Hopf algebras and quivers

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CONTENTS

Throughout, k is a (commutative) field. We denote by $\otimes$ the tensor product $\otimes_k$ over k. Moreover, an algebra is an associative k-algebra with unit.

I. INTRODUCTION TO HOPF ALGEBRAS

1. Motivation

Given an algebra A and two left A-modules M and N, we would like to have a left A-module structure on $M \otimes_k N$.

There are some algebras A for which we know how to do this.

➢ *Group algebras*. Let G be a group. If M and N are two kG-modules, then $M \otimes N$ is a kG-module for the action

$$\forall g \in G,\ \forall m \otimes n \in M \otimes N, \quad g(m \otimes n) = gm \otimes gn.$$

We have used the diagonal map $G \to G \times G$, which induces a k-linear map $\Delta : kG \to k[G \times G] \cong kG \otimes kG$, and the action is defined by the composition

$$kG \otimes M \otimes N \xrightarrow{\Delta} kG \otimes kG \otimes M \otimes N \xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id}} kG \otimes M \otimes kG \otimes N \xrightarrow{\mu_M \otimes \mu_N} M \otimes N$$

where $\tau(g \otimes m) = m \otimes g$ and $\mu_M : kG \otimes M \to M$ and $\mu_N : kG \otimes N \to N$ are the structure maps of M and N.

➢ *Enveloping algebras of Lie algebras*. Let $\mathfrak{g}$ be a Lie algebra and let $U(\mathfrak{g})$ be its enveloping algebra. If M and N are left $U(\mathfrak{g})$-modules, then $M \otimes N$ is a $U(\mathfrak{g})$-module for the action

$$\forall x \in \mathfrak{g},\ \forall m \otimes n \in M \otimes N, \quad x(m \otimes n) = xm \otimes n + m \otimes xn.$$

This action is similar to the previous one, for the map $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ defined on elements $x \in \mathfrak{g}$ by $\Delta(x) = x \otimes 1 + 1 \otimes x$.

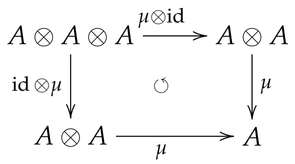
Hopf algebras are algebras endowed with a linear map $\Delta : H \to H \otimes H$ that satisfy some extra properties.

2. Bialgebras

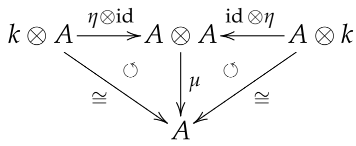
We start by rewriting the axioms of an algebra in terms of commutative diagrams.

An algebra is a k-vector space A endowed with two k-linear maps $\mu : A \otimes A \to A$ (multiplication) and $\eta : k \to A$ (unit: $\eta(1) = 1_A$) that satisfy:

Associativity



Unit

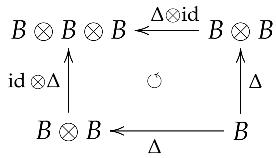


where $k \otimes A \xrightarrow{\cong} A$ and $A \otimes k \xrightarrow{\cong} A$ are the natural isomorphisms, which we view as identifications, so that $\mu \circ (\eta \otimes \mathrm{id}) = \mathrm{id}$ and $\mu \circ (\mathrm{id} \otimes \eta)$.

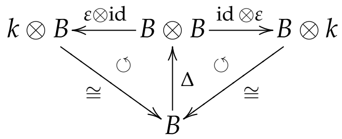
We shall now define bialgebras by formally dualising the structure maps and commutative diagrams.

Definition I.1. *A **bialgebra** is an algebra (B, μ, η) endowed with algebra maps $\Delta : B \to B \otimes B$ and $\varepsilon : B \to k$, respectively called the **comultiplication** and the **counit**, that satisfy*

Coassociativity



Counit



that is, $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ *and* $(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta$.

Notation I.2. Given an element $b \in B$, $\Delta(b)$ is an element of $B \otimes B$, that is, $\Delta(b) = \sum_i a_i \otimes b_i$ for some a_i, b_i in B. We shall use the Sweedler notation for this:

$$\Delta(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)}.$$

The coassociativity and counit axioms then become:

➢ Counit: $\sum_{(b)} \varepsilon(b_{(1)}) b_{(2)} = b = \sum_{(b)} b_{(1)} \varepsilon(b_{(2)})$.

➢ Coassociativity:

$$\sum_{(b)} b_{(1)} \otimes \left(\sum_{(b_{(2)})} (b_{(2)})_{(1)} \otimes (b_{(2)})_{(2)} \right) = \sum_{(b)} \left(\sum_{(b_{(1)})} (b_{(1)})_{(1)} \otimes (b_{(1)})_{(2)} \right) \otimes b_{(2)},$$

and we shall denote this by $\sum_{(b)} b_{(1)} \otimes b_{(2)} \otimes b_{(3)}$.

Example I.3. **(1)** k is a bialgebra (with $\Delta = \mathrm{id} = \varepsilon$).

(2) If G is a group, then the k-vector space kG with basis the elements of G is a bialgebra, in which the multiplication extends the group law, and whose comultiplication and counit are determined by

$$\Delta(g) = g \otimes g \text{ and } \varepsilon(g) = 1 \quad \text{for all } g \in G.$$

(3) Let G be a finite group and let k^G be the set of maps from G to k. This is a vector space (if f and f' are in k^G and $\lambda \in k$, then $(f + f')(g) = f(g) + f'(g)$ and $(\lambda f)(g) = \lambda f(g)$ for all $g \in G$), with basis $\{\delta_g; g \in G\}$ with $\delta_g(h) = 1$ if $h = g$ and $\delta_g(h) = 0$ if $h \neq g$. (If $f \in k^G$ then $f = \sum_{g \in G} f(g)\delta_g$.)

In fact k^G is a bialgebra, whose structure is determined by

$$\delta_g \delta_h = \begin{cases} \delta_g & \text{if } g = h \\ 0 & \text{otherwise,} \end{cases} \quad \Delta(\delta_g) = \sum_{hk = g} \delta_h \otimes \delta_k = \sum_{h \in G} \delta_h \otimes \delta_{h^{-1}g} \text{ and } \varepsilon(\delta_g) = \begin{cases} 1 & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases}$$

for all $g \in G$. The unit element is $\sum_{g \in G} \delta_g$.

(4) Let V be any finite dimensional vector space. Then the tensor algebra $T_k(V)$ is a bialgebra, whose comultiplication and counit are determined by

$$\Delta(v) = 1 \otimes v + v \otimes 1 \text{ if } v \in V, \Delta(1) = 1 \otimes 1$$
$$\varepsilon(v) = 0 \text{ if } v \text{ has positive degree}, \varepsilon(1) = 1.$$

There is a closed formula for $\Delta(x)$ with $x = v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$, given in terms of $(p, n-p)$-shuffles in the symmetric group $\mathfrak{S}_n$, that is, permutations σ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(n)$:

$$\Delta(x) = \sum_{p=0}^{n} \sum_{\sigma \in \mathrm{Sh}_{p,n-p}} \left(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)}\right) \otimes \left(v_{\sigma(p+1)} \otimes \cdots \otimes v_{\sigma(n)}\right).$$

Definition I.4. *Let B be a bialgebra and let $\tau : B \otimes B \to B \otimes B$ be the isomorphism that sends $a \otimes b$ to $b \otimes a$. Set $\Delta^{cop} := \tau \circ \Delta : B \to B \otimes B$.*

*The bialgebra B is **cocommutative** if Δ^{cop} is equal to Δ.*

Example I.5. The bialgebras k, kG and $T_k(V)$ are cocommutative. The bialgebra k^G is cocommutative if and only if G is abelian. If G is not abelian, then the bialgebra $kG \otimes k^G$ (see Lemma I.6 below) is neither commutative nor cocommutative.

Lemma I.6. *Let $(B, \mu, \eta, \Delta, \varepsilon)$ and $(B', \mu', \eta', \Delta', \varepsilon')$ be bialgebras. Then $B \otimes B'$ is a bialgebra, with structure maps given by $\bar{\eta} = \varepsilon \otimes \varepsilon'$, $\bar{\eta}(1) = \eta(1) \otimes \eta'(1)$,*

$$\bar{\mu} = (\mu \otimes \mu') \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) : (a \otimes a') \otimes (b \otimes b') \mapsto (a \otimes a')(b \otimes b') = ab \otimes a'b'$$
$$\bar{\Delta} = (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta') : b \otimes b' \mapsto \sum_{(b),(b')} (b_{(1)} \otimes b'_{(1)}) \otimes (b_{(2)} \otimes b'_{(2)})$$

where $\tau : B \otimes B' \to B' \otimes B$ send $b \otimes b'$ to $b' \otimes b$.

Proof. It is well known that $(B \otimes B', \bar{\mu}, \bar{\eta})$ is an algebra. Let us check the counit axiom and the coassociativity axiom.

$$(\bar{\varepsilon} \otimes \mathrm{id}) \circ \bar{\Delta}(b \otimes b') = \sum_{(b),(b')} \bar{\varepsilon}(b_{(1)} \otimes b'_{(1)}) b_{(2)} \otimes b'_{(2)} = \sum_{(b),(b')} \varepsilon(b_{(1)}) \varepsilon'(b'_{(1)}) b_{(2)} \otimes b'_{(2)} = b \otimes b'$$
$$(\mathrm{id} \otimes \bar{\varepsilon}) \circ \bar{\Delta}(b \otimes b') = \sum_{(b),(b')} \bar{\varepsilon}(b_{(2)} \otimes b'_{(2)}) b_{(1)} \otimes b'_{(1)} = \sum_{(b),(b')} \varepsilon(b_{(2)}) \varepsilon'(b'_{(2)}) b_{(1)} \otimes b'_{(1)} = b \otimes b'$$
$$\begin{aligned}(\bar{\Delta} \otimes \mathrm{id}) \circ \bar{\Delta}(b \otimes b') &= \sum_{(b),(b'),(b_{(1)}),(b'_{(1)})} ((b_{(1)})_{(1)} \otimes (b'_{(1)})_{(1)}) \otimes ((b_{(1)})_{(2)} \otimes (b'_{(1)})_{(2)}) \otimes ((b_{(2)}) \otimes (b'_{(2)})) \\ &= \sum_{(b),(b'),(b_{(2)}),(b'_{(2)})} ((b_{(1)}) \otimes (b'_{(1)})) \otimes ((b_{(2)})_{(1)} \otimes (b'_{(2)})_{(1)}) \otimes ((b_{(2)})_{(2)} \otimes (b'_{(2)})_{(2)}) \\ &= (\mathrm{id} \otimes \bar{\Delta}) \circ \bar{\Delta}(b \otimes b')\end{aligned}$$

using the counit and coassociativity axioms for B and B'.

We must finally show that $\bar{\varepsilon}$ and $\bar{\Delta}$ are algebra maps. We have $\bar{\varepsilon}(1 \otimes 1) = \varepsilon(1)\varepsilon'(1) = 1$ and $\bar{\Delta}(1 \otimes 1) = (1 \otimes 1) \otimes (1 \otimes 1)$, the unit in the algebra $(B \otimes B') \otimes (B \otimes B')$. Moreover, since ε, ε', Δ and Δ' are algebra maps, we have

$$\bar{\varepsilon}((a \otimes a')(b \otimes b')) = (\varepsilon \otimes \varepsilon')(ab \otimes a'b') = \varepsilon(ab)\varepsilon'(a'b') = \varepsilon(a)\varepsilon'(a')\varepsilon(b)\varepsilon(b') = \bar{\varepsilon}(a \otimes a')\bar{\varepsilon}(b \otimes b')$$

$$\begin{aligned}\bar{\Delta}((a \otimes a')(b \otimes b')) = \bar{\Delta}(ab \otimes a'b') &= \sum_{(ab),(a'b')} ((ab)_{(1)} \otimes (a'b')_{(1)}) \otimes ((ab)_{(2)} \otimes (a'b')_{(2)}) \\ &= \sum_{(a),(b),(a'),(b')} (a_{(1)}b_{(1)} \otimes a'_{(1)}b'_{(1)}) \otimes (a_{(2)}b_{(2)} \otimes a'_{(2)}b'_{(2)}) \\ &= \sum_{(a),(b),(a'),(b')} ((a_{(1)} \otimes a'_{(1)})(b_{(1)} \otimes b'_{(1)})) \otimes ((a_{(2)} \otimes a'_{(2)})(b_{(2)} \otimes b'_{(2)})) \\ &= \sum_{(a),(b),(a'),(b')} ((a_{(1)} \otimes a'_{(1)}) \otimes (a_{(2)} \otimes a'_{(2)}))((b_{(1)} \otimes b'_{(1)}) \otimes (b_{(2)} \otimes b'_{(2)})) \\ &= \bar{\Delta}(a \otimes a')\bar{\Delta}(b \otimes b').\end{aligned}$$

□

Lemma I.7. *Let $(B,\mu,\eta,\Delta,\varepsilon)$ be a bialgebra. Then $B^{op} = (B,\mu^{op},\eta,\Delta,\varepsilon)$, $B^{cop} = (B,\mu,\eta,\Delta^{cop},\varepsilon)$ and $B^{op\,cop} = (B,\mu^{op},\eta,\Delta^{cop},\varepsilon)$ are also bialgebras.*

Proof. Exercise. $\square$

We now introduce a new product, useful later on.

Definition I.8. *Let A be an algebra and let B be a bialgebra. Define a bilinear map*

$$\begin{array}{rccc} \star\colon & \mathrm{Hom}_k(B,A)\times\mathrm{Hom}_k(B,A) & \longrightarrow & \mathrm{Hom}_k(B,A) \\ & (f,g) & \longmapsto & f\star g = \mu_A\circ(f\otimes g)\circ\Delta_B. \end{array}$$

With the Sweedler notation, the definition becomes

$$(f\star g)(b) = \sum_{(b)} f(b_{(1)})g(b_{(2)}) \text{ for all } b\in B.$$

Lemma I.9. *The triple $(\mathrm{Hom}_k(B,A),\star,\eta_A\circ\varepsilon_B)$ is an algebra. The product $\star$ is called the* ***convolution product****.*

Proof. The product and unit are k-linear.
The product is associative:

$$\begin{aligned}(f\star(g\star h))(b) &= \sum_{(b)} f(b_{(1)})(g\star h)(b_{(2)}) = \sum_{(b)} f(b_{(1)})(g(b_{(2)})h(b_{(3)})) \\ &= \sum_{(b)}(f(b_{(1)})g(b_{(2)}))h(b_{(3)}) = \sum_{(b)}(f\star g)(b_{(1)})h(b_{(2)}) = ((f\star g)\star h)(b).\end{aligned}$$

The map $\eta_A\circ\varepsilon_B$ is a left and right unit for the product:

$$((\eta_A\circ\varepsilon_B)\star f)(b) = \sum_{(b)}\eta_A(\varepsilon_B(b_{(1)}))f(b_{(2)}) = \sum_{(b)}\eta_A(1)f(\varepsilon_B(b_{(1)})b_{(2)}) = f(b)$$
$$(f\star(\eta_A\circ\varepsilon_B))(b) = \sum_{(b)} f(b_{(1)})\eta_A(\varepsilon_B(b_{(2)})) = \sum_{(b)} f(b_{(1)}\varepsilon_B(b_{(2)}))\eta_A(1) = f(b)$$

$\square$

Definition I.10. *A morphism of bialgebras from $(B,\mu,\eta,\Delta,\varepsilon)$ to $(B',\mu',\eta',\Delta',\varepsilon')$ is a morphism of algebras $f : B\to B'$ that satisfies*

$$\Delta'\circ f = (f\otimes f)\circ\Delta : B\to B'\otimes B' \text{ and } \varepsilon'\circ f = \varepsilon$$

that is, the diagrams

$$\begin{array}{ccc} B & \xrightarrow{f} & B' \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta'} \\ B\otimes B & \xrightarrow[f\otimes f]{} & B'\otimes B' \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \xrightarrow{f} & B' \\ & {\scriptstyle\varepsilon}\searrow \quad \swarrow{\scriptstyle\varepsilon'} & \\ & k & \end{array}$$

commute.

Remark I.11. We denote by V^* the k-dual $\mathrm{Hom}_k(V,k)$ of V. Given two vector spaces V and W, there is a k-linear map $\lambda : V^*\otimes W^*\to(V\otimes W)^*$ which sends $\alpha\otimes\beta\in V^*\otimes W^*$ to $[v\otimes w\mapsto\alpha(v)\beta(w)]$. This map is injective but not surjective unless V or W is finite dimensional.

Proposition I.12. *Let $(B,\mu,\eta,\Delta,\varepsilon)$ be a finite dimensional bialgebra. Then B^* is a bialgebra, with multiplication*

$$\mu_{B^*} : B^*\otimes B^* \xrightarrow{\lambda} (B\otimes B)^* \xrightarrow{\Delta^*} B^*,$$

unit $\eta_{B^*} = \varepsilon^* : k\to B^*$, counit $\varepsilon_{B^*} = \eta^*$ and comultiplication given by

$$\Delta_{B^*} : B^* \xrightarrow{\mu^*} (B\otimes B)^* \xrightarrow{\lambda^{-1}} B^*\otimes B^*,$$

that is, $\Delta_{B^*}(\alpha) : a\otimes b\mapsto\alpha(ab)$.
Moreover, if $f : B\to B'$ is a morphism of bialgebras then $f^* : {B'}^*\to B^*$ is a morphism of bialgebras.

Remark I.13. Using the Sweedler notation, we have $\mu_{B^*}(\alpha\otimes\beta)(x) = (\alpha\beta)(x) = \sum_{(x)}\alpha(x_{(1)})\beta(x_{(2)})$. In fact, μ_{B^*} is the convolution product of the algebra $B^* = \mathrm{Hom}_k(B,k)$.

