb{R})$ satisfy $h(t) = 1$ for all $t \geq 1$ and (4.1). By Lemma 4.2.(a), we have

$$h(\tilde{g}_\epsilon) = 1, \; Cap_{2,2+} - q.e. \text{ over } [\tilde{\phi} < \epsilon] \text{ and } \|h(g_\epsilon)\|_{2,2+} \to 0. \tag{4.5}$$

3.3) Now for any $u \in \mathbb{D}_{2,\infty}$, put

$$v_\epsilon := \phi^{-1}\,(1 - h(g_\epsilon))\,u, \;\; u_\epsilon := \phi v_\epsilon = (1 - h(g_\epsilon))\,u$$

where the convention $0/0 = 0$ is used. Let $a_\epsilon \in C_b^2(\mathbb{R})$ be some function satisfying $a_\epsilon(t) = t^{-1}$ for all $t \geq \epsilon$ and condition (4.1) (for some M depending on ϵ). Then by Lemma 4.2.(a), $a_\epsilon(\phi) \in L^\infty \bigcap \mathbb{D}_{2,2+}$. Next applying twice part (b) of Lemma 4.2, we get

$$v_\epsilon = \phi^{-1}\,(1 - h(g_\epsilon))\,u = a_\epsilon(\phi)\,(1 - h(g_\epsilon))\,u \in L^{\infty-} \bigcap \mathbb{D}_{2,2+}.$$

Therefore $u_\epsilon = \phi v_\epsilon \in \phi(L^{\infty-} \bigcap \mathbb{D}_{2,2+})$. By point 3.1), it remains to show that u_ϵ tends to u in $L^{\infty-} \bigcap \mathbb{D}_{2,2+}$. But

$$u - u_\epsilon = h(g_\epsilon)u$$

tends to zero obviously in $L^{\infty-}$, and it converges to zero in $\mathbb{D}_{2,2+}$ too, by (4.5) and Lemma 4.2.(b). □

Remarks (4.i) : Though in this paper $d \geq 1$ is arbitrary, but for well defining most part of useful physical interaction potentials, we need that $d \leq 2$ (and $d = 1$ is reduced to finite dimensional case). Let us discuss three typical physical models in dimension $d = 2$. In general the interaction potential takes the following formal form on $\mathcal{S}'(\mathbb{R}) \ni S$,

$$\mathcal{V}(S) := \int_{\mathbb{R}} g(x) : V(S(x)) :_\mu dx = \sum_{n=0}^{\infty} a_n \int_{\mathbb{R}} g(x) : S(x)^n :_\mu dx, \tag{4.6}$$

where $0 \leq g \in L^1(\mathbb{R}, dx) \bigcap L^2$ is some space cut-off function, and $V(r) = \sum_{n \geq 0} a_n r^n$ is some holomorphic function on $\mathbb{C}$ representing the classical interaction ($\mathcal{V}$ is a type of quantization of V), and $: S(x)^n :_\mu$ is the Wick ordering associated with the Gaussian measure μ on $\mathcal{S}'(\mathbb{R})$.

Example 4.3. $: P(\phi) :$ **model.** For this model $V(r) = P(r)$ is a lower bounded polynomial of degree $2k \geq 4$. It is well known that the corresponding $\mathcal{V}$ satisfies (1.4) for all $p_0 > 1$ (see e.g. Simon [22]). The essential self-adjointness of the Schrödinger operator $(-\mathcal{L}+\mathcal{V}, \mathcal{F}C_0^\infty)$ was established by Glimm-Jaffe, Rosen, Segal, Hoegh-Krohn and Simon among others during the years 1965-1975 of constructive quantum fields (see [10], [22] for historical comments).

By Corollary 2.6, $(\mathcal{L}-\mathcal{V}, \mathcal{F}C_0^\infty)$ is unique in $L^p(\mu)$ for all $1 \le p < +\infty$. By Theorem 4.1, its associated ground state diffusion generator $(\mathcal{L}^\phi, \mathcal{F}C_0^\infty)$ is essentially self-adjoint in $L^2(\phi^2\mu)$.

We remark that for this model, the L^2-uniqueness of the ground state diffusion follows directly from Theorem 3.5 in [26] and from Theorem 3.7 (without using Lemma 4.2), because $\phi \in \bigcap_{p>1} \mathbb{D}_{2,p}$ (a consequence of Theorem 3.7 by the fact that $\mathcal{V} \in L^{\infty-}$).

Example 4.4. Hoegh-Krohn exponential model. In this model the potential is given by (4.6) for $V(r) = e^{\lambda r}$, where λ is some real number. When $|\lambda| < \sqrt{2\pi}$, it is known that $\mathcal{V}_\lambda$ given by (4.6) is in L^2 and nonnegative, and $(-\mathcal{L}+\mathcal{V}_\lambda, \mathcal{F}C_0^\infty)$ is essentially self-adjoint (see Simon [22], Th.V.25).

