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Bialgebra cohomology of the duals of a class of generalised Taft algebras

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Abstract

We compute explicitly the bialgebra cohomology of the duals of the generalised Taft algebras, which are non-commutative, non-cocommutative finite-dimensional Hopf algebras. In order to do this, we use an identification of this cohomology with an Ext algebra (Taillefer, 2004) and a result describing the Drinfeld double of the dual of a generalised Taft algebra up to Morita equivalence (Erdmann, Green, Snashall, Taillefer, 2006).

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1 Introduction

The object of this note is to present the computation of the bialgebra cohomology of an explicit example of a Hopf algebra that is neither commutative nor cocommutative.

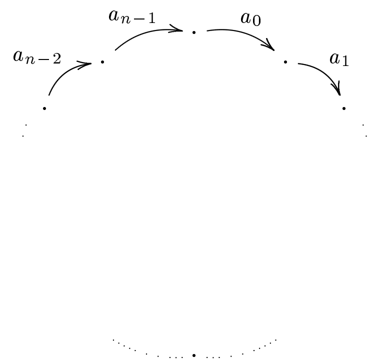
The bialgebra cohomology $\mathrm{H}_b^*(H,H)$ for a Hopf algebra H was defined by M. Gerstenhaber and S.D. Schack in [GS] in order to understand deformations of bialgebras and Hopf algebras (in a similar way that Hochschild cohomology is a tool in the study of deformations of associative algebras). This cohomology has also been useful to establish results on classification or structure of Hopf algebras (see [EG] and [Ş]).

The bialgebra cohomology is endowed with a graded algebra structure which is graded commutative (see [T1, T2]), but it is difficult to compute in general. Some results are known: M. Gerstenhaber and S.D. Schack [GS] showed that if G is a discrete group and k is a field, then $\mathrm{H}_b^*(kG,kG) \cong \mathrm{H}^*(G,k)$ (since kG is coseparable); B. Parshall and J. Wang [PW] established some results for the function algebras of some affine algebraic groups, and various results were obtained for the enveloping algebra of a Lie algebra (see [GS], [G] and [Sh]) that enabled the authors to deduce some information on deformations of the enveloping algebra of a Lie algebra.

In the case of non-commutative non-cocommutative Hopf algebras however, it seems that no computations are known. In this note, we compute $\mathrm{H}_b^*(\Lambda_{n,d},\Lambda_{n,d})$ where $\Lambda_{n,d}$ is the dual of a generalised Taft algebra, which is a finite-dimensional Hopf algebra that is neither commutative nor cocommutative. The dual algebra $\Lambda_{n,d}^*$ is described as follows: let n and d be integers such that d divides n, and let k be an algebraically closed field whose characteristic does not divide n; then $\Lambda_{n,d}^*$ is generated by two elements G and X satisfying the relations $G^n=1$, $X^d=0$ and $GX=\omega^{-1}XG$ where ω is a primitive d^{th} root of unity. The element G is grouplike and $\Delta(X)=G\otimes X+X\otimes 1$. Note that in the case $n=d$, $\Lambda_{n,n}^*$ is the usual Taft algebra, and it is well known that it is selfdual (as a Hopf algebra), hence $\Lambda_{n,n}$ is isomorphic to the usual Taft algebra. The Taft algebras have been very much studied, as they are small enough to be manageable, but their structure is rich enough to provide interesting examples against which to test a theory.

2 Preliminaries

In this section, we define the algebras $\Lambda_{n,d}$ by quiver and relations (we refer to [ARS, III.1] and [B, 4.1] for definitions and properties relating to quivers and relations). Let n and d be integers, and assume that k is an algebraically closed field. The quiver is an oriented cycle,