Moreover, we have identified k with k^* by sending 1 to id_k in defining η_{B^*} and ε_{B^*}. With this identification, the unit element in B^* is $\eta_{B^*}(1) = \varepsilon^*(\mathrm{id}_k) = \varepsilon$ and $\varepsilon_{B^*}(\alpha) = \alpha(1)$.

Moreover, we can multiply an element γ in k^* and an element α in B^* as follows:

$$\begin{array}{rcccccc} \gamma\alpha : B & \stackrel{\sim}{\to} & k\otimes B & \stackrel{\gamma\otimes\alpha}{\to} & k\otimes k & \stackrel{\sim}{\to} & k \\ a & \mapsto & 1\otimes a & \mapsto & \gamma(1)\otimes\alpha(a) & \mapsto & \gamma(1)\alpha(a). \end{array}$$

The product $\alpha\gamma$ is defined similarly.

Proof of Proposition I.12. From the remark above, it is clear that B^* is an algebra (for the convolution product, equal to μ_{B^*}). The unit element is $\eta_k\circ\varepsilon = \mathrm{id}_k\circ\varepsilon = \varepsilon$.

The maps ε_{B^*} and Δ_{B^*} are k-linear. We now check the counit axiom. For $\alpha\in B^*$, the map $((\varepsilon_{B^*}\otimes\mathrm{id})\circ\Delta_{B^*})(\alpha) = \sum_{(\alpha)}\varepsilon_{B^*}(\alpha_{(1)})\alpha_{(2)} = \sum_{(\alpha)}(\alpha_{(1)}\circ\eta_B)\alpha_{(2)}$ sends $b\in B$ to $\sum_{(\alpha)}(\alpha_{(1)}\circ\eta_B)(1)\alpha_{(2)}(b) = \Delta_{B^*}(\alpha)(1\otimes b) = \alpha(b)$ using the remark made before this proof. Therefore $(\varepsilon_{B^*}\otimes\mathrm{id})\circ\Delta_{B^*} = \mathrm{id}$. Similarly, $(\mathrm{id}\otimes\varepsilon_{B^*})\circ\Delta_{B^*} = \mathrm{id}$.

Next, we prove that the coassociativity axiom is satisfied. For $\alpha\in B^*$, the map $((\Delta_{B^*}\otimes\mathrm{id})\circ\Delta_{B^*})(\alpha) = \sum_{(\alpha),(\alpha_{(1)})}(\alpha_{(1)})_{(1)}\otimes(\alpha_{(1)})_{(2)}\otimes\alpha_{(2)}$ sends $a\otimes b\otimes c\in B\otimes B\otimes B$ to $\sum_{(\alpha),(\alpha_{(1)})}\alpha_{(1)}(ab)\alpha_{(2)}(c) = \alpha(abc)$ and the map $((\mathrm{id}\otimes(\Delta_{B^*})\circ\Delta_{B^*})(\alpha) = \sum_{(\alpha),(\alpha_{(2)})}\alpha_{(1)}\otimes(\alpha_{(2)})_{(1)}\otimes(\alpha_{(2)})_{(2)}$ sends $a\otimes b\otimes c\in B\otimes B\otimes B$ to $\sum_{(\alpha),(\alpha_{(2)})}\alpha_{(1)}(a)\alpha_{(2)}(bc) = \alpha(abc)$ so that $(\Delta_{B^*}\otimes\mathrm{id})\circ\Delta_{B^*} = (\mathrm{id}\otimes(\Delta_{B^*})\circ\Delta_{B^*}$.

Finally, we must prove that ε_{B^*} and Δ_{B^*} are algebra maps. For α,β in B^*,

$$\begin{aligned}
\varepsilon_{B^*}(\alpha\beta) &= (\alpha\beta)\circ\eta_B\colon 1\mapsto(\alpha\beta)(1) = \alpha(1)\beta(1) = \varepsilon_{B^*}(\alpha)\varepsilon_{B^*}(\beta)\\
\Delta_{B^*}(\alpha\beta)&\colon a\otimes b\mapsto(\alpha\beta)(ab) = \sum_{(ab)}\alpha((ab)_{(1)})\beta((ab)_{(2)}) = \sum_{(a),(b)}\alpha(a_{(1)}b_{(1)})\beta(a_{(2)}b_{(2)})\\
&= \sum_{(a),(b),(\alpha),(\beta)}\alpha_{(1)}(a_{(1)})\alpha_{(2)}(b_{(1)})\beta_{(1)}(a_{(2)})\beta_{(2)}(b_{(2)})\\
&= \sum_{(a),(b),(\alpha),(\beta)}\alpha_{(1)}(a_{(1)})\beta_{(1)}(a_{(2)})\alpha_{(2)}(b_{(1)})\beta_{(2)}(b_{(2)})\\
&= \sum_{(\alpha),(\beta)}(\alpha_{(1)}\beta_{(1)})(a)(\alpha_{(2)}\beta_{(2)})(b)\\
&= \sum_{(\alpha),(\beta)}\Big((\alpha_{(1)}\otimes\alpha_{(2)})(\beta_{(1)}\otimes\beta_{(2)})\Big)(a\otimes b)\\
&= (\Delta_{B^*}(\alpha)\Delta_{B^*}(\beta))(a\otimes b)
\end{aligned}$$

hence $\Delta_{B^*}(\alpha\beta) = \Delta_{B^*}(\alpha)\Delta_{B^*}(\beta)$.

Now let $f : B\to B'$ be a morphism of bialgebras. Then

$$\begin{aligned}
(f^*(\alpha\beta))(x) &= (\alpha\beta)(f(x)) = \sum_{(f(x))}\alpha((f(x))_{(1)})\beta((f(x))_{(2)})\\
&= \sum_{(x)}\alpha(f(x_{(1)}))\beta(f(x_{(2)})) = \sum_{(x)}(f^*(\alpha))(x_{(1)})(f^*(\beta))(x_{(2)}) = (f^*(\alpha)f^*(\beta))(x)
\end{aligned}$$

so that $f^*(\alpha\beta) = f^*(\alpha)f^*(\beta)$. Moreover, $f^*(\eta_{B'^*}(1)) = f^*(\varepsilon') = \varepsilon'\circ f = \varepsilon$, so that f^* is a morphism of algebras. We also have

$$\begin{aligned}
\varepsilon_{B^*}\circ f^*(\alpha) = \varepsilon_{B^*}(\alpha\circ f) = \alpha\circ f\circ\eta_B\colon 1\mapsto\alpha(f(1)) = \alpha(1) = \varepsilon_{B'^*}(\alpha)(1)\\
\Delta_{B^*}(f^*(\alpha)) = \Delta_{B^*}(\alpha\circ f)\colon a\otimes b\mapsto\alpha\circ f(ab)\\
(f^*\otimes f^*)\circ\Delta_{B^*}(\alpha) = \sum_{(\alpha)}(\alpha_{(1)}\circ f)\otimes(\alpha_{(2)}\circ f)\colon a\otimes b\mapsto\alpha(f(a)f(b)) = \alpha(f(ab))
\end{aligned}$$

so that $\varepsilon_{B^*}\circ f^* = \varepsilon_{B'^*}$ and $\Delta_{B^*}\circ f^* = (f^*\otimes f^*)\circ\Delta_{B^*}$. Therefore, f^* is a morphism of bialgebras. □

Remark I.14. Note that the dual of a bialgebra is always an algebra (even if the bialgebra is note finite dimensional).

Proposition I.15. *Let B be a finite dimensional bialgebra. Then the canonical isomorphism $i : B\to B^{**}$ is an isomorphism of bialgebras.*

Proof. Let a, b be elements in B and let α, β be elements in B^*. Write i_a for $i(a)$ (so that $i_a(\alpha) = \alpha(a)$). Then

$$\begin{aligned}
i_{ab}(\alpha) &= \alpha(ab) = \sum_{(\alpha)} \alpha_{(1)}(a)\alpha_{(2)}(b) = \sum_{(\alpha)} i_a(\alpha_{(1)})i_b(\alpha_{(2)}) = (i_a i_b)(\alpha) \\
i_1(\alpha) &= \alpha(1) = \varepsilon_{B^*}(\alpha) = (\eta_{B^{**}}(1))(\alpha) = (1_{B^{**}})(\alpha) \\
\Delta_{B^{**}}(i_a)(\alpha \otimes \beta) &= i_a(\alpha\beta) = (\alpha\beta)(a) = \sum_{(a)} \alpha(a_{(1)})\beta(a_{(2)}) = \sum_{(a)} i_{a_{(1)}}(\alpha) i_{a_{(2)}}(\beta) = \sum_{(a)} (i_{a_{(1)}} i_{a_{(2)}})(\alpha \otimes \beta) \\
&= ((i \otimes i)(\Delta(a)))\,(\alpha \otimes \beta) \\
\varepsilon_{B^{**}}(i_a) &= i_a(1_{B^*}) = i_a(\varepsilon) = \varepsilon(a)
\end{aligned}$$

and the result follows. ☐

Example I.16. Let G be a finite group. Then the bialgebras kG and k^G are dual to each other (up to isomorphism).

Proof. The set $\{g; g \in G\}$ is a basis for kG, whose dual basis is $\{e_g; g \in G\}$ where $e_g : kG \to k$ is defined on the given basis of kG by $e_g(h) = \begin{cases} 1 & \text{if } h = g \\ 0 & \text{if } h \neq g. \end{cases}$ Define a k-linear isomorphism $\varphi : (kG)^* \to k^G$ by $\varphi(e_g) = \delta_g$ for all $g \in G$.

We must now prove that φ is an isomorphism of bialgebras.

- $\varphi(1_{(kG)^*}) = \varphi(\varepsilon_{kG}) = \varphi(\sum_{g \in G} e_g) = \sum_{g \in G} \delta_g = 1_{k^G}$.
- For g, h, k in G, we have $e_g e_h(k) = \sum_{(k)} e_g(k_{(1)})e_h(k_{(2)}) = e_g(k)e_h(k) = \begin{cases} 1 & \text{if } k = g = h \\ 0 & \text{otherwise} \end{cases}$ hence $e_g e_h = \begin{cases} e_g & \text{if } g = h \\ 0 & \text{otherwise.} \end{cases}$ Therefore $\varphi(e_g e_h) = \begin{cases} \delta_g & \text{if } g = h \\ 0 & \text{otherwise.} \end{cases} = \delta_g \delta_h = \varphi(e_g)\varphi(e_h)$.
- $\varepsilon_{k^G}(\varphi(e_g)) = \varepsilon_{k^G}(\delta_g) = \begin{cases} 1 & \text{if } g = 1 \\ 0 & \text{otherwise} \end{cases}$ and $\varepsilon_{(kG)^*}(e_g) = e_g(1) = \begin{cases} 1 & \text{if } g = 1 \\ 0 & \text{otherwise} \end{cases}$ so that $\varepsilon_{k^G}(\varphi(e_g)) = \varepsilon_{(kG)^*}(e_g)$.
- We have $\Delta_{k^G}(\varphi(e_g)) = \Delta_{k^G}(\delta_g) = \sum_{h,k \in G; kh = g} \delta_h \otimes \delta_k$. Moreover $\Delta_{(kG)^*}(e_g)$ sends $a \otimes b$ to $e_g(ab) = \begin{cases} \sum_{h,k \in G; hk = g} e_h \otimes e_k & \text{if } ab = g \\ 0 & \text{otherwise} \end{cases}$ so that $(\varphi \otimes \varphi)(\Delta_{(kG)^*}(e_g)) = \Delta_{k^G}(\varphi(e_g))$. ☐

Proposition I.17. *Let B be a bialgebra. Then B is a bimodule over the algebra B^*, where the left and right actions are defined by*

$$\alpha \rightharpoonup b = (\mathrm{id} \otimes \alpha) \circ \Delta(b) = \sum_{(b)} \alpha(b_{(2)})b_{(1)} \quad \text{and} \quad b \leftharpoonup \alpha = (\alpha \otimes \mathrm{id}) \circ \Delta(b) = \sum_{(b)} \alpha(b_{(1)})b_{(2)}.$$

Proof.

- Recall that the unit element in B^* is ε. Clearly, $\varepsilon \rightharpoonup b = b = b \leftharpoonup \varepsilon$.
- $(\alpha\beta) \rightharpoonup b = \sum_{(b)} (\alpha\beta)(b_{(2)})b_{(1)} = \sum_{(b)} \alpha(b_{(2)})\beta(b_{(3)})b_{(1)} = \sum_{(b)} \beta(b_{(2)})(\alpha \rightharpoonup b_{(1)}) = \alpha \rightharpoonup \left(\sum_{(b)} \beta(b_{(2)})b_{(1)}\right) = \alpha \rightharpoonup (\beta \rightharpoonup b)$.

  Similarly, $b \leftharpoonup (\alpha\beta) = (b \leftharpoonup \alpha) \leftharpoonup \beta$.
- $(\alpha \rightharpoonup b) \leftharpoonup \beta = \sum_{(b)} \alpha(b_{(2)})b_{(1)} \leftharpoonup \beta = \sum_{(b)} \alpha(b_{(3)})b_{(2)}\beta(b_{(1)}) = \sum_{(b)} \alpha \rightharpoonup b_{(2)}\beta(b_{(1)}) = \alpha \rightharpoonup (b \leftharpoonup \beta)$. ☐

Definition I.18. *Let B be a bialgebra. An element $x \in B$ is called **grouplike** if $x \neq 0$ and $\Delta(x) = x \otimes x$. The set of grouplike elements in B is denoted by $G(B)$.*

Remark I.19. If x is a grouplike element in B, then $\varepsilon(x) = 1$. Indeed, we have $x = (\varepsilon \otimes \mathrm{id})(\Delta(x)) = \varepsilon(x)x$ with $x \neq 0$.

Example I.20.

- In any bialgebra B, 1 is grouplike (since Δ is an algebra map).
- Let G be a group. Then $G(kG) = G$.

Proof. By definition of Δ, the elements in G are grouplike elements in kG.

Let $x = \sum_{g \in G} \lambda_g g$, with $\lambda_g \in k$ for all g, be a grouplike element in kG. The identity $\Delta(x) = x \otimes x$ becomes $\sum_{g \in G} \lambda_g g \otimes g = \sum_{g,h \in G} \lambda_g \lambda_h g \otimes h$. In particular, $\lambda_g^2 = \lambda_g$ for all $g \in G$ so that $\lambda_g \in \{0;1\}$. Moreover, $\varepsilon(x) = 1$ so that $\sum_{g \in G} \lambda_g = 1$. Therefore precisely one λ_g is equal to 1, the others are equal to 0, so that $x \in G$. ☐

- Let G be a finite group. Then $G(k^G) \cong \mathrm{Alg}_k(kG, k)$.

More generally,

Proposition I.21. *Let B be a finite dimensional bialgebra. Then the set $G(B^*)$ is equal to $\mathrm{Alg}_k(B,k)$.*

Proof. Note that both $G(B^*)$ and $\mathrm{Alg}_k(B,k)$ are subsets of B^*.

Let α be an element in B^*. Then $(\Delta_{B^*}(\alpha))(a \otimes b) = \alpha(ab)$ by definition of Δ_{B^*}, $(\alpha \otimes \alpha)(a \otimes b) = \alpha(a)\alpha(b)$ and $\alpha(1) = \varepsilon_{B^*}(\alpha)$ so that α is a grouplike element if and only if α is an algebra map. Hence $G(B^*) = \mathrm{Alg}_k(B,k)$. ☐

Proposition I.22. *Distinct grouplike elements are linearly independent.*

Proof. By induction on the number n of grouplike elements.

- For $n = 2$, if $\lambda_1 g_1 + \lambda_2 g_2 = 0$, applying ε gives $\lambda_2 = -\lambda_1$ so that $\lambda_1(g_1 - g_2) = 0$ and $\lambda_1 = 0 = \lambda_2$.
- Assume the result true for $n-1$ grouplikes. Suppose that $\sum_{i=1}^n \lambda_i g_i = 0$. Then

$$0 = \Delta\left(\sum_{i=1}^n \lambda_i g_i\right) - \sum_{i=1}^n (\lambda_i g_i) \otimes g_n = \sum_{i=1}^{n-1} \lambda_i g_i \otimes (g_i - g_n).$$

Since $\{g_1, \ldots, g_n\}$ is linearly independent, there are $g_i^* \in H^*$ such that $g_i^*(g_j) = \delta_{i,j}$. Apply $g_j^* \otimes \mathrm{id}$ to the last relation for each j with $1 \leqslant j \leqslant n-1$. Then $\lambda_j(g_j - g_n) = 0$ for $1 \leqslant j \leqslant n-1$ so that $\lambda_j = 0$. Finally, $\lambda_n = 0$ also. ☐

3. Hopf algebras

Definition I.23. *A **Hopf algebra** is a bialgebra H endowed with a linear map $S : H \to H$ that satisfies*

$$S \star \mathrm{id}_H = \eta \circ \varepsilon = \mathrm{id}_H \star S$$

or equivalently

$$\forall x \in H, \sum_{(x)} S(x_{(1)})x_{(2)} = \varepsilon(x)1 = \sum_{(x)} x_{(1)}S(x_{(2)}).$$

*The map S is called the **antipode** of H.*

Remark I.24. The antipode is unique. Indeed, S is the inverse of id_H for the convolution product, and the inverse (when it exists) is unique.

Examples I.25.

- k is a Hopf algebra, with antipode id_k.
- For any finite group G, the bialgebra kG is a Hopf algebra with antipode defined by $S(g) = g^{-1}$.
- For any group G, the bialgebra k^G is a Hopf algebra with antipode defined by $S(\delta_g) = \delta_{g^{-1}}$.
- For any finite dimensional vector space V, the bialgebra $T(V)$ is a Hopf algebra with antipode determined by $S(v) = -v$ for all $v \in V$.

