Mathematical models for eddy current problems in thin conductors

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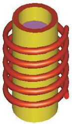
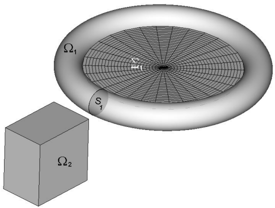
Motivation

- Typical configuration of an eddy current setup:
  A set of conductors of different thicknesses.
- *Typically:* An inductor (thin conductor: coil, wire, ...) that carries electric current + a "thick" conductor.
- This results in difficulties in numerical simulation (needs a fine mesh for the inductors, ill conditioning, ...)
- Remedy: Asymptotic analysis to obtain models that couple field equations and circuit equations.





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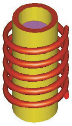

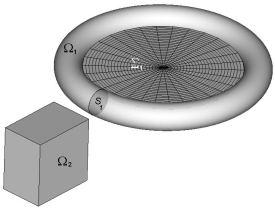

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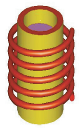

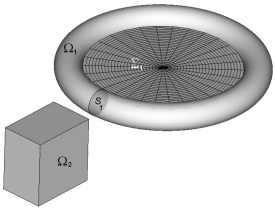

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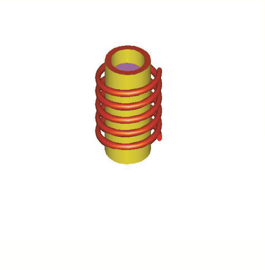
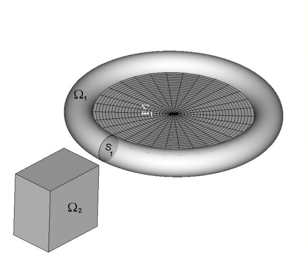




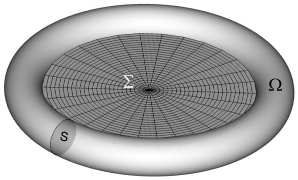
Eddy current equations in time harmonic regime:

$$\Omega : \text{ union of conductors}, \quad \Omega' = \mathbb{R}^3 \setminus \overline{\Omega}.$$

$$\begin{cases} \mathbf{curl}\,\mathbf{H} = \mathbf{J} & \text{in } \mathbb{R}^3 \\ i\omega\mu\mathbf{H} + \mathbf{curl}\,\mathbf{E} = 0 & \text{in } \Omega \cup \Omega' \\ \mathbf{J} = \sigma\mathbf{E} & \text{in } \Omega \\ \mathbf{J} = 0 & \text{in } \Omega' \end{cases}$$

To simplify, we assume μ (magnetic permeability) and σ (electric conductivity) constant.

The problem is not completely solved.
We aim at computing the inductance coefficient of toroidal thin conductor.



Let Ω_ϵ denote a toroidal domain with inner radius $\epsilon > 0$, $\Omega'_\epsilon = \mathbb{R}^3 \setminus \overline{\Omega}_\epsilon$, Σ a cut in $\mathbb{R}^3$ ($\Omega'_\epsilon \setminus \Sigma$ is simply connected).

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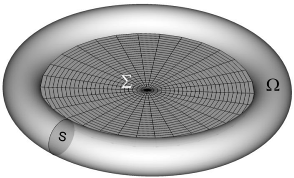

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We have

$$\mathbf{curl}\,\mathbf{H}^\epsilon = 0 \qquad \text{in } \Omega'_\epsilon.$$

Therefore

$$\mathbf{H}^\epsilon = \nabla\varphi^\epsilon + \alpha\nabla p^\epsilon \qquad \text{in } \Omega'_\epsilon \setminus \Sigma,$$

where $\alpha \in \mathbb{C}$ and

$$\begin{cases} \Delta p^\epsilon = 0 & \text{in } \Omega'_\epsilon \setminus \Sigma \\ [p^\epsilon]_\Sigma = 1, \ \left[\dfrac{\partial p^\epsilon}{\partial n}\right]_\Sigma = 0 & \\ \dfrac{\partial p^\epsilon}{\partial n} = 0 & \text{on } \partial\Omega_\epsilon \\ p^\epsilon(x) = O(|x|^{-1}) & |x| \to \infty \end{cases}$$

The inductance coefficient is defined by

$$L^\epsilon = \mu \int_{\Omega'_\epsilon \setminus \Sigma} |\nabla p^\epsilon|^2 \, dx = \mu \int_\Sigma \frac{\partial p^\epsilon}{\partial n} \, ds$$

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Remark

When $\epsilon \longrightarrow 0$, we have $L^\epsilon \longrightarrow +\infty$.