with n vertices $\epsilon_0, \dots, \epsilon_{n-1}$ and n arrows $a_0, \dots, a_{n-1}$, where the arrow a_i goes from the vertex ϵ_i to the vertex ϵ_{i+1}, and we factor by the ideal generated by all paths of length $d \geqslant 2$. The indices are viewed as elements in the cyclic group $\mathbb{Z}_n$ and are written mod n. Denote by γ_i^m the path $a_{i+m-1} \dots a_{i+1}a_i$ (read from right to left), that is, the path of length m starting at the vertex ϵ_i. In particular, $\gamma_i^0 = \epsilon_i$ and $\gamma_i^1 = a_i$. The algebra $\Lambda_{n,d}$ is then equal to the vector space over k whose basis is the set of paths γ_i^m for $i \in \mathbb{Z}_n$ and $0 \leqslant m \leqslant d-1$, and its product is defined by $\gamma_j^\ell \gamma_i^m = \begin{cases} \gamma_i^{m+\ell} & \text{if } i+m \equiv j \pmod{n} \text{ and } m+\ell < d \\ 0 & \text{otherwise} \end{cases}$ on the paths, and extended linearly.

When d divides n, this algebra is a Hopf algebra, and in fact the condition $d \mid n$ is a necessary and sufficient condition for $\Lambda_{n,d}$ to be a Hopf algebra when char$k = 0$ (see [C, CHYZ]). This Hopf algebra can actually be considered over more general fields, and in this paper, we assume only that the characteristic of k does not divide n (to ensure existence of roots of unity).

We fix a primitive d^{th} root of unity ω in k. The formulae

$$\begin{array}{lll} \varepsilon(\epsilon_i) = \delta_{i0} & \Delta(\epsilon_i) = \sum_{j+\ell=i} \epsilon_j \otimes \epsilon_\ell & S(\epsilon_i) = \epsilon_{-i} \\ \varepsilon(a_i) = 0 & \Delta(a_i) = \sum_{j+\ell=i} (\epsilon_j \otimes a_\ell + \omega^\ell a_j \otimes \epsilon_\ell) & S(a_i) = -\omega^{i+1} a_{-i-1}, \end{array}$$

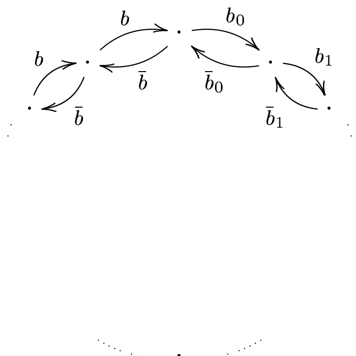
where δ is the Kronecker symbol, determine the Hopf algebra structure of $\Lambda_{n,d}$ (the indices are written in $\mathbb{Z}_n$).

3 Computation of $\mathrm{H}_b^*(\Lambda_{n,d}, \Lambda_{n,d})$

The proof relies on the two following results:

Theorem 3.1 ([T1] Remark, end of Section 4, and [T2] Remark 3.10) *Let H be a finite-dimensional Hopf algebra and let $\mathcal{D}(H)$ be its Drinfeld double. Denote by k the trivial simple left module over $\mathcal{D}(H)$. Then $\mathrm{H}_b^*(H,H)$ is isomorphic to $\mathrm{Ext}^*_{\mathcal{D}(H)}(k,k)$ as a graded algebra (the product on the algebra $\mathrm{Ext}^*_{\mathcal{D}(H)}(k,k)$ is the Yoneda product).*

Theorem 3.2 ([EGST] Theorem 2.26) *Let $\Lambda_{n,d}$ be as in Section 2. Then $\mathcal{D}(\Lambda_{n,d})$ is Morita equivalent to the algebra $A = k^{\frac{n^2}{d}} \times (\frac{kQ}{I})^{\frac{n(d-1)}{2}}$, where Q is the quiver*



(the underlying non-oriented graph is a cycle, and for each edge we put two arrows in opposing directions) with $\frac{2n}{d}$ vertices and $\frac{4n}{d}$ arrows indexed by the cyclic group $\mathbb{Z}_{\frac{2n}{d}}$. The ideal I is generated by the elements $b_{i+1}b_i$, $\bar{b}_{i-1}\bar{b}_i$ and $b_i\bar{b}_i - \bar{b}_{i+1}b_{i+1}$ for all $i \in \mathbb{Z}_{\frac{2n}{d}}$ (there are $\frac{6n}{d}$ relations on each of these quivers). The vertices in the isolated copies of k correspond to simple $\mathcal{D}(\Lambda_{n,d})$-modules of dimension n, the vertices in the copies of Q correspond to simple $\mathcal{D}(\Lambda_{n,d})$-modules of dimensions between 1 and $n-1$.