Proposition I.26. *Let H be a Hopf algebra. Then $S : (H, \mu, \eta, \Delta, \varepsilon) \to (H, \mu^{op}, \eta, \Delta^{cop}, \varepsilon)$ is a morphism of bialgebras. In other words, for all x, y in H, we have*

$$S(xy) = S(y)S(x),\ S(1) = 1,\ \varepsilon(S(x)) = \varepsilon(x) \text{ and } \sum_{(x)} S(x_{(1)}) \otimes S(x_{(2)}) = \sum_{(S(x))} (S(x))_{(2)} \otimes (S(x))_{(1)}.$$

Proof. ➢ Let $\sigma,\nu : H\otimes H \to H$ be the linear maps defined by $\sigma(x\otimes y) = S(xy)$ and $\nu(x\otimes y) = S(y)S(x)$. Then

$$\begin{aligned}
(\sigma\star\mu)(x\otimes y) &= \sum_{(x\otimes y)} \sigma((x\otimes y)_{(1)})\mu((x\otimes y)_{(2)}) = \sum_{(x),(y)} \sigma(x_{(1)}\otimes y_{(1)})\mu(x_{(2)}\otimes y_{(2)})\\
&= \sum_{(x),(y)} S(x_{(1)}y_{(1)})x_{(2)}y_{(2)} = \sum_{(xy)} S((xy)_{(1)})(xy)_{(2)} = \varepsilon(xy)1 = \eta_H\circ\varepsilon_{H\otimes H}(x\otimes y)\\
(\mu\star\nu)(x\otimes y) &= \sum_{(x\otimes y)} \mu((x\otimes y)_{(1)})\nu((x\otimes y)_{(2)}) = \sum_{(x),(y)} \mu(x_{(1)}\otimes y_{(1)})\nu(x_{(2)}\otimes y_{(2)})\\
&= \sum_{(x),(y)} x_{(1)}y_{(1)}S(y_{(2)})S(x_{(2)}) = \sum_{(x)} x_{(1)}S(x_{(2)})\varepsilon(y) = \varepsilon(x)\varepsilon(y)1\\
&= \varepsilon(xy)1 = \eta_H\circ\varepsilon_{H\otimes H}(x\otimes y).
\end{aligned}$$

Therefore μ is invertible for the convolution product on $\mathrm{Hom}_k(H\otimes H,H)$, and by uniqueness of the inverse, $\sigma=\mu$ as required. Moreover, $\Delta(1) = 1\otimes 1$ so that $1 = \varepsilon(1)1 = 1S(1) = S(1)$ and therefore $S(1) = 1$.

➢ We must prove that $\Delta^{cop}\circ S = (S\otimes S)\circ\Delta$, which is equivalent to $\Delta\circ S = (S\otimes S)\circ\Delta^{cop}$. Set $\sigma = \Delta\circ S$ and $\nu = (S\otimes S)\circ\Delta^{cop}$. We have

$$\begin{aligned}
(\sigma\star\Delta)(x) &= \sum_{(x)}\sigma(x_{(1)})\Delta(x_{(2)}) = \sum_{(x)}\Delta(S(x_{(1)}))\Delta(x_{(2)}) = \sum_{(x)}\Delta(S(x_{(1)})x_{(2)}) = \Delta(\varepsilon(x)1)\\
&= \varepsilon(x)\,1\otimes 1 = \eta_{H\otimes H}\circ\varepsilon_H(x)\\
(\Delta\star\nu)(x) &= \sum_{(x)}\Delta(x_{(1)})\nu(x_{(2)}) = \sum_{(x)}\Delta(x_{(1)})\left((S\otimes S)(x_{(3)}\otimes x_{(2)})\right) = \sum_{(x)}\Delta(x_{(1)})(S(x_{(3)})\otimes S(x_{(2)}))\\
&= \sum_{(x)}(x_{(1)}\otimes x_{(2)})(S(x_{(4)})\otimes S(x_{(3)})) = \sum_{(x)}x_{(1)}S(x_{(4)})\otimes x_{(2)}S(x_{(3)})\\
&= \sum_{(x)}x_{(1)}S(x_{(3)})\otimes\varepsilon(x_{(2)})1 = \sum_{(x)}x_{(1)}S(x_{(2)})\otimes 1 = \varepsilon(x)1\otimes 1 = \eta_{H\otimes H}\circ\varepsilon_H(x).
\end{aligned}$$

Therefore Δ is invertible for the convolution product on $\mathrm{Hom}_k(H,H\otimes H)$, with inverse σ and ν so that $\sigma=\nu$ as required. Finally,

$$\varepsilon(S(x)) = \sum_{(x)}\varepsilon(S(\varepsilon(x_{(1)})x_{(2)})) = \sum_{(x)}\varepsilon(x_{(1)})\varepsilon(S(x_{(2)})) = \sum_{(x)}\varepsilon(x_{(1)}S(x_{(2)})) = \varepsilon(\varepsilon(x)1) = \varepsilon(x). \qquad \square$$

Definition-Proposition I.27. *A morphism of Hopf algebras is a morphism $f : H\to H'$ of the underlying bialgebras. It satisfies the identity $S'\circ f = f\circ S$.*

Proof. Fix $x\in H$. We have

$$\begin{aligned}
((f\circ S)\star f)(x) &= \sum_{(x)}(f\circ S)(x_{(1)})f(x_{(2)}) = f\left(\sum_{(x)}S(x_{(1)})x_{(2)}\right) = f(\varepsilon(x)1) = \varepsilon(x)1 = \eta\circ\varepsilon(x)\\
(f\star(S'\circ f))(x) &= \sum_{(x)}f(x_{(1)})(S'\circ f)(x_{(2)}) = \sum_{(f(x))} f(x)_{(1)}S'(f(x)_{(2)}) = \varepsilon'(f(x))1 = \varepsilon(x)1 = \eta\circ\varepsilon(x).
\end{aligned}$$

Therefore f is invertible for the convolution product, with inverse $S'\circ f = f\circ S$. $\square$

Proposition I.28. *Let H be a finite dimensional Hopf algebra. Then H^* is a Hopf algebra, whose antipode is the transpose S^* of the antipode S of H.*
Moreover, the canonical isomorphism $i : H\to H^{**}$ is an isomorphism of Hopf algebras.

Proof. We already know that H^* is a bialgebra. We need only check that S^* is the antipode, that is, that $S^*\star\mathrm{id}_{H^*} = \eta_{H^*}\circ\varepsilon_{H^*} : \alpha\mapsto\alpha(1)\varepsilon$ and that $\mathrm{id}_{H^*}\star S^* = \eta_{H^*}\circ\varepsilon_{H^*}$.

For any $\alpha\in H^*$, we have $(S^*\star\mathrm{id}_{H^*})(\alpha) = \sum_{(\alpha)} S^*(\alpha_{(1)})\alpha_{(2)} = \sum_{(\alpha)}(\alpha_{(1)}\circ S)\alpha_{(2)}$. For any $x\in H$ we then have $((S^*\star\mathrm{id}_{H^*})(\alpha))\,(x) = \sum_{(\alpha),(x)}\alpha_{(1)}(S(x_{(1)}))\alpha_{(2)}(x_{(2)}) = \sum_{(x)}\alpha(S(x_{(1)})x_{(2)}) = \alpha(\varepsilon(x)1) = \alpha(1)\varepsilon(x)$ as required. The other identity is similar.

We know that i is an isomorphism of bialgebras, therefore it is an isomorphism of Hopf algebras. $\square$

Example I.29. Let G be a finite group. Then kG and k^G are dual Hopf algebras.

Indeed, we already know that they are dual bialgebras. Moreover, the antipode on $(kG)^*$ is S^* which sends δ_g to $S^*(\delta_g) = \delta_g \circ S : h \mapsto \delta_g(h^{-1}) = \delta_{g^{-1}}(h)$ so that $S^*(\delta_g) = \delta_{g^{-1}}$. Therefore S^* is the antipode of k^G.

Proposition I.30. *Let H be a finite dimensional Hopf algebra. Then the set $G(H^*)$ of grouplike elements in H^* is a group.*

Proof. We prove that $G(H^*)$ is a group for the convolution product $\alpha \star \beta : x \mapsto \sum_{(x)} \alpha(x_{(1)})\beta(x_{(2)})$ of H^*. Recall that $G(H^*) = \mathrm{Alg}_k(H, k)$.

- ➢ The law is associative since H^* is an associative algebra.
- ➢ The counit ε is in $G(H^*)$ since it is an algebra map, and it is the unit element in H^*, hence it is the unit element in $G(H^*)$.
- ➢ Let α be an element in $G(H^*)$. Then

$$((\alpha \circ S) \star \alpha)(h) = \sum_{(h)} \alpha(S(h_{(1)}))\alpha(h_{(2)}) = \sum_{(h)} \alpha(S(h_{(1)})h_{(2)}) = \alpha(\varepsilon(h)1) = \varepsilon(h)\alpha(1) = \varepsilon(h)$$

so that $(\alpha \circ S) \star \alpha = \varepsilon$. Similarly, $\alpha \star (\alpha \circ S) = \varepsilon$. Therefore, $\alpha \circ S$ is the inverse of α in $G(H^*)$. □

Theorem I.31. *Let H be a finite dimensional Hopf algebra that is isomorphic to k^n as an algebra. Then there exists a finite group G such that $H \cong k^G$ as Hopf algebras.*

Proof. Set $G = G(H^*) = \mathrm{Alg}_k(H, k)$. We know that G is a group, whose law is the restriction of the product of H^* to G.

- ➢ Let $\{e_1, \dots, e_n\}$ be the canonical basis of $k^n \cong H$. Let $\{e_1^*, \dots, e_n^*\}$ be the dual basis. Then $e_i^* \in G$ for all i; we need only check that $e_i^*(e_j e_k) = e_i^*(e_j)e_i^*(e_k)$ for all j, k and that $e_i^*(1) = 1$:
  - ✧ We have $e_i^*(e_j e_k) = e_i^*(\delta_{j,k} e_j) = \delta_{i,j}\delta_{j,k}$ and $e_i^*(e_j)e_i^*(e_k) = \delta_{i,j}\delta_{i,k} = \delta_{i,j}\delta_{j,k}$.
  - ✧ $e_i^*(1) = e_i^*(\sum_{j=1}^n e_j) = \sum_{j=1}^n e_i^*(e_j) = 1$.

  Consequently, $\operatorname{span}\{e_1^*, \dots, e_n^*\} \subset kG \subset H^* = \operatorname{span}\{e_1^*, \dots, e_n^*\}$, so that $kG = H^*$ as vector spaces.
- ➢ Since kG and H^* have the same unit element and the same product, kG and H^* are equal as algebras. Moreover, they have the same comultiplication and counit (it is enough to check this on the basis elements e_i^*), therefore they are equal as bialgebras.
- ➢ Dualising gives $H \cong k^G$. □

Definition I.32. *Let H be a Hopf algebra.*

*A **Hopf ideal** in H is a two-sided ideal I in the algebra H such that*

$$\Delta(I) \subseteq I \otimes H + H \otimes I, \quad \varepsilon(I) = 0 \quad \text{and} \quad S(I) \subseteq I.$$

*A **Hopf subalgebra** of H is a subalgebra K of H that satisfies*

$$\Delta(K) \subseteq K \otimes K \quad \text{and} \quad S(K) \subseteq K.$$

Lemma I.33. *Let H be a Hopf algebra and let E be a subset of H that satisfies $\Delta(E) \subseteq E \otimes H + H \otimes E$, $\varepsilon(E) = 0$ and $S(E) \subseteq E$. Then the ideal in H generated by E is a Hopf ideal.*

Proof. Let I be the ideal generated by E. Let h be an element in I, then $h = \sum_{j \in J} u_j e_j v_j$ where J is a finite set, the e_j are in E and the u_j, v_j are in H. Clearly, since ε is a morphism of algebras such that $\varepsilon(E) = 0$, we have $\varepsilon(I) = 0$.

By assumption, $\Delta(e_j) \in E \otimes H + H \otimes E$ so that $\Delta(e_j) = \sum_{r \in R} e_r \otimes x_{rj} + \sum_{t \in T} y_{tj} \otimes e_t$ where R, T are finite sets, the e_r, e_t are in E and the x_{rj}, y_{tj} are in H. We then have

$$\begin{aligned}
\Delta(h) &= \sum_{j \in J} \Delta(u_j)\Delta(e_j)\Delta(v_j) \\
&= \sum_{j \in J} \left(\sum_{(u_j)} (u_j)_{(1)} \otimes (u_j)_{(2)} \right) \left(\sum_{r \in R} e_r \otimes x_{rj} + \sum_{t \in T} y_{tj} \otimes e_t \right) \left(\sum_{(v_j)} (v_j)_{(1)} \otimes (v_j)_{(2)} \right) \\
&= \sum_{j \in J} \sum_{(u_j),(v_j)} \sum_{r \in R} (u_j)_{(1)} e_r (v_j)_{(1)} \otimes (u_j)_{(2)} x_{rj} (v_j)_{(2)} \\
&\quad + \sum_{j \in J} \sum_{(u_j),(v_j)} \sum_{t \in TR} (u_j)_{(1)} y_{tj} (v_j)_{(1)} \otimes (u_j)_{(2)} e_t (v_j)_{(2)} \\
&\in H \otimes I + I \otimes H
\end{aligned}$$

and $S(h) = \sum_{j \in J} S(v_j) S(e_j) S(u_j)$ is in I since $S(e_j)$ is in E by assumption.

Therefore $\Delta(I) \subseteq I \to H + H \otimes I$ and $S(I) \subseteq I$ as required. □

Example I.34. Let $\mathfrak{g}$ be a Lie algebra and let I be the ideal in $T(\mathfrak{g})$ generated by the elements $xy - yx - [x, y]$ for all x, y in $\mathfrak{g}$. Then I is a Hopf ideal in $T(\mathfrak{g})$.

Set $E = \{xy - yx - [x, y]; x \in \mathfrak{g}, y \in \mathfrak{g}\}$. We need only check that $\Delta(E) \subseteq E \otimes H + H \otimes E$, $\varepsilon(E) = 0$ and $S(E) \subseteq E$. The fact that $\varepsilon(E) = 0$ is clear since ε vanishes on all elements of positive degree. Moreover,

$$\begin{aligned}
\Delta(xy - yx - [x, y]) &= \Delta(x)\Delta(y) - \Delta(y)\Delta(x) - \Delta([x, y]) \\
&= (1 \otimes x + x \otimes 1)(1 \otimes y + y \otimes 1) - (1 \otimes y + y \otimes 1)(1 \otimes x + x \otimes 1) \\
&\quad - (1 \otimes [x, y] + [x, y] \otimes 1) \\
&= xy \otimes 1 - yx \otimes 1 - [x, y] \otimes 1 + 1 \otimes xy - 1 \otimes yx - 1 \otimes [x, y] \\
&= (xy - yx - [x, y]) \otimes 1 + 1 \otimes (xy - yx - [x, y]) \in E \otimes H + H \otimes E \\
S(xy - yx - [x, y]) &= S(y)S(x) - S(x)S(y) - S([x, y]) = (-y)(-x) - (-x)(-y) - (-[x, y]) \\
&= yx - xy - [y, x] \in E.
\end{aligned}$$

Lemma I.35. *Let $f \colon U \to U'$ and $g \colon V \to V'$ be linear maps. Then* $\operatorname{Ker}(f \otimes g) = \operatorname{Ker}(f) \otimes V + U \otimes \operatorname{Ker}(g)$.

Proof. The inclusion $\operatorname{Ker}(f) \otimes V + U \otimes \operatorname{Ker}(g) \subseteq \operatorname{Ker}(f \otimes g)$ is clear.

Let $\{x_i; i \in I\}$ be a basis of $\operatorname{Ker}(f)$, that we complete to obtain a basis $\{x_i; i \in I\} \cup \{y_j; j \in J\}$ of U. The restriction of f to $W = \operatorname{span}\{y_j; j \in J\}$ is injective. If $X \in \operatorname{Ker}(f \otimes g)$, then X can be written uniquely $X = \sum_{i \in I} x_i \otimes z_i + \sum_{j \in J} y_j \otimes t_j$ for some z_i, t_j in V. We then have $\sum_{j \in J} f(y_j) \otimes g(t_j) = 0$ with the $f(y_j)$ linearly independent, therefore $g(t_j) = 0$ for all $j \in J$ (for $j \in J$, let $\alpha_j \in U^*$ be equal to 1 on $f(y_j)$ and 0 on all other elements of a basis of U containing $\{f(y_j); j \in J\}$, then apply $\alpha_j \otimes \mathrm{id}_V$). Finally $X \in \operatorname{Ker}(f) \otimes V + U \otimes \operatorname{Ker}(g)$. □

Proposition I.36. *Let $f : H \to H'$ be a morphism of Hopf algebras. Then* $\operatorname{Ker} f$ *is a Hopf ideal in H and* $\operatorname{Im} f$ *is a Hopf subalgebra of H'.*

Proof. Since f is a morphism of algebras, $\operatorname{Ker} f$ is an ideal in H and $\operatorname{Im} f$ is a subalgebra of H.

Take $x \in \operatorname{Ker} f$. Then $(f \otimes f)(\Delta(x)) = \Delta'(f(x)) = \Delta'(0) = 0$ so that $\Delta(x) \in \operatorname{Ker}(f \otimes f) = \operatorname{Ker} f \otimes H + H \otimes \operatorname{Ker} f$ by Lemma I.35. Moreover, $f(S(x)) = S'(f(x)) = S'(0) = 0$ so that $S(x) \in \operatorname{Ker} f$. Finally, $\varepsilon(x) = \varepsilon'(f(x)) = \varepsilon'(0) = 0$, therefore $\operatorname{Ker} f$ is a Hopf ideal of H.