Theorem

We have the expansion

$$L^\epsilon = -\frac{\mu\ell}{2\pi} \log \epsilon + a + O(\epsilon^{\frac{5}{3}-\eta}) \qquad \eta > 0,$$

where a is a real number given explicitly and ℓ is the length of the curve that "generates" Ω_ϵ.

Remarks

- This result is given in the book of LANDAU and LIFSHITZ (without proof) without expliciting the term a.
- In the RL circuit equation, we have

$$(i\omega L + R)\, I = V.$$

This suggests that $I^\epsilon = O(1/\log \epsilon)$.

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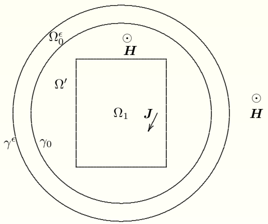

If $\mathbf{J} = (J_1, J_2, 0)$, we prove that $\mathbf{H} = (0, 0, u^\epsilon)$ where u^ϵ is solution of the problem:

$$\begin{cases} -\sigma^{-1}\Delta u^\epsilon + i\omega\mu u^\epsilon = 0 & \text{in } \Omega^\epsilon := \overline{\Omega}_1 \cup \overline{\Omega}' \cup \Omega_0^\epsilon \\ u^\epsilon = 0 & \text{on } \gamma^\epsilon \\ u^\epsilon = \text{Const.} & \text{in } \Omega' \end{cases}$$

We define the space

$$V = \{v \in H_0^1(\Omega^\epsilon);\ v_{|\Omega'} = \text{Const.}\}.$$

We have the variational formulation:

$$\begin{cases} u^\epsilon \in V, \\ \sigma^{-1} \int_{\Omega_1} \nabla u^\epsilon \cdot \nabla \overline{v}\, dx + \sigma^{-1} \int_{\Omega_0^\epsilon} \nabla u^\epsilon \cdot \nabla \overline{v}\, dx + i\omega\mu \int_{\Omega^\epsilon} u^\epsilon \overline{v}\, dx = f\, \overline{v}_{|\Omega'} \end{cases}$$

where $f \in \mathbb{C}$ if the current voltage.

Let us assume that $\sigma^{-1} = a\epsilon$ (or $\sigma^{-1} = O(\epsilon)$).

We then prove that ($\Omega = \overline{\Omega}_1 \cup \Omega'$)

$$u^\epsilon \longrightarrow u \qquad \text{in } H^1(\Omega) \quad \text{when } \epsilon \longrightarrow 0$$

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where ℓ is the length of γ_0.

Remark

If $\Omega_1 = \emptyset$ and $\varphi = u_{|\partial\Omega}$, we obtain the algebraic equation

$$(i\omega\mu|\Omega'| + a\,\ell)\,\varphi = f.$$

If we set

$$L \equiv \mu\,|\Omega'|, \qquad R \equiv \frac{\ell\epsilon}{\sigma},$$

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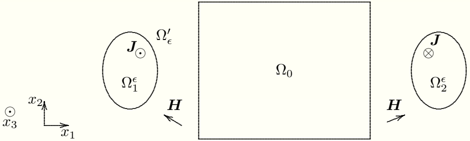

We define $\Omega^\epsilon = \Omega_0 \cup \Omega_1^\epsilon \cup \Omega_2^\epsilon$. We then look for a solution that satisfies $\mathbf{J} = (0, 0, J)$. From this, we deduce $\mathbf{H} = (H_1, H_2, 0)$ with

$$\begin{cases} \operatorname{curl} \mathbf{H} = J & \text{in } \mathbb{R}^2 \\ J = 0 & \text{in } \Omega'_\epsilon \\ i\omega\mu\mathbf{H} + \mathbf{curl}\,(\sigma^{-1} J) = 0 & \text{in } \Omega_\epsilon \\ \operatorname{div}(\mu\mathbf{H}) = 0 & \text{in } \mathbb{R}^2 \end{cases}$$

A transversal 2-D model

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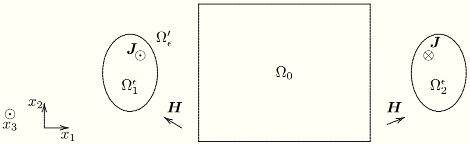