For applying Theorem 4.1, we have to verify that $\mathcal{V}_\lambda \in L^{2+}(\mu)$. To this purpose let (O_t) be the standard Ornstein-Uhlenbeck semigroup acting on $L^2(\mu)$, i.e., $O_t u = e^{-nt}u$ for u belonging to the $n-th$ chaos of $L^2(\mu)$. We have the following relation :

$$O_t \mathcal{V}_{e^t\lambda} = \mathcal{V}_\lambda.$$

But by the hypercontractivity of Nelson for (O_t),

$$O_t : L^2(\mu) \to L^{q(t)}(\mu)$$

is contractive where $q(t) = 1 + e^{2t}$. We deduce that if $\lambda^2 < 2\pi$:

$$\mathcal{V}_\lambda \in L^p, \ \forall p < p(\lambda) = 1 + \frac{2\pi}{\lambda^2}. \tag{4.7}$$

Hence Theorem 4.1 is applicable and it gives the $L^2(\phi^2\mu)$-uniqueness of the associated ground state diffusion generator restricted to $\mathcal{F}C_0^\infty$. Moreover by Corollary 2.6, $(-\mathcal{L}+\mathcal{V}_\lambda, \mathcal{F}C_0^\infty)$ is unique in L^p for all $1 \le p < p(\lambda)$, extending the L^2-uniqueness result mentioned above.

Example 4.5. Sine-Gordon trigonometric model. In this model $V(r) = \cos(\lambda r + \theta_0)$ where θ_0 is a constant. Then

$$\mathcal{V}_\lambda(S) = \int_{\mathbb{R}} g(x) \left(\cos\theta_0 : \cos(\lambda S(x)) :_\mu - \sin\theta_0 : \sin(\lambda S(x)) :_\mu\right) dx.$$

From the exponential model and the chaos decomposition, we see that $\mathcal{V}_\lambda \in L^2$ as long as $\lambda^2 < 2\pi$. By the same argument as in Example 4.4, $\mathcal{V}_\lambda$ satisfies still (4.7).

The second condition in (1.4) is also satisfied by this potential (due to Frölich). Then Theorem 4.1 and Corollary 2.6 are applicable, and yield the same results as for the exponential model above.

For the last two models, Theorem 3.5 in [26] is not applicable for the L^2-uniqueness of the associated ground state diffusion generators.

Remarks (4.ii): The $L^2(\mu_\phi)$-uniqueness in Theorem 4.1 for the ground state diffusion implies the $L^p(\mu_\phi)$-uniqueness for all $1 \le p < 2$ (because μ_ϕ is a probability). Notice that some general results about the $L^p(\mu_\phi)$-uniqueness for $1 \le p < 2$ are obtained by Eberle [5] (Corollary 5.4), but one of his conditions being

$$\phi^{(2-p)/p} \in \mathbb{D}_{1,2p/(2-p)},$$

is not well adapted to the ground or excited state diffusion (except for $p = 1$). Note also that the results of Liskevitch and Röckner [15] are neither applicable here.

What happens for the $L^p(\mu_\phi)$-uniqueness for $p > 2$? This is a challenging open question. Our approach is not at all well adapted for $p > 2$, because the ground state representation used in the proof above is only an isomorphism in L^2, not in L^p for $p \neq 2$.

Remarks (4.iii): We conclude this paper by recalling that $(\mathcal{L}^\phi, \mathcal{F}C_b^\infty)$ is **not** essentially self-adjoint on $L^2(\mathcal{S}'(\mathbb{R}^{d-1}, \mu_\phi)$ for any excited state ϕ, in contrast with Theorem 4.1 for the ground state, see [29] where two explicit and different "physical" self-adjoint extensions are given. We can also guess this negative result from the proof above, in which two essential properties are not satisfied by excited states :

1) $\tilde{\phi} > 0$, $Cap_{2,2+} - q.e.$;

2) Lemma 4.2(a) holds only for $\phi \ge 0$ (see [31]).

References

[1] S. Albeverio, J.G. Kondratiev, M. Röckner: An approximate criterion of essential self-adjointness of Dirichlet operators. *Potential Analysis*, **1**, p307-317, 1992.

[2] W. Arendt : The abstract Cauchy problem, special semigroups and perturbation. In *One parameter semigroups of positive operators,* ed. by R. Nagel. Lect. Notes in Math. **1184**, Springer, Berlin 1986.

[3] D. Bakry : *L'hypercontractivité et son utilisation en théorie des semigroupes.* Ecole d'Eté de Probabilités de Saint-Flour (1992), LNM 1581, Springer-Verlag 1994.