Now let $y = f(x)$ be an element of $\operatorname{Im} f$. Then $\Delta'(y) = \Delta'(f(x)) = (f \otimes f)(\Delta(x)) \in \operatorname{Im}(f \otimes f) = \operatorname{Im} f \otimes \operatorname{Im} f$ and $S'(y) = S'(f(x)) = f(S(x)) \in \operatorname{Im} f$. Therefore $\operatorname{Im} f$ is a Hopf subalgebra of H'. □

Proposition I.37. *Let H be a Hopf algebra and let I be a Hopf ideal in H. Then there exists a unique structure of Hopf algebra on the algebra H/I such that the natural projection $\pi : H \to H/I$ is a morphism of Hopf algebras.*

Proof. The algebra map $(\pi \otimes \pi) \circ \Delta : H \to H/I \otimes H/I$ vanishes on the ideal I, therefore it induces a unique algebra map $\bar{\Delta} : H/I \to H/I \otimes H/I$ such that $\bar{\Delta} \circ \pi = (\pi \otimes \pi) \circ \Delta$. Similarly, ε induces a unique algebra map $\bar{\varepsilon} : H/I \to k$ such that $\bar{\varepsilon} \circ \pi = \varepsilon$ and S induces a unique algebra map $\bar{S} : H/I \to (H/I)^{op}$ such that $\bar{S} \circ \pi = \pi \circ S$.

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi\otimes\pi} \\ H/I & \xrightarrow[\bar{\Delta}]{} & H/I \otimes H/I \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{\varepsilon} & k \\ {\scriptstyle \pi}\downarrow & \nearrow_{\bar{\varepsilon}} & \\ H/I & & \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{S} & H^{op} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ H/I & \xrightarrow[\bar{S}]{} & (H/I)^{op} \end{array}$$

Note that the product and unit maps on H/I satisfy $\bar{\mu} \circ (\pi \otimes \pi) = \pi \circ \mu$ and $\bar{\eta} = \pi \circ \eta$. We have

$$\begin{aligned}
(\bar{\Delta} \otimes \mathrm{id}) \circ \bar{\Delta} \circ \pi &= (\bar{\Delta} \otimes \mathrm{id}) \circ (\pi \otimes \pi) \circ \Delta = (\pi \otimes \pi \otimes \pi) \circ (\Delta \otimes \mathrm{id}) \circ \Delta \\
&= (\pi \otimes \pi \otimes \pi) \circ (\mathrm{id} \otimes \Delta) \circ \Delta = (\mathrm{id} \otimes \bar{\Delta}) \circ (\pi \otimes \pi) \circ \Delta = (\mathrm{id} \otimes \bar{\Delta}) \circ \bar{\Delta} \circ \pi \\
(\bar{\varepsilon} \otimes \mathrm{id}) \circ \bar{\Delta} \circ \pi &= (\bar{\varepsilon} \otimes \mathrm{id}) \circ (\pi \otimes \pi) \circ \Delta = (\varepsilon \otimes \pi) \circ \Delta = \pi \\
(\mathrm{id} \otimes \bar{\varepsilon}) \circ \bar{\Delta} \circ \pi &= \mathrm{id} \otimes (\bar{\varepsilon}) \circ (\pi \otimes \pi) \circ \Delta = (\pi \otimes \varepsilon) \circ \Delta = \pi \\
\bar{\mu} \circ (\bar{S} \otimes \mathrm{id}) \circ \bar{\Delta} \circ \pi &= \bar{\mu} \circ (\bar{S} \otimes \mathrm{id}) \circ (\pi \otimes \pi) \circ \Delta = \bar{\mu} \circ (\pi \otimes \pi) \circ (S \otimes \mathrm{id})\Delta = \pi \circ \mu \circ (S \otimes \mathrm{id})\Delta \\
&= \pi \circ \eta \circ \varepsilon = \bar{\eta} \circ \bar{\varepsilon} \circ \pi \\
\bar{\mu} \circ (\mathrm{id} \otimes \bar{S}) \circ \bar{\Delta} \circ \pi &= \bar{\mu} \circ (\mathrm{id} \otimes \bar{S}) \circ (\pi \otimes \pi) \circ \Delta = \bar{\mu} \circ (\pi \otimes \pi) \circ (\mathrm{id} \otimes S)\Delta = \pi \circ \mu \circ (\mathrm{id} \otimes S)\Delta \\
&= \pi \circ \eta \circ \varepsilon = \bar{\eta} \circ \bar{\varepsilon} \circ \pi
\end{aligned}$$

and since π is surjective, we get $(\bar{\Delta} \otimes \mathrm{id}) \circ \bar{\Delta} = (\mathrm{id} \otimes \bar{\Delta}) \circ \bar{\Delta}$, $(\bar{\varepsilon} \otimes \mathrm{id}) \circ \bar{\Delta} = \mathrm{id} = (\mathrm{id} \otimes \bar{\varepsilon}) \circ \bar{\Delta}$ and $\bar{\mu} \circ (\bar{S} \otimes \mathrm{id}) \circ \bar{\Delta} = \bar{\eta} \circ \bar{\varepsilon} = \bar{\mu} \circ (\mathrm{id} \otimes \bar{S}) \circ \bar{\Delta}$ so that H/I is a bialgebra with structure maps $\bar{\Delta}$, $\bar{\varepsilon}$ and $\bar{S}$.

It is clear from the formulas (or diagrams) above that π is a morphism of Hopf algebras. □

Example I.38. Let $\mathfrak{g}$ be a Lie algebra. Let $U(\mathfrak{g}) = T(\mathfrak{g}) / (\{xy - yx - [x, y]; x \in \mathfrak{g}, y \in \mathfrak{g}\})$. Then $U(\mathfrak{g})$ is a Hopf algebra, whose comultiplication and counit are determined by

$$\begin{aligned}
\Delta(x) &= x \otimes 1 + 1 \otimes x \quad \text{for all } x \in \mathfrak{g} \\
\varepsilon(x) &= 0 \quad \text{for all } x \in \mathfrak{g} \\
\varepsilon(1) &= 1.
\end{aligned}$$

Indeed, we have already seen that $(\{xy - yx - [x, y]; x \in \mathfrak{g}, y \in \mathfrak{g}\})$ is a Hopf ideal in $T(\mathfrak{g})$.

II. Introduction to Hopf bimodules.

Let A be an algebra. Recall that a left A-module is a vector space M endowed with a k-linear map $\mu_M : A \otimes M \to M$ that satisfies

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{\mu_M \otimes \mathrm{id}} & A \otimes M \\ {\scriptstyle \mathrm{id} \otimes \mu_M}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \mu_M} \\ A \otimes M & \xrightarrow[\mu_M]{} & M \end{array} \qquad \begin{array}{ccc} k \otimes M & \xrightarrow{\eta \otimes \mathrm{id}} & A \otimes M \\ & {\scriptstyle \cong}\searrow \;\circlearrowleft & \downarrow{\scriptstyle \mu_M} \\ & & M \end{array}$$

and a right A-module is a vector space M endowed with a k-linear map $\mu_M : M \otimes A \to M$ that satisfies

$$\begin{array}{ccc} M \otimes A \otimes A & \xrightarrow{\mathrm{id} \otimes \mu_M} & M \otimes A \\ {\scriptstyle \mu_M \otimes \mathrm{id}}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \mu_M} \\ M \otimes A & \xrightarrow[\mu_M]{} & M \end{array} \qquad \begin{array}{ccc} M \otimes A & \xleftarrow{\mathrm{id} \otimes \eta} & M \otimes k \\ {\scriptstyle \mu_M}\downarrow & \circlearrowleft \;\swarrow{\scriptstyle \cong} & \\ M & & \end{array}$$

Finally, an A-bimodule is a left module and a right module M with structure maps $\mu_\ell : A \otimes M \to M$ and $\mu_r : M \otimes A \to M$ that satisfy

$$\begin{array}{ccc} A \otimes M \otimes A & \xrightarrow{\mu_\ell \otimes \mathrm{id}} & M \otimes A \\ {\scriptstyle \mathrm{id} \otimes \mu_r}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \mu_r} \\ A \otimes M & \xrightarrow[\mu_\ell]{} & M \end{array}$$

(that is, the left and right actions commute).

We will now formally dualise these definitions.

Definition II.1. *Let B be a bialgebra. A* ***left comodule*** *over B is a pair (V, ρ_V) where V is a vector space and $\rho_V : V \to B \otimes V$ is a linear map that satisfies*

$$\begin{array}{ccc} B \otimes B \otimes V & \xleftarrow{\rho_V \otimes \mathrm{id}} & B \otimes V \\ {\scriptstyle \mathrm{id} \otimes \rho_V} \uparrow & \circlearrowleft & \uparrow {\scriptstyle \rho_V} \\ B \otimes V & \xleftarrow[\rho_V]{} & V \end{array} \qquad \begin{array}{ccc} k \otimes V & \xleftarrow{\varepsilon \otimes \mathrm{id}} & B \otimes V \\ & \overset{\circlearrowleft}{\nwarrow}{\scriptstyle \cong} & \uparrow {\scriptstyle \rho_V} \\ & & V \end{array}$$

The map ρ_V is called the left coaction. A ***right comodule*** *over B is a pair (V, ρ_V) where V is a vector space and $\rho_V : V \to V \otimes B$ is a linear map that satisfies*

$$\begin{array}{ccc} V \otimes B \otimes B & \xleftarrow{\mathrm{id} \otimes \rho} & V \otimes B \\ {\scriptstyle \rho_V \otimes \mathrm{id}} \uparrow & \circlearrowleft & \uparrow {\scriptstyle \rho_V} \\ V \otimes B & \xleftarrow[\rho_V]{} & V \end{array} \qquad \begin{array}{ccc} V \otimes B & \xrightarrow{\mathrm{id} \otimes \varepsilon} & V \otimes k \\ {\scriptstyle \rho_V} \uparrow & \overset{\circlearrowleft}{\nearrow}{\scriptstyle \cong} & \\ V & & \end{array}$$

The map ρ_V is called the right coaction. A ***bicomodule*** *over B is a left comodule and right comodule V with structure maps $\rho_\ell : V \to B \otimes V$ and $\rho_r : V \to V \otimes B$ that commute:*

$$\begin{array}{ccc} B \otimes V \otimes B & \xleftarrow{\rho_\ell \otimes \mathrm{id}} & V \otimes B \\ {\scriptstyle \mathrm{id} \otimes \rho_r} \uparrow & \circlearrowleft & \uparrow {\scriptstyle \rho_r} \\ B \otimes V & \xleftarrow[\rho_\ell]{} & V \end{array}$$

Notation II.2. There is also a Sweedler notation for comodules.

➢ If V is a right B-comodule, we put $\rho_V(v) = \sum_{(v)} v_{(0)} \otimes v_{(1)}$. The axioms become, for all $v \in V$,

$$\begin{aligned} \sum_{(m),(m_{(1)})} m_{(0)} \otimes (m_{(1)})_{(1)} \otimes (m_{(1)})_{(2)} &= \sum_{(m),(m_{(0)})} (m_{(0)})_{(0)} \otimes (m_{(0)})_{(1)} \otimes m_{(1)} \\ &=: \sum_{(m)} m_{(0)} \otimes m_{(1)} \otimes m_{(2)}. \end{aligned}$$

➢ If V is a left B-comodule, we put $\rho_V(v) = \sum_{(v)} v_{(-1)} \otimes v_{(0)}$. The axioms become, for all $v \in V$,

$$\begin{aligned} \sum_{(m),(m_{(-1)})} (m_{(-1)})_{(1)} \otimes (m_{(-1)})_{(2)} \otimes m_{(0)} &= \sum_{(m),(m_{(0)})} m_{(-1)} \otimes (m_{(0)})_{(-1)} \otimes (m_{(0)})_{(0)} \\ &=: \sum_{(m)} m_{(-2)} \otimes m_{(-1)} \otimes m_{(0)}. \end{aligned}$$

Example II.3. ➢ H is a bicomodule over itself, using Δ.

➢ k is a bicomodule over H, using η: for any $\lambda \in k$, $\rho_\ell(\lambda) = 1_H \otimes \lambda$ and $\rho_r(\lambda) = \lambda \otimes 1_H$. Using the identifications $k \otimes H \cong H \cong H \otimes k$, both coactions are given by η. This is called the **trivial** bicomodule (or comodule if we forget one of the structures).

Examples of constructions of new H-(co)modules over a Hopf algebra H.

➢ Let M be a left H-module. Then M is a right H-module via S, that is,

$$\forall h \in H, \forall m \in M, \quad m \triangleleft h = S(h)m.$$

Similarly, every right H-module is a left H-module via S.

➢ Let M be a left H-module. It is well known that the k-dual M^* is a right H-module:

$$\forall \alpha \in M^*, \forall h \in H, \forall m \in M, \quad (h \cdot \alpha)(m) = \alpha(mh)$$

Hence M^* is a left H-module via S.

➢ Let M and N be two left H-modules. Then $M \otimes N$ is a left H-module via Δ, that is,

$$\forall h \in H, \forall m \in M, \forall n \in N, \quad h(m \otimes n) = \sum_{(h)} h_{(1)} m \otimes h_{(2)} n.$$

This action of H on $M \otimes N$ is called **diagonal**.

➢ We can dualise the previous construction. Let M and N be two left H-comodules. Then $M \otimes N$ is a left H-comodule with coaction

$$\rho_{M\otimes N} = (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta),$$

that is, $\rho_{M\otimes N}(m \otimes n) = \sum_{(m),(n)} m_{(-1)}n_{(-1)} \otimes m_{(0)} \otimes n_{(0)}$. This coaction is called **codiagonal**.

Definition II.4. *Let M and N be two left comodules over B. A **morphism of left comodules** from M to N is a linear map $f : M \to N$ such that $\rho_n \circ f = (\mathrm{id} \otimes f) \circ \rho_M$, that is,*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle \rho_M}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \rho_N} \\ B \otimes M & \xrightarrow[\mathrm{id}\otimes f]{} & B \otimes N \end{array}$$

*A **morphism of right comodules** is defined similarly. A **morphism of bicomodules** is a morphism of left and right comodules.*

We shall now combine module and comodule structures.

Definition II.5. *Let H be a Hopf algebra. A **left Hopf module** over H is a left H-module M that is also a left comodule whose structure map $\rho_M : M \to H \otimes M$ is a morphism of left H-modules, where the left H-module structure on $H \otimes M$ is the diagonal structure given above (with the Sweedler notation, this can be written $\sum_{(hm)} (hm)_{(-1)} \otimes (hm)_{(0)} = \sum_{(m),(h)} h_{(1)}m_{(-1)} \otimes h_{(2)}m_{(0)}$).*

*A **morphism of left Hopf modules** is a morphism of left H-modules that is also a morphism of left H-comodules.*

*The definitions of a **right Hopf module** over H and of a **morphism of right Hopf modules** are similar.*

*A **Hopf bimodule** over H is an H-bimodule M that is also a bicomodule whose structure maps $\rho_\ell : M \to H \otimes M$ and $\rho_r : M \to M \otimes H$ are morphisms of H-bimodules. A **morphism of Hopf bimodules** is a morphism of H-bimodules that is also a morphism of H-bicomodules.*

Example II.6. ➢ Let H be a Hopf algebra. Then H is a left (*resp.* right) Hopf module with coaction Δ.

➢ Let M be any left H-module. Then $H \otimes M$ is a left H-module for the diagonal action. It is moreover a left Hopf module with coaction $\Delta \otimes \mathrm{id}$.

Proof. The fact that $H \otimes M$ is a left comodule follows from the properties of Δ. We must check that $\rho = \Delta \otimes \mathrm{id}$ is a morphism of left H-modules. Let a, h be elements in H and m be an element of M. Then

$$a\rho(h \otimes m) = a\left(\sum_{(h)} h_{(1)} \otimes h_{(2)} \otimes m\right) = \sum_{(h),(a)} a_{(1)}h_{(1)} \otimes a_{(2)}h_{(2)} \otimes a_{(3)}m$$

$$\rho(a(h \otimes m)) = \rho(\sum_{(a)} a_{(1)}h \otimes a_{(2)}m) = \sum_{(a)} \Delta(a_{(1)}h) \otimes a_{(2)}m = \sum_{(h),(a)} a_{(1)}h_{(1)} \otimes a_{(2)}h_{(2)} \otimes a_{(3)}m$$

so that $a\rho(h \otimes m) = \rho(a(h \otimes m))$. □

➢ Let V be a vector space. Then V is a left H-module via ε (that is, $h \cdot v = \varepsilon(h)v$ for $v \in V$ and $h \in H$, we say the V is a trivial left H-module). Therefore $M = H \otimes V$ is a left Hopf module, with $\mu_M = \mu \otimes \mathrm{id}$ and $\rho_M = \Delta \otimes \mathrm{id}$.

Indeed, $a(h \otimes v) = \sum_{(a)} a_{(1)}h \otimes \varepsilon(a_{(2)})v = \sum_{(a)} a_{(1)}\varepsilon(a_{(2)})h \otimes v = ah \otimes v$.

This will be called a **trivial Hopf module**.

➢ H is a Hopf bimodule for the multiplication and comultiplication of H.

➢ Let M and N be Hopf bimodules (*eg.* $M = H = N$) and let V be a bicomodule. Then $M \otimes V \otimes N$ is a Hopf bimodule with the following structure maps:

$$h \cdot (m \otimes v \otimes n) = (hm) \otimes v \otimes n, \quad \rho_\ell(m \otimes v \otimes n) = \sum_{(m),(v),(n)} m_{(-1)}v_{(-1)}n_{(-1)} \otimes (m_{(0)} \otimes v_{(0)} \otimes n_{(0)}),$$

$$(m \otimes v \otimes n) \cdot h = m \otimes v \otimes (nh), \quad \rho_r(m \otimes v \otimes n) = \sum_{(m),(v),(n)} (m_{(0)} \otimes v_{(0)} \otimes n_{(0)}) \otimes m_{(1)}v_{(1)}n_{(1)}$$

for any $h \in H$, $m \in M$, $n \in N$ and $v \in V$ (the coactions are codiagonal).