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The identity $\operatorname{div}(\mu \mathbf{H}) = 0$ implies

$$\mu \mathbf{H} = \mathbf{curl}\, u \qquad \text{in } \mathbb{R}^2.$$

We deduce $\operatorname{curl} \mathbf{curl}\, u = -\Delta u = \mu J$ in Ω.
Moreover $\mathbf{curl}\,(i\omega u + \sigma^{-1} J) = 0$ in Ω_k^ϵ. This results in

$$i\omega\sigma u + J = \sigma C_k \qquad \text{in } \Omega_k^\epsilon, \quad C_k \in \mathbb{C}.$$

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$$\begin{cases} -\Delta u + i\omega\mu\sigma u = \mu\sigma C_k & \text{in } \Omega_k^\epsilon,\ k = 0, 1, 2 \\ \Delta u = 0 & \text{in } \Omega'_\epsilon \\ [u] = \left[\dfrac{\partial u}{\partial n}\right] = 0 & \text{on } \Gamma_\epsilon \\ u(x) = \beta + O(|x|^{-1}) & |x| \to \infty \end{cases}$$

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How to determine C_k ?

Let I stand for the total current in the inductor and assume

$$I = \int_{\Omega_1^\epsilon} J\,dx = -\int_{\Omega_2^\epsilon} J\,dx, \qquad \int_{\Omega_0^\epsilon} J\,dx = 0.$$

Let in addition $\theta = \omega\mu\sigma$ and

$$\chi_k^\epsilon = \mathbf{1}_{\Omega_k^\epsilon}, \quad \tilde{v}_k^\epsilon = \frac{1}{|\Omega_k^\epsilon|}\int_{\Omega_k^\epsilon} v\,dx, \quad k = 0, 1, 2.$$

We obtain after some calculations:

$$\begin{cases} -\Delta u^\epsilon + i\theta \displaystyle\sum_{k=0}^{2} \chi_k^\epsilon (u^\epsilon - \tilde{u}_k^\epsilon) = \mu I \left(\frac{\chi_1^\epsilon}{|\Omega_1^\epsilon|} - \frac{\chi_2^\epsilon}{|\Omega_2^\epsilon|} \right) & \text{in } \mathbb{R}^2 \\ \tilde{u}_0^\epsilon = 0 & \\ u^\epsilon(x) = \beta + O(|x|^{-1}) & |x| \to \infty \end{cases}$$

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$$\begin{cases} -\Delta u^\epsilon + i\theta \displaystyle\sum_{k=0}^{2} \chi_k^\epsilon\,(u^\epsilon - \tilde{u}_k^\epsilon) = \mu I \left(\frac{\chi_1^\epsilon}{|\Omega_1^\epsilon|} - \frac{\chi_2^\epsilon}{|\Omega_2^\epsilon|}\right) & \text{in } \mathbb{R}^2 \\ \tilde{u}_0^\epsilon = 0 & \\ u^\epsilon(x) = \beta + O(|x|^{-1}) & |x| \to \infty \end{cases}$$

The condition $\tilde{u}_0^\epsilon = 0$ ensures uniqueness of solutions.

We define the spaces

$$W^1(\mathbb{R}^2) = \{\, v;\ \rho v \in L^2(\mathbb{R}^2),\ \nabla v \in L^2(\mathbb{R}^2)^2\}$$
$$V = \{v \in W^1(\mathbb{R}^2);\ \tilde{v}_0 = 0\}$$

where $\rho(x) = \frac{1}{(1+|x|)\log(2+|x|)}$

We have the variational formulation (note the dependency on ϵ):

$$\begin{cases} u^\epsilon \in V, \\ \displaystyle\int_{\mathbb{R}^2} \nabla u^\epsilon \cdot \nabla \overline{v}\, dx + i\theta \sum_{k=0}^{2} \int_{\Omega_k^\epsilon} (u^\epsilon - \tilde{u}_k^\epsilon)\, \overline{v}\, dx = \mu I(\overline{\tilde{v}}_1^\epsilon - \overline{\tilde{v}}_2^\epsilon) \qquad \forall\, v \in V. \end{cases}$$

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The limit problem

We define $\Omega_k^\epsilon = z_k + \epsilon\,\widehat{\Omega}_k$, $k = 1, 2$ where $\epsilon \ll 1$, $z_k \in \mathbb{R}^2$, $\widehat{\Omega}_k \subset \mathbb{R}^2$.