[4] K.L. Chung and Z.X. Zhao : *From Brownian Motion to Schrödinger's Equations.* Grundlehren der mathematischen Wissenschaften 312, Springer-Verlag 1995.

[5] A. Eberle : Uniqueness and non-uniqueness of singular diffusion operators. Ph.D Dissertation, Bielefeld 1997.

[6] D. Feyel and A. de la Pratelle : capacités gaussiennes. *Ann. Inst. Fourier* **41**, *p49-76* (Grenoble) 1991.

[7] D. Feyel and A. de la Pratelle : Opérateurs linéaires gaussiennes. *Preprint 1991*

[8] Fukushima and Kaneko : On (r, p)-capacity for general Markovian semigroups. In *Infinite dimensional analysis and stochastic processes, p41-47,* ed. by S. Albeverio, Pitman, 1985.

[9] J. Glimm and A. Jaffe : *Quantum Physics - A Functional Integral Point of View.* Second Edition, Springer-Verlag 1987.

[10] R. Hoegh-Krohn and B. Simon : Hypercontractive semigroups and two dimensional self-coupled Bose fields. *J. Funct. Anal.* **9**, *p121-180, 1972.*

[11] T. Kato : *Perturbation Theory For Linear Operators,* 2nd ed. (2nd corrected printing), Springer-Berlin, 1984.

[12] T. Kazumi and I. Shigekawa : Measures of finite (r, p)-energy and potentials on a separable metric space. *Lect. Notes in Math.* **1526**, *Sém. Probab. XXVI, p415-444,* Springer-Verlag 1992.

[13] V. Liskevitch : On the uniqueness problem for Dirichlet operators. *J. Funct. Anal.* **162**, *p1-13*, 1999.

[14] V. Liskevitch and Y. Semenov : Dirichlet operators : a-priori estimates and the uniqueness problem. *J. Funct. Anal.* **109**, *p199-213*, 1992.

[15] V. Liskevitch and M. Röckner : Strong uniqueness for a class of infinite dimensional Dirichlet operators and applications to stochastic quantization. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **(4) 27**, *No.1, p69-91*, 1998.

[16] Z.M. Ma and M. Röckner: *Introduction to the Theory of (non symmetric) Dirichlet Forms.* Springer-Verlag, 1992.

[17] P. Malliavin : Implicit functions in finite co-rank on the Wiener space. In *Proc. Taniguchi Symp. on Stochastic Analysis,* Kinokuniya, 1984.

[18] P. Malliavin : *Stochastic Analysis*, Grundlehren der mathematischen Wissenschaften 313, Springer-Verlag 1997.

[19] E. Nelson : The free Markov field. *J. Funct. Anal.* **12**, *p211-227, 1973.*

[20] A. Pazy : *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Applied Math. Sciences **44**, Springer-Verlag 1983.

[21] M. Reed and B. Simon: *Methods of Modern Mathematical Physics, II, Fourier Analysis, Self-adjointness.* Academic Press, 1975.

[22] B. Simon: *The $P(\phi)_2$ Euclidean (Quantum) Field Theory.* Princeton Series in Physics, Princeton Univ. Press 1974.

[23] S.Q. Song : Markov uniqueness and essential self-adjointness of perturbed Ornstein-Uhlenbeck operators. *Osaka J. Math.* **32**, p823-832, 1995.

[24] W. Stannat : The abstract Cauchy problem, special semigroups and perturbation. The abstract Cauchy problem, special semigroups and perturbation. Preprint (Bielefeld), 1997.

[25] N. Wielens: On the essential self-adjointness of generalized Schrödinger operators. *J. Funct. Anal. 61, pp98-115 (1985).*

[26] L.M. WU: Feynman-Kac semigroups, ground state diffusions and large deviations. *J. Funct. Anal. 123, Number 1, p202-231,1994.*

[27] L.M. WU: Uniqueness of Schödinger operateurs restricted to a domain. *J. Funct. Anal.* **153**, *p276-319 (1998)*

[28] L.M. WU: Uniqueness of Nelson's diffusions. *Probab. Theory Rel. Fields* **114**, *p549-585 (1999)*

[29] L.M. WU: Uniqueness of Nelson's diffusions (II) : infinite dimensional setting and applications. *to appear in Potential Analysis (2000)*

[30] L.M. WU: Uniformly integrable operators and large deviations for Markov processes. *to appear in JFA* **170** *(2000).*

[31] L.M. WU: Two inequalities for symmetric diffusion Markov semigroups under $\Gamma_3 \geq 0$. *Preprint 2000, to appear in JFA.*

[32] L.M. WU: L^p-uniqueness for second order differential operators. *in preparation*

[33] W.P. Ziemer : *Weakly differentiable functions.* Springer-Verlag 1989