In particular, taking $V = k$, the tensor product of Hopf bimodules is a Hopf bimodule as above. For instance, $H^{\otimes n}$ is a Hopf bimodule with codiagonal coactions for any $n \in \mathbb{N}$.

➢ Let M and N be Hopf bimodules (*eg.* $M = H = N$) and let W be a comodule. Then $M \otimes W \otimes N$ is a Hopf bimodule with the following structure maps:

$$h \cdot (m \otimes w \otimes n) = \sum_{(h)} (h_{(1)}m) \otimes (h_{(2)}w) \otimes (h_{(3)}n), \qquad \rho_\ell(m \otimes w \otimes n) = \sum_{(m)} m_{(-1)} \otimes (m_{(0)} \otimes w \otimes n),$$

$$(m \otimes w \otimes n) \cdot h = \sum_{(h)} (mh_{(1)}) \otimes (wh_{(2)}) \otimes (nh_{(3)}), \qquad \rho_r(m \otimes w \otimes n) = \sum_{(n)} (m \otimes w \otimes n_{(0)}) \otimes n_{(1)}$$

for any $h \in H$, $m \in M$, $n \in N$ and $w \in W$ (the actions are diagonal).

In particular, taking $W = k$, the tensor product of Hopf bimodules is a Hopf bimodule as above (the structure is not the same as in the previous example). For instance, $H^{\otimes n}$ is a Hopf bimodule for any with diagonal actions $n \in \mathbb{N}$.

We shall now see that every Hopf module is isomorphic to a trivial Hopf module $H \otimes V$. For this we need the following definition.

Definition II.7. *Let B be a bialgebra and let M be a left B-comodule. The space of (left)* ***coinvariants*** *of M is the vector space ${}^HM := \{m \in M; \rho(m) = 1 \otimes m\}$.*

Theorem II.8 (Fundamental Theorem for Hopf modules)**.** *Let H be a Hopf algebra and let M be a left Hopf module. Then $M \cong H \otimes {}^HM$ as left Hopf modules, where $H \otimes {}^HM$ is a trivial Hopf module.*
In particular, M is a free left H-module of rank $\dim_k {}^HM$.

Proof. Define $\varphi : H \otimes {}^HM \to M$ by $\varphi(h \otimes v) = hv$ and $\psi : M \to H \otimes M$ by $\psi(m) = \sum_{(m)} m_{(-2)} \otimes S(m_{(-1)})m_{(0)}$.

➢ We first check that $\psi(M) \subseteq H \otimes {}^HM$.

$$\begin{aligned}\sum_{(m)} \rho(S(m_{(-1)})m_{(0)}) &= \sum_{(m)} S(m_{(-1)})\rho(m_{(0)}) = \sum_{(m)} S(m_{(-2)})(m_{(-1)} \otimes m_{(0)}) \\ &= \sum_{(m),(S(m))} (S(m_{(-2)}))_{(1)}m_{(-1)} \otimes (S(m_{(-2)}))_{(2)}m_{(0)} \\ &= \sum_{(m)} S(m_{(-2)})m_{(-1)} \otimes S(m_{(-3)})m_{(0)} = \sum_{(m)} \varepsilon(m_{(-1)})1 \otimes S(m_{(-2)})m_{(0)} \\ &= \sum_{(m)} 1 \otimes S(m_{(-1)})m_{(0)}\end{aligned}$$

so that $S(m_{(-1)})m_{(0)} \in {}^HM$.

➢ We now check that φ is a bijection.

$$\psi \circ \varphi(h \otimes v) = \psi(hv) = \sum_{(hv)} (hv)_{(-2)} \otimes S((hv)_{(-1)})(hv)_{(0)} = \sum_{(h),(v)} h_{(1)}v_{(-2)} \otimes S(h_{(2)}v_{(-1)})h_{(3)}v_{(0)}$$

$$\overset{(v \in {}^HM)}{=} \sum_{(h)} h_{(1)} \otimes S(h_{(2)})h_{(3)}v = \sum_{(h)} h_{(1)} \otimes \varepsilon(h_{(2)})v = h \otimes v$$

$$\begin{aligned}\varphi \circ \psi(m) &= \sum_{(m)} \varphi(m_{(-2)} \otimes S(m_{(-1)})m_{(0)}) = \sum_{(m)} m_{(-2)}S(m_{(-1)})m_{(0)} \\ &= \sum_{(m),(m_{(-1)})} (m_{(-1)})_{(1)}S((m_{(-1)})_{(2)})m_{(0)} = \sum_{(m)} \varepsilon(m_{(-1)})m_{(0)} = m\end{aligned}$$

so that $\psi \circ \varphi = \mathrm{id}$ and $\varphi \circ \psi = \mathrm{id}$.

➢ We finally prove that φ is a morphism of Hopf modules. It is clearly an H-module morphism, and, since $v \in {}^HM$,

$$\begin{aligned}\rho \circ \varphi(h \otimes v) = \rho(hv) &= \sum_{(h),(v)} h_{(1)}v_{(-1)} \otimes h_{(2)}v_{(0)} = \sum_{(h)} h_{(1)} \otimes h_{(2)}v \\ &= \sum_{(h)} h_{(1)} \otimes \varphi(h_{(2)} \otimes v) = (\mathrm{id} \otimes \varphi)(\sum_{(h)} h_{(1)} \otimes h_{(2)} \otimes v) = (\mathrm{id} \otimes \varphi)(\rho(h \otimes v))\end{aligned}$$

so that $\rho \circ \varphi = (\mathrm{id} \otimes \varphi) \circ \rho$. □

1. Path algebra and identification with a tensor algebra

Definition III.1. *Recall from Patrick Le Meur's lectures that a **quiver** is an oriented graph Γ. We denote by Γ_0 the set of vertices, Γ_1 the set of arrows and more generally Γ_n the set of paths of length n in Γ. We shall always assume that the quiver is **finite**, that is, that Γ_0 and Γ_1 are finite sets. There are two maps $\mathfrak{s}, \mathfrak{t} : \Gamma_1 \to \Gamma_0$, which associate to an arrow in Γ its source and its target respectively.*

*The **path algebra** $k\Gamma$ is the k-vector space with basis the set $\bigcup_{n\in\mathbb{N}} \Gamma_n$ of paths in Γ, and the product of two paths p and q is given by* $pq = \begin{cases} \text{concatenation of } p \text{ and } q & \text{if } \mathfrak{t}(p) = \mathfrak{s}(q) \\ 0 & \text{otherwise.} \end{cases}$

The algebra $k\Gamma$ is graded, with $(k\Gamma)_n = k\Gamma_n$.

We will show that $k\Gamma$ is isomorphic to a tensor algebra.

Definition III.2. *Let R be a k-algebra and let M be an R-bimodule. The **tensor algebra** of M over R is the R-bimodule $T_R(M) = \bigoplus_{n\in\mathbb{N}} T^n_R(M) := R \oplus \bigoplus_{n\in\mathbb{N}^*} M^{\otimes_R n}$ in which the product is defined by*

$$(x_1 \otimes_R \cdots \otimes_R x_p) \cdot (y_1 \otimes_R \cdots \otimes_R y_q) = x_1 \otimes_R \cdots \otimes_R x_p \otimes_R y_1 \otimes_R \cdots \otimes_R y_q$$

for $x_1 \otimes_R \cdots \otimes_R x_p \in T^p_R(M)$, $y_1 \otimes_R \cdots \otimes_R y_q \in T^q_R(M)$.

First recall the universal property of the tensor algebra $T_R(M)$ (where R is a k-algebra and M is an R-bimodule).

Proposition III.3. *For any k-algebra A and any homomorphisms $\varphi_R : R \to A$ of k-algebras and $\varphi_M : M \to A$ of R-bimodules, where A is an R-bimodule via φ_R, there exists a unique homomorphism $\Phi : T_R(M) \to A$ of k-algebras such that $\Phi_{|R} = \varphi_R$ and $\Phi_{|M} = \varphi_M$.*
If moreover $A = \bigoplus_{n\in\mathbb{N}} A_n$ is graded, $\operatorname{Im} \varphi_R \subseteq A_0$ and $\operatorname{Im} \varphi_M \subseteq A_1$, then Φ is graded.

Proof.
- ➢ Uniqueness. If $\Phi : T_R(M) \to A$ is an algebra map such that $\Phi_{|R} = \varphi_R$ and $\Phi_{|M} = \varphi_M$, then $\Phi_{|R} = \varphi_R = \Phi_{|R}$ and, for $n \geqslant 1$ and $x_1, \ldots, x_n$ in M,

$$\Phi(x_1 \otimes_R \cdots \otimes_R x_n) = \Phi(x_1) \cdots \Phi(x_n) = \varphi_M(x_1) \cdots \varphi_M(x_n) = \Phi(x_1 \otimes_R \cdots \otimes_R x_n).$$

  Since the $x_1 \otimes_R \cdots \otimes_R x_n$ generate $T^{\geqslant 1}_R(M)$ as an abelian group, Φ is completely determined in a unique way.

- ➢ Existence. Let Φ be the additive map defined by $\Phi_{|R} = \varphi_R$ and $\Phi(x_1 \otimes_R \cdots \otimes_R x_n) = \Phi(x_1) \cdots \Phi(x_n)$ for $n \geqslant 1$ and $x_1, \ldots, x_n$ in M. Then $\Phi_{|M} = \varphi_M$ so that we need only prove that Φ is a map of algebras. Note that $\Phi(1) = \varphi_R(1) = 1$ and, if x and y have degree 0 we have $\Phi(xy) = \varphi_R(xy) = \varphi_R(x)\varphi_R(y) = \Phi(x)\Phi(y)$. Morever, if x has degree 0 and $y = y_1 \otimes_R \cdots \otimes_R y_q$ has degree at least 1, we have

$$\begin{aligned}\Phi(xy) &= \Phi(xy_1 \otimes_R y_2 \otimes_R \cdots \otimes_R y_q) = \varphi_M(xy_1)\varphi_M(y_2) \cdots \varphi_M(y_q) = \varphi_R(x)\varphi_M(y_1) \cdots \varphi_M(y_q) \\ &= \Phi(x)\Phi(y).\end{aligned}$$

  Similarly, if y has degree 0 and x has degree at least 1, then $\Phi(xy) = \Phi(x)\Phi(y)$. Finally, if both $x = x_1 \otimes_R \cdots \otimes_R x_p$ and $y = y_1 \otimes_R \cdots \otimes_R y_q$ have degree at least 1, then

$$\begin{aligned}\Phi(xy) &= \Phi(x_1 \otimes_R \cdots \otimes_R x_p \otimes_R y_1 \otimes_R \cdots \otimes_R y_q) \\ &= \varphi_M(x_1) \cdots \varphi_M(x_p)\varphi_M(y_1) \cdots \varphi_M(y_q) = \Phi(x)\Phi(y).\end{aligned}$$

- ➢ In the graded case, $\varphi_R(r) \in A_0$ and $\varphi_M(x_i) \in A_1$ for all i so that for $n \geqslant 1$ we have $\Phi(x_1 \otimes_R \cdots \otimes_R x_n) \in A_n$ (since A is graded) so that $\Phi(T^n_R(M)) \subseteq A_n$ for all n. □

Corollary III.4. *Let Γ be a quiver. Let Γ_0 be the set of vertices in Γ and let Γ_1 be the set of arrows in $k\Gamma$. Let $k\Gamma_0$ be the semisimple commutative k-subalgebra of $k\Gamma$ with basis Γ_0.*
Then $k\Gamma_1$ is a $k\Gamma_0$-bimodule and the graded k-algebra $k\Gamma$ is isomorphic to $T_{k\Gamma_0}(k\Gamma_1)$.

Proof. Set $R := k\Gamma_0$ and $M = k\Gamma_1$. Then the inclusions $\varphi_0 : R = k\Gamma_0 \hookrightarrow k\Gamma$ and $\varphi_1 : M = k\Gamma_1 \hookrightarrow k\Gamma$ are respectively a k-algebra map and an R-bimodule morphism. Therefore there is a unique k-algebra map $\Phi : T_R(M) \to k\Gamma$ such that $\Phi_{|R} = \varphi_0$ and $\Phi_{|M} = \varphi_1$. Moreover, this map is graded.

To prove that Φ is bijective, we need only prove that the restriction $\Phi_n : T_R^n(M) \to (k\Gamma)_n = k\Gamma_n$ is bijective. This is clearly true for $n = 0$ and $n = 1$. Since Γ_1 is a k-basis of M, we have, for $n \geqslant 2$,

$$T_rR^n(M) = M^{\otimes_R n} = (k\Gamma_1)^{\otimes_R n} = \bigoplus_{\alpha_1,\dots,\alpha_n \in \Gamma_1} k\alpha_1 \otimes_R \cdots \otimes_R k\alpha_n = \bigoplus_{\substack{\alpha_1,\dots,\alpha_n \in \Gamma_1 \\ t(\alpha_i)=\mathfrak{s}(\alpha_{i+1}), i=1,\dots,n-1}} k\alpha_1 \otimes_R \cdots \otimes_R k\alpha_n$$

$$= \bigoplus_{\alpha_1 \cdots \alpha_n \in \Gamma_n} k\alpha_1 \otimes_R \cdots \otimes_R k\alpha_n$$

so that $\mathcal{B} = \{\alpha_1 \otimes_R \cdots \otimes_R \alpha_n; \alpha_1 \cdots \alpha_n \in \Gamma_n\}$ is a basis of $T_R^n(M)$. Since $\Phi(\mathcal{B}) = \Gamma_n$, it is a basis of $(k\Gamma)_n$ and Φ_n is bijective as required. □

2. Conditions for a tensor algebra to be a graded Hopf algebra

Definition III.5. *A bialgebra H is **graded** if $H = \bigoplus_{n\in\mathbb{N}} H_n$ is graded as an algebra and*

$$\varepsilon = \varepsilon_{|H_0}$$

$$\Delta(H_n) \subseteq \bigoplus_{p=0}^{n} H_p \otimes H_{n-p}$$

*If H is a Hopf algebra, then it is **graded** if it is graded as a bialgebra and*

$$S(H_n) \subseteq H_n.$$

Proposition III.6. *Let R be a k-algebra and let M be an R-bimodule. If $T_R(M)$ is a graded Hopf algebra, then R is a Hopf subalgebra of $T_R(M)$ and M is a Hopf bimodule over R.*

Proof. Assume that $T_R(M)$ is a graded Hopf algebra with comultiplication Δ, counit ε and antipode S. These structure maps induce the following k-linear maps:

$$\varepsilon_R = \varepsilon_R : R = T_R^0(M) \to k$$
$$\Delta_R = \Delta_{|R} : R = T_R^0(M) \to T_R^0(M) \otimes T_R^0(M) = R \otimes R$$
$$S_R = S_{|R} : R = T_R^0(M) \to T_R^0(M) = R$$

and the subalgebra R of $T_R(M)$ endowed with these maps is clearly a Hopf subalgebra of $T_R(M)$. Moreover, we also have

$$\Delta_{|M} : M = T_R^1(M) \to T_R^0(M) \otimes T_R^1(M) \oplus T_R^1(M) \otimes T_R^0(M) = (R \otimes M) \oplus (M \otimes R).$$

Let $p_1 : (R \otimes M) \oplus (M \otimes R) \to R \otimes M$ and $p_2 : (R \otimes M) \oplus (M \otimes R) \to M \otimes R$ be the natural projections, and define

$$\rho_\ell : M \to R \otimes M \text{ by } \rho_\ell = p_1 \circ \Delta_{|M}$$
$$\rho_r : M \to M \otimes R \text{ by } \rho_r = p_2 \circ \Delta_{|M}.$$

We have $(\varepsilon_R \otimes \mathrm{id}) \circ p_2 = 0$ and $(\mathrm{id} \otimes \varepsilon_R) \circ p_1 = 0$ so that

$$(\varepsilon_R \otimes \mathrm{id}) \circ \rho_\ell = (\varepsilon_R \otimes \mathrm{id}) \circ \Delta_{|M} - (\varepsilon R \otimes \mathrm{id}) \circ p_2 \circ \Delta_{|M} = \mathrm{id}_M$$

and similarly $(\mathrm{id} \otimes \varepsilon_R) \circ \rho_r = \mathrm{id}_M$.

The maps $(\Delta \otimes \mathrm{id}) \circ \Delta$ and $(\mathrm{id} \otimes \Delta) \circ \Delta$ restricted to M take values in $(R \otimes R \otimes M) \oplus (R \otimes M \otimes R) \oplus (M \otimes R \otimes R)$. Let

$$\pi_1 \colon (R \otimes R \otimes M) \oplus (R \otimes M \otimes R) \oplus (M \otimes R \otimes R) \to R \otimes R \otimes M$$
$$\pi_2 \colon (R \otimes R \otimes M) \oplus (R \otimes M \otimes R) \oplus (M \otimes R \otimes R) \to R \otimes M \otimes R$$
$$\pi_3 \colon (R \otimes R \otimes M) \oplus (R \otimes M \otimes R) \oplus (M \otimes R \otimes R) \to M \otimes R \otimes R$$

be the natural projections. Then applying π_1, π_2 and π_3 to the identity $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ gives, in that order,

$$(\Delta_R \otimes \mathrm{id}_M) \circ \rho_\ell = (\mathrm{id}_R \otimes \rho_\ell) \circ \rho_\ell$$
$$(\rho_\ell \otimes \mathrm{id}_R) \circ \rho_r = (\mathrm{id} \otimes \rho_r) \circ \rho_\ell$$
$$(\rho_r \otimes \mathrm{id}_R) \circ \rho_r = (\mathrm{id}_M \otimes \rho_r) \circ \rho_r$$

so that M is a Hopf bimodule over R. □

The converse is also true. We shall need the following result.