Limit problem (?)

$$\begin{cases} -\Delta u + i\theta\chi_0\, u = \mu I\,(\delta_{z_1} - \delta_{z_2}) & \text{in } \mathbb{R}^2 \\ u(x) = \beta + O(|x|^{-1}) & |x| \to \infty \end{cases}$$

Remarks

- We cannot have convergence in V because of Dirac masses.
- The "limit" problem has at most one solution in

$$L^2_\rho(\mathbb{R}^2) := \{v : \mathbb{R}^2 \to \mathbb{C};\ \rho v \in L^2(\mathbb{R}^2)\}$$

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Convergence in L^2

We use duality techniques (Lions, Stampacchia, ...).
By choosing $v = \varphi \in V \cap H^2_{\mathrm{loc}}(\mathbb{R}^2)$ we find

$$-\int_{\mathbb{R}^2} u^\epsilon \, \Delta\overline{\varphi} \, dx + i\theta \sum_{k=1}^{2} \int_{\Omega_k^\epsilon} (u^\epsilon - \tilde{u}_k^\epsilon) \, \overline{\varphi} \, dx + \int_{\Omega_k^\epsilon} u^\epsilon \, \overline{\varphi} \, dx = \mu I (\overline{\tilde{\varphi}}_1^\epsilon - \overline{\tilde{\varphi}}_2^\epsilon)$$

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This can be written in the equivalent form:

$$\int_{\mathbb{R}^2} \rho^2 u^\epsilon \, \overline{\psi} \, dx = \mu I (\overline{\tilde{\varphi}}_1^\epsilon - \overline{\tilde{\varphi}}_2^\epsilon)$$

where φ^ϵ is the solution, in $V \cap H^2_{\mathrm{loc}}(\mathbb{R}^2)$, of

$$-\Delta\varphi^\epsilon + i\theta \sum_{k=1}^{2} \chi_k^\epsilon (\varphi^\epsilon - \tilde{\varphi}_k^\epsilon) + i\theta \chi_0 \varphi^\epsilon = \rho^2 \psi \qquad \text{in } \mathbb{R}^2$$

Note that the right-hand side does not depend on ϵ.
We consider the "limit" problem:

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$$\|\varphi^\epsilon\|_{H^2(B)} \leq C\,\|\rho\psi\|_{L^2(\mathbb{R}^2)} \qquad \text{for all compact sets } B \text{ of } \mathbb{R}^2$$

Theorem

The sequence (φ^ϵ) converges in $W^1(\mathbb{R}^2)$ to φ.

We want now to take the limit in the equation

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Theorem

We have $\|\rho\,(u - u^{\epsilon})\|_{L^2(\mathbb{R}^2)} \leq C\epsilon^{\alpha} \qquad 0 \leq \alpha < \frac{1}{2}$

A sharper convergence result

We want to prove the convergence in $W^{1,p}$ for $1 \leq p < 2$.
For this, we use truncation or renormalization techniques (Lions, Murat, Boccardo-Gallout,...)
We first prove the estimates

$$\|\rho u^\epsilon\|_{L^2(\mathbb{R}^2)} + \epsilon^{-\frac{1}{2}} \sum_{k=1}^{2} \|u^\epsilon - \tilde{u}_k^\epsilon\|_{L^2(\Omega_k^\epsilon)} \leq C$$

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Remark

In the case where $\Omega_0 = \emptyset$, we have the limit problem:

$$\begin{cases} -\Delta u_0 = \mu I \left(\delta_{z_1} - \delta_{z_2}\right) & \text{in } \mathbb{R}^2 \\ u_0(x) = \beta + O(|x|^{-1}) & |x| \to \infty. \end{cases}$$

The solution is given by

$$u_0(x) = \frac{\mu I}{2\pi} \log \frac{|x - z_2|}{|x - z_1|} \qquad x \in \mathbb{R}^2 \setminus \{z_1, z_2\}.$$

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In the general case, we obtain the solution by superposition:

$$u = w + u_0$$

where $w \in W^1(\mathbb{R}^2)$ and

$$\begin{cases} -\Delta w + i\theta\chi_0 w = -i\theta\chi_0 u_0 & \text{in } \mathbb{R}^2 \\ w(x) = \beta + O(|x|^{-1}) & |x| \to \infty \end{cases}$$