We now have enough information to prove the following result:

Proposition 3.3 *There is a graded algebra isomorphism:*

$$\mathrm{H}^*_b(\Lambda_{n,d}, \Lambda_{n,d}) \cong \frac{k[x,y,z]}{(z^{\frac{2n}{d}} - xy)}$$

with x and y of degree $\frac{2n}{d}$ and z of degree 2.

Proof: We know from Theorem 3.1 that there is a graded algebra isomorphism $\mathrm{H}^*_b(\Lambda_{n,d}, \Lambda_{n,d}) \cong \mathrm{Ext}^*_{\mathcal{D}(\Lambda_{n,d})}(k,k)$. Moreover, by Theorem 3.2, $\mathrm{Ext}^*_{\mathcal{D}(\Lambda_{n,d})}(k,k)$ is isomorphic as a graded algebra to $\mathrm{Ext}^*_A(S,S)$ where S is a simple A-module to be determined.

The trivial $\mathcal{D}(\Lambda_{n,d})$-module k is a simple $\mathcal{D}(\Lambda_{n,d})$-module of dimension 1. Therefore the simple A-module S cannot be a simple A-module at an isolated vertex in the quiver of A (they correspond to simple $\mathcal{D}(\Lambda_{n,d})$-modules of dimension n), so it must be a simple A-module at a vertex in one of the copies of Q. These are indistinguishable, so we choose S to be any one of them. Moreover, $\mathrm{Ext}^*_A(S,S) \cong \mathrm{Ext}^*_{\frac{kQ}{I}}(S,S)$ (the other components of A act as zero on S, so that S can be viewed as a module over $\frac{kQ}{I}$).

Finally, we have a graded algebra isomorphism $\mathrm{H}^*_b(\Lambda_{n,d}, \Lambda_{n,d}) \cong \mathrm{Ext}^*_{\frac{kQ}{I}}(S,S)$, so we only need to compute the algebra $\mathrm{Ext}^*_{\frac{kQ}{I}}(S,S)$.

Let e_i be the i^{th} vertex in Q, with i in the cyclic group of order $\frac{2n}{d}$. We denote by P_i the projective module $\frac{kQ}{I}e_i$ and by $S_i = \frac{kQ}{I}e_i/\mathrm{rad}\left(\frac{kQ}{I}e_i\right)$ the simple module at the vertex e_i. Choose S to be the simple $\frac{kQ}{I}$-module S_0.

It is easy to check that there is a minimal projective left $\frac{kQ}{I}$-module resolution $\mathbf{P}^\bullet$ of S_0 given by $\mathbf{P}^r = \bigoplus_{\ell=0}^r P_{r-2\ell}$ with differential $\partial^n : \mathbf{P}^n \to \mathbf{P}^{n-1}$ defined on the generators by $\partial^n(e_{n-2\ell}) = b_{n-2\ell-1}e_{n-2\ell-1} + \bar{b}_{n-2\ell}e_{n-2\ell+1}$. Denote by $e_i^{(j)}$ the generator of the j^{th} copy of P_i in $\mathbf{P}^r$ ($-r \leqslant j \leqslant r$). Applying the functor $\mathrm{Hom}_{\frac{kQ}{I}}(-, S_0)$ gives the complex $\mathrm{Hom}_{\frac{kQ}{I}}(\mathbf{P}^\bullet, S_0)$.