Theorem III.7 (Takeuchi)**.** *Let $H = \bigoplus_{n\in\mathbb{N}} H_n$ be a graded bialgebra such that H_0 is a Hopf algebra. Then H is a graded Hopf algebra.*

Proof. We must prove that H has an antipode, that is, that id_H is $\star$-invertible.

➢ Take $f \in \mathrm{End}_k(H)$ a graded map such that $f_{|H_0}$ is the unit of $\mathrm{End}_k(H_0)$ for the convolution product of H_0. Then f is invertible for the convolution product of H.

Indeed, consider $h = \eta\circ\varepsilon - f$. Then $h_{|H_0} = 0$. By induction, $h^{\star n}$ vanishes on $\bigoplus_{s\leqslant n} H_s$ so that $\eta\circ\varepsilon + \sum_{n\in\mathbb{N}^*} h^{\star n}$ is well-defined on H. Moreover, it is the convolution inverse of $\eta\circ\varepsilon - h = f$, and it is graded since each $h^{\star n}$ is graded.

➢ Now consider the antipode S of H_0. Let $\bar{S} : H \to H$ be any graded k-linear extension of S to H. Then $\mathrm{id}_H \star\bar{S}$ and $\mathrm{id}_H \star\bar{S}$ restrict to $\eta\circ\varepsilon$ on H_0, hence are convolution invertible with graded inverse. Therefore id_H has a graded convolution inverse. □

Theorem III.8 (Nichols)**.** *Let R be a Hopf algebra and M a Hopf bimodule. Then $T_R(M)$ is a bialgebra.*

Proof. Denote by $\rho_\ell : M \to R\otimes M$ and $\rho_r : M \to M\otimes R$ the R-bicomodule structures on M.

Consider the graded algebra $T_R(M)\otimes T_R(M)$, where $(T_R(M)\otimes T_R(M))_n = \bigoplus_{i=0}^n T_R^i(M)\otimes T_R^{n-i}(M)$. The comultiplication $\Delta_R : R \to R\otimes R$ of R is a morphism of algebras whose image is contained in $(T_R(M)\otimes T_R(M))_0$ and the map $\Delta_1 : M \to T_R(M)\otimes T_R(M)$ defined by $\Delta_1 = \rho_\ell + \rho_r$ is a morphism of R-bimodules whose image is contained in $(T_R(M)\otimes T_R(M))_1$. Therefore they induce a graded algebra morphism $\Delta : T_R(M) \to T_R(M)\otimes T_R(M)$.

The field k may be viewed as a graded algebra, concentrated in degree 0. The counit $\varepsilon_R : R \to k$ is a morphism of algebras whose image is contained in the degree 0 part of k and the map $\varepsilon : M \to k$ defined by $\varepsilon = 0$ is a morphism of R-bimodules whose image is contained in the degree 1 part of k. Therefore they induce a graded algebra morphism $\varepsilon : T_R(M) \to k$.

Moreover, the R-bimodule maps $(\Delta\otimes\mathrm{id})\circ\Delta$ and $(\mathrm{id}\otimes\Delta)\circ\Delta$ are equal on M:

$$\begin{aligned}(\Delta\otimes\mathrm{id})\circ\Delta(m) &= (\Delta\otimes\mathrm{id})\circ\rho_\ell(m) + (\Delta\otimes\mathrm{id})\circ\rho_r(m)\\ &= (\mathrm{id}\otimes\rho_\ell)\circ\rho_\ell(m) + (\rho_\ell\otimes\mathrm{id})\circ\rho_r(m) + (\rho_r\otimes\mathrm{id})\circ\rho_r(m)\\ &= (\mathrm{id}\otimes\rho_\ell)\circ\rho_\ell(m) + (\mathrm{id}\otimes\rho_r)\circ\rho_\ell(m) + (\mathrm{id}\otimes\Delta)\circ\rho_r(m)\\ &= (\mathrm{id}\otimes\Delta)\circ\rho_\ell(m) + (\mathrm{id}\otimes\Delta)\circ\rho_r(m) = (\mathrm{id}\otimes\Delta)\circ\Delta(m).\end{aligned}$$

Therefore $(\Delta\otimes\mathrm{id})\circ\Delta$ and $(\mathrm{id}\otimes\Delta)\circ\Delta$ are equal on $T_R(M)$ by the uniqueness in the universal property. The R-bimodule maps $(\varepsilon\otimes\mathrm{id})\circ\Delta$, id and $(\mathrm{id}\otimes\varepsilon)\circ\Delta$ are equal on M (*eg.* $(\varepsilon\otimes\mathrm{id})\circ\Delta(m) = (\varepsilon\otimes\mathrm{id})\circ\rho_\ell(m) + (\varepsilon\otimes\mathrm{id})\circ\rho_r(m) = m + 0 = m$), hence equal on $T_R(M)$ by uniqueness.

Therefore $T_R(M)$ is a graded bialgebra. □

Corollary III.9. *Let R be a Hopf algebra and M a Hopf bimodule. Then $T_R(M)$ is a graded Hopf algebra.*

Proof. By Nichols' theorem, $T_R(M)$ is a graded bialgebra. Since $T_R^0(M) = R$ is a Hopf algebra, by Takeuchi's theorem, $T_R(M)$ is a graded Hopf algebra. □

IV. Conditions for a path algebra to be a graded Hopf algebra.

We follow essentially the paper [GS], and explain at the end of this section how [CR] ties in with this.

First assume that $k\Gamma$ is a graded Hopf algebra. Then $k\Gamma_0$ (the degree 0 part) is a Hopf algebra. Since it is isomorphic to k^n with $n = \#\Gamma_0$ as an algebra, it is isomorphic to k^G for a group G with $\#G = \#\Gamma$ by Theorem I.31. Therefore we may set $\Gamma_0 = \{v_g; g\in G\}$ and the structure maps of $k\Gamma_0$ are given by

$$\begin{aligned}&\Delta(v_g) = \sum_{h\in G} v_h\otimes v_{h^{-1}g} && v_g v_h = \begin{cases} v_g & \text{if } g = h\\ 0 & \text{otherwise}\end{cases}\\ &\varepsilon(v_g) = \begin{cases} 1 & \text{if } g = 1\\ 0 & \text{otherwise}\end{cases} && \eta(1) = \sum_{g\in G} v_g\\ &S(v_g) = v_{g^{-1}}\end{aligned} \tag{1}$$

Now set $R = k\Gamma_0$ and $M = k\Gamma_1$. There is a projection $k\Gamma \twoheadrightarrow k\Gamma_0 = R \cong k^G \cong (kG)^*$ of Hopf algebras so that dualising gives an algebra embedding $kG \cong (kG)^{**} \hookrightarrow (k\Gamma)^*$. Since $k\Gamma$ is a $(k\Gamma)^*$-bimodule by Proposition I.17, it is also a kG-bimodule via this embedding.

We have $g \rightharpoonup v_h = \sum_{k\in G} g(v_{k^{-1}h})v_k = v_{hg^{-1}}$ and $v_h \leftharpoonup g = \sum_{k\in G} g(v_k)v_{k^{-1}h} = v_{g^{-1}h}$ (view G as the dual basis of $\{v_g; g \in G\}$).

Since M is a Hopf bimodule over R by Proposition III.6, we have $\Delta(M) \subseteq (R\otimes M) \oplus (M \otimes R)$. Therefore, for $x \in M$ we can write $\Delta(x) = \sum_{g\in G}(v_g \otimes y_g + z_g \otimes v_g)$. Therefore,

$$g \rightharpoonup x = \sum_{h\in G}(g(y_h)v_h + g(v_h)z_h) = z_g$$
$$x \leftharpoonup g = \sum_{h\in G}(g(v_h)y_h + g(z_h)v_h) = y_g$$

so that $\Delta(x) = \sum_{g\in G}(v_g \otimes (x \leftharpoonup g) + (g \rightharpoonup x)\otimes v_g)$.

For d, f in G, set ${}_dM_f := v_d M v_f$ (this is the notation used in [CR], it is denoted by V_f^d in [GS]). Then $M = \bigoplus_{d,f\in G} {}_dM_f$. Now take $x \in {}_dM_f$. we have $x = v_d x v_f$ so that

$$\begin{aligned}\Delta(x) &= \Delta(v_d)\Delta(x)\Delta(v_f)\\ &= \sum_{h,k,\ell\in G}(v_h \otimes v_{h^{-1}d})(v_k \otimes x \leftharpoonup k + k \rightharpoonup x \otimes v_k)(v_\ell \otimes v_{\ell^{-1}f})\\ &= \sum_{h\in G} v_h \otimes v_{h^{-1}d}(x \leftharpoonup h)v_{h^{-1}f} + \sum_{k\in G} v_{dk^{-1}}(k \rightharpoonup x)v_{fk^{-1}} \otimes v_k\\ &= \sum_{g\in G}(v_g \otimes (x \leftharpoonup g) + (g \rightharpoonup x)\otimes v_g).\end{aligned}$$

Identifying the terms in $R\otimes M$ and applying $g \otimes \mathrm{id}_M$ gives $x \leftharpoonup g = v_{g^{-1}d}(x \leftharpoonup g)v_{g^{-1}f}$ so that $x \leftharpoonup g \in {}_{g^{-1}d}M_{g^{-1}f}$. Similarly, $g \rightharpoonup x \in {}_{dg^{-1}}M_{fg^{-1}}$.

Therefore, the left action of kG on $k\Gamma$ induces k-linear maps

$${}_dL_f(g) : {}_dM_f \to {}_{dg^{-1}}M_{fg^{-1}}$$

for $g, f, d \in G$. They are isomorphisms, with ${}_dL_f(g)^{-1} = {}_{dg^{-1}}L_{fg^{-1}}(g^{-1})$.

Now fix a basis of ${}_1M_h$ for each $h \in G$ (*eg.* the set of arrows from 1 to h). Since ${}_dM_f = {}_1L_{fd^{-1}}(d^{-1})({}_1M_{fd^{-1}})$, we can choose a basis of ${}_dM_f$ such that the matrix of ${}_dL_f(g)$ is the identity matrix for all d, f, g.

In particular, the left action of G on $k\Gamma$ induces an action of G on Γ: it sends arrow to arrow and, if $p = a_1 \dots a_n$ is a path, then $g \rightharpoonup p = (g \rightharpoonup a_1)\cdots(g \rightharpoonup a_n)$. Indeed,

$$\begin{aligned}g \rightharpoonup (ab) &= \sum_{(a),(b)} g(a_{(2)}b_{(2)})a_{(1)}b_{(1)}\\ &= \sum_{(a),(b),(g)} g_{(1)}(a_{(2)})g_{(2)}(a_{(2)})a_{(1)}b_{(1)}\\ &= \sum_{(a),(b)} g(a_{(1)})g(a_{(2)})a_{(1)}a_{(2)} = (g \rightharpoonup a)(g \rightharpoonup b)\end{aligned}$$

and conclude by induction. Note that $g \rightharpoonup p$ is a path from $v_{\mathfrak{s}(a_1)g^{-1}}$ to $v_{\mathfrak{t}(a_n)g^{-1}}$.

Similarly, the right action of kG on $k\Gamma$ induces k-linear isomorphisms

$${}_dR_f(g) : {}_dM_f \to {}_{g^{-1}d}M_{g^{-1}f}$$

for $g, f, d \in G$ (whose matrices are not the identity in general).

These isomorphisms satisfy the following relations:

$${}_{dg^{-1}}R_{fg^{-1}}(h)\,{}_dL_f(g) = {}_{h^{-1}d}L_{h^{-1}f}(g)\,{}_dR_f(h) \tag{2}$$
$${}_{g^{-1}d}R_{g^{-1}f}(h)\,{}_dR_f(g) = {}_dR_f(gh). \tag{3}$$

Definition IV.1 ([GS]). *Let Γ be a quiver with $\Gamma_0 = \{v_g; g \in G\}$ for some group G. Set $M = k\Gamma_1$ and for d, f in G set ${}_dM_f = v_d M v_f$. A kG-bimodule structure on $k\Gamma$ is **allowable** if*

- *G acts on the vertices via $g \rightharpoonup v_h = v_{hg^{-1}}$ and $v_h \leftharpoonup g = v_{g^{-1}h}$,*
- *G acts on the left on Γ (that is, if $\alpha \in \Gamma_1$ is an arrow from d to f, then $g \rightharpoonup \alpha$ is an arrow from dg^{-1} to fg^{-1} and if $p = a_1\cdots a_n$ is a path then $g \rightharpoonup p = (g \rightharpoonup a_1)\cdots(g \rightharpoonup a_n)$); this induces isomorphisms ${}_dL_f(g)\colon {}_dM_f \to {}_{dg^{-1}}M_{fg^{-1}}$,*

- *the right action induces isomorphisms* ${}_dR_f(g)\colon {}_dM_f \to {}_{g^{-1}d}M_{g^{-1}f}$,
- *Equations* (2) *and* (3) *are satisfied.*

Remark IV.2. Note that the left action of G on Γ is free.

Theorem IV.3 ([GS]). *Let Γ be a quiver with $\Gamma_0 = \{v_g; g \in G\}$ for some group G. Then $k\Gamma$ is a Hopf algebra if and only if there is an allowable kG-bimodule structure on $k\Gamma$.*

Proof. We have already proved that if $k\Gamma$ is a Hopf algebra then there is an allowable kG-bimodule structure on $k\Gamma$.

Conversely, assume that there is an allowable kG-bimodule structure on $k\Gamma$. Then the formulas (1) define a Hopf algebra structure on $R = k\Gamma_0$. Moreover, $M = k\Gamma_1$ is a Hopf bimodule for the actions given by the multiplication in $k\Gamma$ and coactions

$$\rho_\ell(x) = \sum_{g\in G} v_g \otimes (x \leftharpoonup g) \quad \text{and} \quad \rho_r(x) = \sum_{g\in G} (g \rightharpoonup x) \otimes v_g$$

for $x \in M$. Therefore $k\Gamma \cong T_R(M)$ is a Hopf algebra by Corollary III.9. □

Proposition IV.4. *[GS, Proposition 3.5] Let Γ be a quiver whose vertex set is indexed by a finite group G and assume that there is an allowable kG-bimodule structure on $k\Gamma$. Then*

(i) $k\Gamma \otimes k\Gamma$ is a kG-bimodule via $g \rightharpoonup (x \otimes y) = x \otimes (g \rightharpoonup y)$ and $(x \otimes y) \leftharpoonup g = (x \leftharpoonup g) \otimes y$ for $g \in G$ and $x, y \in k\Gamma$;

(ii) the comultiplication $\Delta\colon k\Gamma \to k\Gamma \otimes k\Gamma$ is a kG-bimodule morphism;

(iii) the antipode $S : k\Gamma \to k\Gamma$ is determined by

$$\forall x \in {}_d(k\Gamma_1)_f,\ S(x) = -d \rightharpoonup x \leftharpoonup f$$

and satisfies $S(x \leftharpoonup g) = g^{-1} \rightharpoonup S(x)$ and $S(g \rightharpoonup x) = S(x) \leftharpoonup g^{-1}$ for $g \in G$ and $x \in k\Gamma$.

Proof. *(i)* Straightforward verification.

(ii) Note that $k\Gamma$ is a Hopf algebra. Since Δ is an algebra map, we need only prove the result on the vertices and arrows. Take $h \in G$ and let a be an arrow in Γ.

$$\begin{aligned}
\Delta(g \rightharpoonup v_h) &= \Delta(v_{hg^{-1}}) = \sum_{t\in G} v_t \otimes v_{t^{-1}hg^{-1}} = \sum_{t\in G} v_t \otimes g \rightharpoonup v_{t^{-1}h} = g \rightharpoonup \left(\sum_{t\in G} v_t \otimes v_{t^{-1}h}\right)\\
&= g \rightharpoonup \Delta(v_h)\\
\Delta(v_h \leftharpoonup g) &= \Delta(v_{g^{-1}h}) = \sum_{t\in G} v_{g^{-1}ht^{-1}} \otimes v_t = \sum_{t\in G} v_{ht^{-1}} \leftharpoonup g \otimes v_t = \left(\sum_{t\in G} v_{ht^{-1}} \otimes v_t\right) \leftharpoonup g\\
&= \Delta(v_h) \leftharpoonup g\\
\Delta(g \rightharpoonup a) &= \sum_{t\in G} (tg \rightharpoonup a \otimes v_t + v_t \otimes g \rightharpoonup a \leftharpoonup t) = \sum_{t\in G} \left(tg \rightharpoonup a \otimes v_{tgg^{-1}} + v_t \otimes g \rightharpoonup a \leftharpoonup t\right)\\
&= \sum_{t\in G} (g \rightharpoonup (tg \rightharpoonup a \otimes v_{tg}) + g \rightharpoonup (v_t \otimes a \leftharpoonup t))\\
&= g \rightharpoonup \left(\sum_{s\in G} (s \rightharpoonup a \otimes v_s + v_s \otimes a \leftharpoonup s)\right) = g \rightharpoonup \Delta(a)\\
\Delta(a \leftharpoonup g) &= \sum_{t\in G} (t \rightharpoonup a \leftharpoonup g \otimes v_t + v_t \otimes a \leftharpoonup gt) = \sum_{t\in G} \left(t \rightharpoonup a \leftharpoonup g \otimes v_t + v_{g^{-1}gt} \otimes a \leftharpoonup gt\right)\\
&= \sum_{s\in G} ((a \otimes v_t) \leftharpoonup g + (v_s \otimes a \leftharpoonup s) \leftharpoonup g) = \Delta(a) \leftharpoonup g.
\end{aligned}$$

(iii) Recall that $S\colon k\Gamma \to k\Gamma^{op}$ is an algebra map. Set $M = k\Gamma_1$ and let x be an element in ${}_dM_f$. Then $S(x) = S(v_d x v_f) = S(v_f)S(x)S(v_d) = v_{f^{-1}}S(x)v_{d^{-1}}$ so that $S(x) \in {}_{f^{-1}}M_{d^{-1}}$.

Therefore, given an element $g \in G$ we have $g \rightharpoonup x \in {}_{dg^{-1}}M_{fg^{-1}}$, $a \leftharpoonup g \in {}_{g^{-1}d}M_{g^{-1}f}$, $S(g \rightharpoonup x) \in {}_{gf^{-1}}M_{gd^{-1}}$ and $S(a \leftharpoonup g) \in {}_{f^{-1}g}M_{d^{-1}g}$. Now consider $y = d \rightharpoonup x \in {}_1M_{fd^{-1}}$. Since $\Delta(y) = \sum_{g\in G}(g \rightharpoonup y \otimes v_g + v_g \otimes y \leftharpoonup g)$ we have

$$\begin{aligned} 0 = \varepsilon(y)1 &= \sum_{g\in G}(S(g \rightharpoonup y)v_g + S(v_g)(y \leftharpoonup g)) = \sum_{g\in G}(S(g \rightharpoonup y)v_g + v_{g^{-1}}(y \leftharpoonup g)) \\ &= \sum_{g\in G}(S(g \rightharpoonup y) + (y \leftharpoonup g)) = \sum_{g\in G}(S(g \rightharpoonup y) + y \leftharpoonup fd^{-1}g^{-1}) \in \bigoplus_{g\in G} {}_{gdf^{-1}}M_g \end{aligned}$$

so that $S(g \rightharpoonup y) = -y \leftharpoonup fd^{-1}g^{-1}$.

Now $x = d^{-1} \rightharpoonup y$, so $S(x) = S(d^{-1} \rightharpoonup y) = -y \leftharpoonup fd^{-1}d = -y \leftharpoonup f = -d \rightharpoonup x \leftharpoonup f$ and therefore $S(g \rightharpoonup x) = S(gd^{-1} \rightharpoonup y) = -y \leftharpoonup fg^{-1} = S(x)g^{-1}$.

Moreover, $x \leftharpoonup g \in {}_{g^{-1}d}M_{g^{-1}f}$ so that $S(x \leftharpoonup g) = -g^{-1}d \rightharpoonup (x \leftharpoonup g) \leftharpoonup g^{-1}f = g^{-1} \rightharpoonup (-d \rightharpoonup x \leftharpoonup f) = g^{-1} \rightharpoonup S(x)$.

To conclude, we need only prove that the required property is true on vertices:

$$\begin{aligned} S(g \rightharpoonup v_h) &= S(v_{hg^{-1}}) = v_{gh^{-1}} = v_{h^{-1}} \leftharpoonup g^{-1} = S(v_h) \leftharpoonup g^{-1} \\ S(v_h \leftharpoonup g) &= S(v_{g^{-1}h}) = v_{h^{-1}g} = g^{-1} \rightharpoonup v_{h^{-1}} = g^{-1} \rightharpoonup S(v_h) \end{aligned}$$

□

Definition IV.5 ([GS]). *Let G be a finite group and let $W = \{w_1, \ldots, w_n\}$ be a sequence of elements of G (there may be repetitions). Define a quiver $\Gamma_G(W)$, called* ***covering quiver****, whose vertices are $\{v_g; g \in G\}$ indexed by G and whose arrows are*

$$\left\{(a_i, g) : v_{g^{-1}} \to v_{w_ig^{-1}}; i = 1, \ldots, n; g \in G\right\}.$$

Remark IV.6. The covering quiver $\Gamma_G(W)$ is endowed with a left action of G given by $g \rightharpoonup v_h = v_{hg^{-1}}$ and $g \rightharpoonup (a_i, h) = (a_i, gh)$.

Indeed, we have

$$\begin{aligned} &1 \rightharpoonup v_g = v_g, && g \rightharpoonup (h \rightharpoonup v_k) = g \rightharpoonup v_{kh^{-1}} = v_{kh^{-1}g^{-1}} = v_{k(gh)^{-1}} = (gh) \rightharpoonup v_k \\ &1 \rightharpoonup (a_i, g) = (a_i, g), && g \rightharpoonup (h \rightharpoonup (a_i, k)) = g \rightharpoonup (a_i, hk) = (a_i, ghk) = (gh) \rightharpoonup (a_i, k). \end{aligned}$$

The aim of the rest of this section is to prove that $k\Gamma$ is a Hopf algebra if and only if Γ is G-isomorphic to $\Gamma_G(W)$ for some finite group G and some specific type of W.

Definition IV.7. *Let $a \in \Gamma_1$ be an arrow from v_d to v_f. Set $\ell(a) = fd^{-1}$ and $r(a) = d^{-1}f$.*

Lemma IV.8. *[GS, Proposition 4.1] Let G be a finite group and $W = \{w_1, \ldots, w_n\}$ a sequence of elements of G. Then there is an allowable kG-bimodule structure on $k\Gamma_G(W)$ extending the left action of G on $\Gamma_G(W)$ above if and only if W is a* ***weight sequence****, that is, for all $g \in G$ the set $\{gw_1g^{-1}, \ldots, gw_ng^{-1}\}$ is equal to W up to permutation.*

Proof. ➢ First assume that there is an allowable kG-bimodule structure on $k\Gamma$. Then, for any $f \in G$, let $\mathcal{B}_f$ be a k-basis of ${}_fM_1$ and let $\mathcal{B} = \cup_{f\in G}\mathcal{B}_f$. Note that since ${}_{fg^{-1}}R_1(g^{-1}) \circ {}_fL_1(g) : {}_fM_1 \to {}_{gfg^{-1}}M_1$ is an isomorphism, we have $\#\mathcal{B}_{gfg^{-1}} = \#B_f$ for all $f, g \in G$.

Set $W = \{\ell(b); b \in \mathcal{B}\}$ in some order (with repetitions, that is, $\#W = \#\mathcal{B}$). Then W is a weight sequence. Indeed, we have

$$\begin{aligned} \{\ell(b); b \in \mathcal{B}\} &= \bigcup_{f\in G; {}_fM_1\neq 0} \amalg_{[\#\mathcal{B}_f]} \{f\} = \bigcup_{f\in G; {}_{gfg^{-1}}M_1\neq 0} \amalg_{[\#\mathcal{B}_{gfg^{-1}}]} \left\{gfg^{-1}\right\} \\ &= \bigcup_{f\in G; {}_fM_1\neq 0} \amalg_{[\#\mathcal{B}_f]} \left\{gfg^{-1}\right\} = \left\{g\ell(b)g^{-1}; b \in \mathcal{B}\right\}. \end{aligned}$$

➢ Conversely, assume that $W = \{w_1, \ldots, w_n\}$ is a weight sequence. Then, for any $g \in G$ there is a permutation $\sigma_g \in \mathfrak{S}_n$ such that $gw_ig^{-1} = w_{\sigma_g(i)}$ for all i. This induces a group morphism $\theta : G^{op} \to \mathfrak{S}_n$ defined by $\theta(g) = \sigma_{g^{-1}}$. Then $w_{\theta(g)(i)} = g^{-1}w_ig$ for all i.

Define a right action of kG on $k\Gamma$ as follows: $v_h \leftharpoonup g = v_{g^{-1}h}$ and $(a_i, h) \leftharpoonup g = (a_{\theta(g)(i)}, hg)$. Then we have an allowable kG-bimodule structure on $k\Gamma$, as shown in Example IV.9 below (with the f_i identically equal to 1). □

Example IV.9. [GS, Theorem 5.6.(a)] Let G be a finite group and let $W = \{w_1, \dots, w_n\}$ be a non-empty weight sequence. Choose a group morphism $\Theta : G^{op} \to \mathfrak{S}_n$ such that $w_{\Theta(g)(i)} = g^{-1}w_i g$ and choose group morphisms $f_i = f_{\Theta(g)(i)} : G \to k^\times$, for all $i = 1, \dots, n$ and $g \in G$.

Then the formulas

$$g \rightharpoonup v_h = v_{hg^{-1}} \qquad g \rightharpoonup (a_i, h) = (a_i, gh)$$
$$v_h \leftharpoonup g = v_{g^{-1}h} \qquad (a_i, h) \leftharpoonup g = f_i(g)(a_{\Theta(g)(i)}, hg)$$

define an allowable kG-bimodule structure on $k\Gamma$.

Proof. We have already seen that the left action is indeed a left action on the graph $\Gamma_G(W)$. It is easy to check that $1 \in G$ acts trivially on the right. Moreover,

$$(v_k) \leftharpoonup h \leftharpoonup g = v_{h^{-1}k} \leftharpoonup g = v_{g^{-1}h^{-1}k} = v_{(hg)^{-1}k} = v_k \leftharpoonup (hg)$$
$$g \rightharpoonup (v_k \leftharpoonup h) = g \rightharpoonup v_{h^{-1}k} = v_{h^{-1}kg^{-1}} = v_{kg^{-1}} \leftharpoonup h = (g \rightharpoonup v_k) \leftharpoonup h$$
$$((a_i, t) \leftharpoonup h) \leftharpoonup g = f_i(h)(a_{\Theta(h)(i)}, th) \leftharpoonup g = f_i(h) f_{\Theta(h)(i)}(g)(a_{\Theta(g)(\Theta(h)(i))}, thg)$$
$$= f_i(h) f_i(g)(a_{\Theta(hg)(i)}, thg) = f_i(hg)(a_{\Theta(hg)(i)}, thg) = (a_i, t) \leftharpoonup (hg)$$
$$g \rightharpoonup ((a_i, t) \leftharpoonup h) = f_i(h) g \rightharpoonup (a_{\Theta(h)(i)}, th) = f_i(h)(a_{\Theta(h)(i)}, gth) = (a_i, gt) \leftharpoonup h = (g \rightharpoonup (a_i, t)) \leftharpoonup h. \square$$

Definition IV.10 ([GS]). *We say that two quivers Γ and Γ', endowed with free left G-actions, are G-**isomorphic** if there is an isomorphism $\varphi : \Gamma \to \Gamma'$ of quivers such that, for all $g \in G$ and all $x \in \Gamma_0 \cup \Gamma_1$, we have $\varphi(g \rightharpoonup x) = g \rightharpoonup \varphi(x)$.*

Proposition IV.11. *[GS, Proposition 4.2] Let Γ be a quiver with vertex set indexed by a finite group G and on which G acts freely on the left. Let $*$ denote this action and assume that the action on vertices is given by $g * v_h = v_{hg^{-1}}$. Then there is a sequence of elements W of G such that Γ is G-isomorphic to $\Gamma_G(W)$.*

Proof. Let $\{a_1, \dots, a_n\}$ be the set of arrows in Γ starting at v_1. Let W be defined as in the proof of Lemma IV.8, that is, $W = \{\ell(a_i); i = 1, \dots, n\}$. Define $\phi : \Gamma \to \Gamma_G(W)$ on vertices by $\phi(v_g) = v_g$. Now let $a : v_d \to v_f$ be an arrow in Γ. Then $d * a : v_1 \to v_{fd^{-1}}$ so that there exists i such that $d * a = a_i$. Therefore $a = d^{-1} * a_i$. Define $\phi(a) = (a_i, d^{-1}) \in \Gamma_G(W)$. Then ϕ is a G-isomorphism of graphs:

- The maps ϕ defined on vertices and on arrows are compatible: if a is an arrow from v_g to v_f, then $a_i = d * a$ is an arrow from v_1 to $v_{fd^{-1}}$ so that $w_i = fd^{-1}$, therefore $\phi(a) = (a_i, d^{-1})$ goes from $v_d = \phi(v_d)$ to $v_{w_i d} = v_f = \phi(v_f)$ as required.
- $\phi(g * v_h) = \phi(v_{hg^{-1}}) = v_{hg^{-1}} = g \rightharpoonup v_h$.
- Take $g \in G$ and $a : v_d \to v_f$ an arrow in Γ. Then $d * a = a_i$ for some i and $a = d^{-1} * a_i$, therefore $g * a = gd^{-1} * a_i$, so that $\phi(g * a) = (a_i, gd^{-1}) = g \rightharpoonup (a_i, d^{-1}) = g \rightharpoonup \phi(a)$.
- Define $\psi : \Gamma_G(W) \to \Gamma$ by $\psi(v_g) = v_g$ and $\psi(a_i, h) = h * a_i$. Then ψ and ϕ are inverse isomorphisms. $\square$

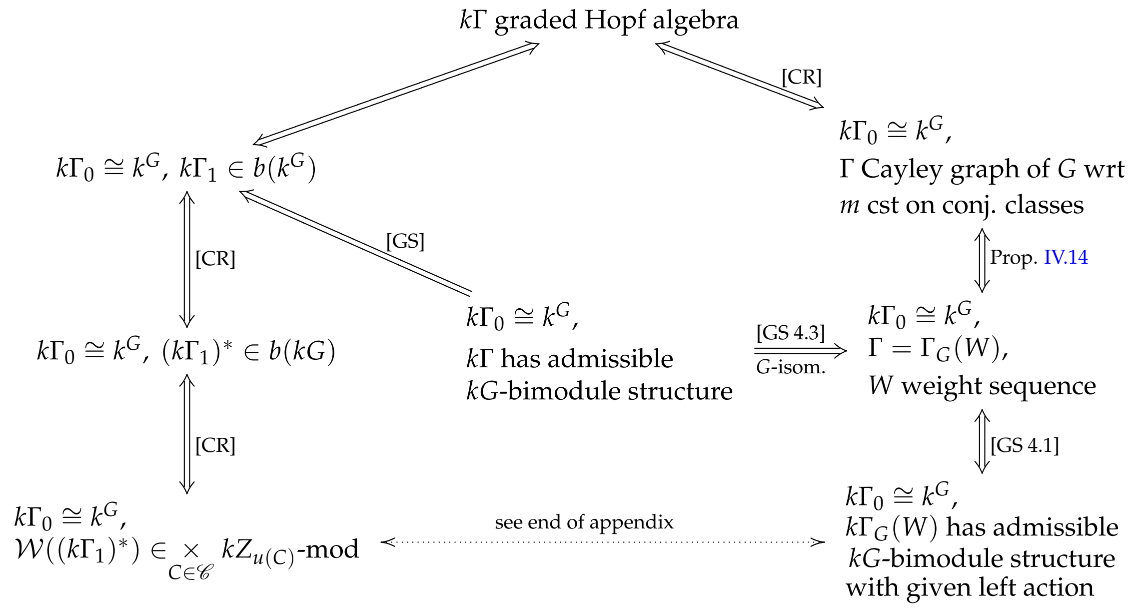
Corollary IV.12 ([GS]). *The path algebra $k\Gamma$ is a Hopf algebra if and only if there exist a finite group G and a weight sequence W such that Γ is G-isomorphic to $\Gamma_G(W)$.*

Proof. Assume that $k\Gamma$ is a Hopf algebra. Then we know that the vertex set of Γ is indexed by a finite group G and that there is an allowable kG-bimodule structure on $k\Gamma$. In particular, there is a free left G-action on Γ. Therefore there is a sequence of elements W of G such that Γ is G-isomorphic to $\Gamma_G(W)$. Moreover, since there is an allowable kG-bimodule structure on $k\Gamma$ extending the free left G-action, W is a weight sequence.

Conversely, it follows from the proof of Lemma IV.8 that there is an allowable kG-bimodule structure on $k\Gamma_G(W)$ when W is a weight sequence. $\square$

In their paper [CR], C. Cibils and M. Rosso consider the same problem (among other things), which they prove in terms of category theory.

The diagram below summarises the results in [GS] and [CR] related to the question of when $k\Gamma$ is a Hopf algebra. If H is a Hopf algebra, $b(H)$ denotes the category of Hopf bimodules over H that are finite dimensional over k. Moreover, $\mathscr{C}$ is the set of conjugacy classes in G, $u(C) \in G$ is a representative of the conjugacy class $C \in \mathscr{C}$, the group $Z_{u(C)}$ is the centraliser of $u(C)$ in G and $\mathcal{W} : b(kG) \to \underset{C\in\mathscr{C}}{\times} kZ_{u(C)}\text{-mod}$ is a functor (equivalence of categories).



Definition IV.13. *[CR] The Cayley graph of a group G with respect to a marking map $m : G \to \mathbb{N}$ is an oriented graph Γ whose vertices are indexed by the elements of the group, $\Gamma_0 = \{v_g; g \in G\}$, and such that the number of arrows from v_d to v_f is $m(fd^{-1})$.*

Proposition IV.14. *Γ is the Cayley graph of G with respect to $m : G \to \mathbb{N}$ constant on conjugacy classes if and only if $\Gamma = \Gamma_G(W)$ for some weight sequence W.*

Proof.

➢ Assume that Γ is the Cayley graph of G with respect to $m : G \to \mathbb{N}$ constant on conjugacy classes. By definition, $\Gamma_0 = \{v_g; g \in G\}$ is indexed by the elements of G and for any $(g, h) \in G^2$ we have $\dim_k v_d(k\Gamma_1)v_f = m(fd^{-1})$.

Define $W = \amalg_{g\in G, m(g\neq 0)} \left(\amalg_{m(g)} \{g\}\right) =: \{w_1, \dots, w_n\}$. In other words, an element $g \in G$ occurs exactly $m(g)$ times in W. Hence $m(g) = \#\{i; w_i = g\}$. Note that $m(g)$ is also the number of arrows in Γ from v_1 to v_g.

The number of w_i such that $h^{-1} = w_i g^{-1}$ is $m(h^{-1}g)$, that is, the number of arrows from $v_{g^{-1}}$ to $v_{h^{-1}}$. Therefore the arrows are the $(a_i, g) : v_{g^{-1}} \to v_{w_i g^{-1}}$ for $g \in G$ and $w_i \in W$.