Using the fact that $\mathrm{Hom}_{\frac{kQ}{I}}(P_i, S_0) \cong \mathrm{Hom}_{\frac{kQ}{I}}(S_i, S_0) \cong \delta_{i0}k$ (the radical of P_i must go to 0), we see in particular that the differentials in the complex $\mathrm{Hom}_{\frac{kQ}{I}}(\mathbf{P}^\bullet, S_0)$ are zero and we get:

$$\dim_k \mathrm{Ext}^r_{\frac{kQ}{I}}(S_0, S_0) = \begin{cases} 2\alpha + 1 & \text{if } r = \alpha\frac{2n}{d} + \beta \text{ with } 0 \leqslant \beta < \frac{2n}{d} \text{ and } \beta \text{ even,} \\ 0 & \text{if } r = \alpha\frac{2n}{d} + \beta \text{ with } 0 \leqslant \beta < \frac{2n}{d} \text{ and } \beta \text{ odd.} \end{cases}$$

We now choose a basis of cochain maps for $\mathrm{Ext}^r_{\frac{kQ}{I}}(S_0, S_0)$: for each integer t with $-\alpha \leqslant t \leqslant \alpha$, define

$$\begin{aligned} \gamma_{r,t} : \mathbf{P}^{m+r} & \to \mathbf{P}^m \\ e_i^{(j)} & \mapsto \begin{cases} e_i^{(j-t)} & \text{if } t - m \leqslant j \leqslant t + m \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

for every $m \in \mathbb{N}$ (where $e_i^{(p)} = 0$ if $p < 0$).

It is not difficult to check that $\gamma_{r,t}$ is a chain map, and that the set $\{\gamma_{r,t} \mid -\alpha \leqslant t \leqslant \alpha\}$ is a basis for $\mathrm{Ext}^r_{\frac{kQ}{I}}(S_0, S_0)$.

The cup-product is then obtained by composing these maps. We can easily see that $\gamma_{r',t'} \circ \gamma_{r,t} = \gamma_{r+r',t+t'}$ and hence that $z := \gamma_{2,0}$, $x := \gamma_{\frac{2n}{d},-1}$ and $y := \gamma_{\frac{2n}{d},1}$ generate $\mathrm{Ext}^*_{\frac{kQ}{I}}(S_0, S_0)$. Moreover, the $\gamma_{r,t}$ commute, and $xy = z^{\frac{2n}{d}}$. Therefore $\mathrm{Ext}^*_{\frac{kQ}{I}}(S_0, S_0)$ is isomorphic to a quotient of the algebra $\dfrac{k[x, y, z]}{(z^{\frac{2n}{d}} - xy)}$. Both are graded algebras, so to prove that they are isomorphic, it is enough to check that the dimensions for each graded part are equal. This is easily seen (a basis of the degree $\alpha\frac{2n}{d} + \beta$ part is empty if β is odd, and is given by $\{x^p z^{(\alpha-p)\frac{n}{d}+\frac{\beta}{2}}; y^p z^{(\alpha-p)\frac{n}{d}+\frac{\beta}{2}} \mid 0 \leqslant p \leqslant \alpha\}$ if β is even). ■

Corollary 3.4 *The dimensions of the* $\mathrm{H}^r_b(\Lambda_{n,d}, \Lambda_{n,d})$ *for* $r \in \mathbb{N}$ *are given by:*

$$\dim_k \mathrm{H}^r_b(\Lambda_{n,d}, \Lambda_{n,d}) = \begin{cases} 0 & \text{if } r \text{ is odd} \\ 2\lfloor \frac{rd}{2n} \rfloor + 1 & \text{if } r \text{ is even} \end{cases}$$

where $\lfloor\ \rfloor$ *denotes the lower integer part.*

Remark 3.5 Since the $\mathrm{H}^r_b(\Lambda_{n,d}, \Lambda_{n,d})$ vanish when r is odd, any Gerstenhaber algebra structure on $\mathrm{H}^*_b(\Lambda_{n,d}, \Lambda_{n,d})$ would have to be trivial (a Gerstenhaber algebra structure includes a graded Lie bracket: $\mathrm{H}^r_b(\Lambda_{n,d}, \Lambda_{n,d}) \times \mathrm{H}^s_b(\Lambda_{n,d}, \Lambda_{n,d}) \to \mathrm{H}^{r+s-1}_b(\Lambda_{n,d}, \Lambda_{n,d})$ so that if r and s are even, $r+s-1$ is odd, and therefore the bracket must be zero). See [GS] and [T2] for questions and definitions relating to Gerstenhaber algebras.

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