Therefore $\Gamma = \Gamma_G(W)$.

Moreover, since m is constant on conjugacy classes, we have $W = \amalg_{C\in\mathscr{C}, m(C)\neq 0} \amalg_{m(C)} C$. Therefore $\{gw_ig^{-1}; i = 1, \dots, n\} = \amalg_{C\in\mathscr{C}, m(C)\neq 0} \amalg_{m(C)} gCg^{-1} = \amalg_{C\in\mathscr{C}, m(C)\neq 0} \amalg_{m(C)} C = W$, so that W is a weight sequence.

➢ Assume that $\Gamma = \Gamma_G(W)$ where $W = \{w_1, \dots, w_n\}$ is a weight sequence. By definition, $\Gamma_0 = \{v_g; g \in G\}$ is indexed by the elements of G and $\Gamma_1 = \left\{(a_i, g) : v_{g^{-1}} \to v_{w_i g^{-1}}; 1 \leqslant i \leqslant n, g \in G\right\}$.

Define $m(g) = \dim_k v_1(k\Gamma_1)v_g = \#\{i; w_i = g\}$. Then, for any $h \in G$, we have $m(hgh^{-1}) = \#\{i; w_i = hgh^{-1}\} = \#\{i; h^{-1}w_ih = g\}$. Since $\varphi_h : W \to W$ defined by $\varphi_h(w) = h^{-1}wh$ is a bijection, $m(hgh^{-1}) = \#\{i; \varphi_h(w_i) = g\} = \#\{i; w_i = g\} = m(g)$. Therefore m is constant on conjugacy classes.

Finally, the number of arrows from v_g to v_h is $\#\{(a_i, k); k^{-1} = g,\ w_ik^{-1} = h\} = \#\{i; w_i = hg^{-1}\} = m(hg^{-1})$.

Therefore, Γ is the Cayley graph of G with respect to m, and m is constant on conjugacy classes. □

The appendix gives some details on the results in [CR] related to the question of when $k\Gamma$ is a Hopf algebra.

1. Quiver of a finite dimensional basic Hopf algebra

In [GS], the authors consider a finite dimensional Hopf algebra H such that $H \cong k\Gamma/I$, where I is an admissible ideal in the path algebra $k\Gamma$. Let $\mathfrak{r}$ denote the Jacobson radical of H. They prove that $\mathfrak{r}$ is a Hopf ideal in H, so that $H/\mathfrak{r}$ is a Hopf algebra isomorphic to k^n for some $n \geqslant 1$. Therefore, there is a group G such that $H/\mathfrak{r} \cong k^G$.

They then describe the Hopf algebra structure of H modulo $\mathfrak{r}^2$.

Their final result on finite dimensional basic Hopf algebras is [GS, Theorem 2.3], which states that there is an admissible sequence W such that $\Gamma \cong \Gamma_G(W)$.

We shall now turn to the end of their paper.

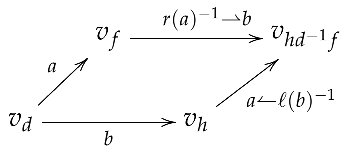
2. Construction of finite dimensional Hopf algebras

In the last section of the paper [GS], given a Hopf algebra $k\Gamma_G(W)$, the authors construct explicit ideals I in $k\Gamma_G(W)$ such that $k\Gamma_G(W)/I$ is finite dimensional, and give necessary and sufficient conditions for this ideal I to be a Hopf ideal.

a) The ideal I_q

Definition V.1. *[GS] For a and b two distinct arrows in $\Gamma_G(W)$ with the same source, set*

$$q(a,b) = a(r(a)^{-1} \rightharpoonup b) - b(a \leftharpoonup \ell(b)^{-1}).$$



The ideal I_q in $k\Gamma_G(W)$ is the ideal generated by the $q(a,b) \leftharpoonup g$ where (a,b) are pairs of distinct arrows with same source and $g \in G$.

Lemma V.2. *[GS, Lemma 5.1] The ideal I_q is a Hopf ideal if, and only if, both conditions below are satisfied.*

(i) The subgroup of G generated by W is abelian.

(ii) For any pair (a,b) of distinct arrows with the same source, there exists a scalar $c_b(a) \in k^\times$ such that $a \leftharpoonup \ell(b) = c_b(a) r(b) \rightharpoonup a$ satisfying $c_a(b) = c_b(a)^{-1}$.

Remark V.3. Note that if we have an admissible kG-bimodule structure on $k\Gamma$ in general, the right action does not necessarily send an arrow to a scalar multiple of an arrow (there is an example illustrating this at the end of the paper [GS]). However, if I_q is a Hopf ideal, then the right action of $\ell(b)$ on a, where a and b are two distinct arrows with same source, is a scalar multiple of an arrow.

Proof of Lemma V.2. ➢ Let $a : v_d \to v_f$ and $b : v_d \to v_h$ be two distinct arrows in $\Gamma_G(W)$ with the same source. Then $\varepsilon(q(a,b) \leftharpoonup g) = 0$ for all $g \in G$. Moreover,

$$\begin{aligned}
\Delta(q(a,b)) &= \Delta(a)\Delta(r(a)^{-1} \rightharpoonup b) - \Delta(b)\Delta(a \leftharpoonup \ell(b)^{-1}) \\
&= \Delta(a)\left(\Delta(b) \leftharpoonup r(a)^{-1}\right) - \Delta(b)\left(\ell(b)^{-1} \rightharpoonup \Delta(a)\right) \\
&= \left(\sum_{t\in G} t \rightharpoonup a \otimes v_t + v_t \otimes a \leftharpoonup t\right)\left(\sum_{g\in G} (g \rightharpoonup b \otimes r(a)^{-1} v_g + v_g \otimes r(a)^{-1} \rightharpoonup b \leftharpoonup g)\right) \\
&\quad - \left(\sum_{t\in G} t \rightharpoonup b \otimes v_t + v_t \otimes b \leftharpoonup t\right)\left(\sum_{g\in G} (g \rightharpoonup a \leftharpoonup \ell(b)^{-1} \otimes v_g + v_g \leftharpoonup \ell(b)^{-1} \otimes a \leftharpoonup g)\right) \\
&= \sum_{g\in G}\Big[(gr(a) \rightharpoonup a)(g \rightharpoonup b) \otimes v_{gr(a)} + (g \rightharpoonup b) \otimes (a \leftharpoonup dg^{-1}) \\
&\qquad + (g^{-1}f \rightharpoonup a) \otimes (r(a)^{-1} \rightharpoonup b \leftharpoonup g) + v_g \otimes (a \leftharpoonup g)(r(a)^{-1} \rightharpoonup b \leftharpoonup g) \\
&\qquad - (g \rightharpoonup b)(g \rightharpoonup a \leftharpoonup \ell(b)^{-1}) \otimes v_g - (g \rightharpoonup a \leftharpoonup \ell(b)^{-1}) \otimes (b \leftharpoonup hg^{-1}) \\
&\qquad - (g^{-1}d \rightharpoonup b) \otimes (a \leftharpoonup g) - (v_g \leftharpoonup \ell(b)^{-1}) \otimes (b \leftharpoonup \ell(b)g)(a \leftharpoonup g)\Big]
\end{aligned}$$

$$= \sum_{g\in G} \Big[(g \rightharpoonup a)(gr(a)^{-1} \rightharpoonup b) \otimes v_g + (g \rightharpoonup b) \otimes (a \leftharpoonup dg^{-1})$$
$$+ (g^{-1}f \rightharpoonup a) \otimes (r(a)^{-1} \rightharpoonup b \leftharpoonup g) + v_g \otimes (a \leftharpoonup g)(r(a)^{-1} \rightharpoonup b \leftharpoonup g)$$
$$- (g \rightharpoonup b)(g \rightharpoonup a \leftharpoonup \ell(b)^{-1}) \otimes v_g - (g \rightharpoonup a \leftharpoonup \ell(b)^{-1}) \otimes (b \leftharpoonup hg^{-1})$$
$$-(g \rightharpoonup b) \otimes (a \leftharpoonup dg^{-1}) - v_g \otimes (b \leftharpoonup g)(a \leftharpoonup \ell(b)^{-1}g)\Big]$$
$$= \sum_{g\in G} \left(q(g \rightharpoonup a, g \rightharpoonup b) \otimes v_g + v_g \otimes q(a,b) \leftharpoonup g\right) + X$$

where

$$X = \sum_{g\in G} (g^{-1}f \rightharpoonup a) \otimes (f^{-1}d \rightharpoonup b \leftharpoonup g) - \sum_{g\in G} (g \rightharpoonup a \leftharpoonup dh^{-1}) \otimes (b \leftharpoonup hg^{-1})$$
$$= \sum_{g\in G} (g^{-1}f \rightharpoonup a)v_g \otimes (f^{-1}d \rightharpoonup b \leftharpoonup g)v_{g^{-1}hd^{-1}f}$$
$$- \sum_{g\in G} (g^{-1}hd^{-1}f \rightharpoonup a \leftharpoonup dh^{-1})v_g \otimes (b \leftharpoonup hf^{-1}dh^{-1}g)v_{g^{-1}hd^{-1}f}.$$

We have $\Delta(q(a,b)) - X \in I_q \otimes k\Gamma_G(W) + k\Gamma_G(W) \otimes I_q$ and, for all $g \in G$, $\Delta(q(a,b) \leftharpoonup g) = \Delta(q(a,b)) \leftharpoonup g$. Therefore, if $X = 0$ for all distinct arrows a, b with same source, then I_q is a bi-ideal. Conversely, since $X \in k\Gamma_G(W)_1 \otimes k\Gamma_G(W)_1$ and $I_q \in k\Gamma_G(W)_{\geqslant 2}$, if I_q is a bi-ideal then $X = 0$.

Hence I_q is a bi-ideal if and only if $X = 0$ for all distinct arrows a, b with same source. Now $X = 0$ if, and only if, for all $g \in G$, we have

$$(g^{-1}f \rightharpoonup a) \otimes (f^{-1}d \rightharpoonup b \leftharpoonup g) - (g^{-1}hd^{-1}f \rightharpoonup a \leftharpoonup dh^{-1}) \otimes (b \leftharpoonup hf^{-1}dh^{-1}g) = 0$$

(multiply on the right by $v_g \otimes v_{g^{-1}hd^{-1}f}$ for each $g \in G$). Now multiplying on the left by $v_{df^{-1}g} \otimes v_{g^{-1}f}$ shows that we must have $df^{-1} = hf^{-1}dh^{-1}$, that is, $\ell(b)f = fr(b)$.

Therefore, if I_q is a bi-ideal, then we must have $\ell(b)\mathfrak{t}(a) = \mathfrak{t}(a)r(b)$ for every pair of distinct arrows a, b with same source.

➢ We now prove that $\ell(b)\mathfrak{t}(a) = \mathfrak{t}(a)r(b)$ for every distinct arrows a, b with same source if, and only if, the subgroup of G generated by W is abelian.

Assume that the subgroup of G generated by W is abelian. Let a and b be two distinct arrows in $\Gamma_G(W)$ with same source v_d. Then there exist distinct i and j such that $a = (a_i, d^{-1})$ and $b = (a_j, d^{-1})$ so that $f = w_i d$ and $h = w_j d$. Hence $\ell(a) = fd^{-1} = w_i \in W$ and $\ell(b) = hd^{-1} = w_j \in W$ commute. Therefore

$$\ell(b)f = \ell(b)fd^{-1}d = \ell(b)\ell(a)d = \ell(a)\ell(b)d = fd^{-1}hd^{-1}d = fr(b).$$

Conversely, let w_i and w_j be distinct elements in W. Then $a = (a_i, 1)$ and $b = (a_j, 1)$ are two distinct arrows with same source v_1. Therefore $w_j w_i = \ell(b)\mathfrak{t}(a) = \mathfrak{t}(a)r(b) = w_i w_j$. All elements in W commute, therefore they generate an abelian subgroup of G.

Therefore, if I_q is a bi-ideal, then W generates an abelian subgroup of G.

➢ Now assume that *(i)* and *(ii)* hold. Then

$$\begin{aligned} hd^{-1}f \rightharpoonup a \leftharpoonup dh^{-1} \otimes b \leftharpoonup hf^{-1}dh^{-1} &= \ell(b)f \rightharpoonup a \leftharpoonup \ell(b)^{-1} \otimes b \leftharpoonup hf^{-1}\ell(b)^{-1} \\ &= fr(b) \rightharpoonup a \leftharpoonup \ell(b)^{-1} \otimes b \leftharpoonup hr(b)^{-1}f^{-1} \\ &= f \rightharpoonup c_b(a)^{-1}a \otimes b \leftharpoonup df^{-1} \\ &= c_a(b)f \rightharpoonup a \otimes c_b(a)r(a)^{-1} \rightharpoonup b \\ &= f \rightharpoonup a \otimes f^{-1}d \rightharpoonup b. \end{aligned}$$

Therefore, for any $g \in G$, we have

$$g^{-1}f \rightharpoonup a \otimes f^{-1}d \rightharpoonup b \leftharpoonup g = g^{-1}hd^{-1}d \rightharpoonup a \leftharpoonup dh^{-1} \otimes b \leftharpoonup hf^{-1}dh^{-1}g$$

so that $X = 0$.

Therefore, if *(i)* and *(ii)* hold, then I_q is a bi-ideal.

➢ Now assume that I_q is a bi-ideal, that is, $X = 0$ for any pair of distinct arrows a, b with same source. We then know that *(i)* holds. Replacing g by fg^{-1} in X and using *(i)* gives, for all $g \in G$,

$$\begin{aligned} g \rightharpoonup a \otimes r(a)^{-1} \rightharpoonup b \leftharpoonup fg^{-1} &= gf^{-1}\ell(b)f \rightharpoonup a \leftharpoonup \ell(b)^{-1} \otimes b \leftharpoonup hf^{-1}\ell(b)^{-1}fg^{-1} \\ &= gf^{-1}fr(b) \rightharpoonup a \leftharpoonup \ell(b)^{-1} \otimes b \leftharpoonup hr(b)^{-1}f^{-1}fg^{-1} \\ &= gr(b) \rightharpoonup a \leftharpoonup \ell(b)^{-1} \otimes b \leftharpoonup hh^{-1}dg^{-1} \\ &= gr(b) \rightharpoonup a \leftharpoonup \ell(b)^{-1} \otimes b \leftharpoonup dg^{-1}. \end{aligned}$$

For $g = d$ this gives

$$d \rightharpoonup a \otimes r(a)^{-1} \rightharpoonup b \leftharpoonup \ell(a) = dr(b) \rightharpoonup a \leftharpoonup \ell(b)^{-1} \otimes b. \qquad (\dagger)$$

Since $a \leftharpoonup \ell(b)^{-1} \in {}_{\ell(b)d}M_{\ell(b)f}$ we can write $a \leftharpoonup \ell(b)^{-1} = \sum_{i=1}^{s} \alpha_i a_i$ for some scalars α_i where $\{a_1, \ldots, a_s\}$ is part of a basis of arrows of ${}_{\ell(b)d}M_{\ell(b)f}$. Similarly, $b \leftharpoonup \ell(a) = \sum_{i=1}^{t} \beta_i b_i$ for some scalars β_i where $\{b_1, \ldots, b_t\}$ is part of a basis of arrows of ${}_{\ell(a)^{-1}d}M_{\ell(a)^{-1}h}$. Hence equation (†) is equivalent to

$$\sum_{i=1}^{t} \beta_i (d \rightharpoonup a \otimes r(a)^{-1} \rightharpoonup b_i) = \sum_{i=1}^{s} \alpha_i (dr(b) \rightharpoonup a_i \otimes b). \qquad (\ddagger)$$

Since $d \rightharpoonup a$, $r(a)^{-1}b_i$, $dr(b) \rightharpoonup a_i$ and b are all arrows (using the running assumption on the left action of G), they can be chosen as part of a basis of $k\Gamma_G(W)$. This implies that for all i we have $r(a)^{-1} \rightharpoonup b_i = b$ so that $b_i = r(a) \rightharpoonup b$ and therefore, up to reordering, $b_1 = r(a) \rightharpoonup b$ and $\beta_i = 0$ for $i > 1$. Similarly, $a_1 = r(b)^{-1} \rightharpoonup a$ and $\alpha_j = 0$ for $j > 1$. Replacing in equation (‡) gives $\beta_1 d \rightharpoonup a \otimes b = \alpha_1 d \rightharpoonup a \otimes b$ so that $\alpha_1 = \beta_1$. It then follows that $b \leftharpoonup \ell(a) = \alpha_1 b_1 = \alpha_1 (r(a) \rightharpoonup b)$ so that we may set $c_a(b) = \alpha_1$, and $a \leftharpoonup \ell(b)^{-1} = \alpha_1 a_1 = c_a(b) r(b)^{-1} \rightharpoonup a$ hence $c_a(b) = c_b(a)^{-1}$ as required.

Therefore *(ii)* is satisfied.

➢ We have now proved that I_q is a bi-ideal if and only if *(i)* and *(ii)* hold. It remains to be shown that, assuming *(i)* and *(ii)* are satisfied, I_q is a Hopf ideal, that is, $S(I_q) \subseteq I_q$. Let a and b be two distinct arrows with the same source as before. We have

$$\begin{aligned} d^{-1} \rightharpoonup S(q(a,b)) \leftharpoonup f^{-1}\ell(b)^{-1} &= d^{-1} \rightharpoonup ((S(b) \leftharpoonup \ell(a))S(a) - (\ell(b) \rightharpoonup S(a))S(b)) \leftharpoonup f^{-1}\ell(b)^{-1} \\ &= d^{-1} \rightharpoonup ((-d \rightharpoonup b \leftharpoonup hr(a))(-d \rightharpoonup a \leftharpoonup f) \\ &\quad -(-\ell(b)d \rightharpoonup a \leftharpoonup f)(-d \rightharpoonup b \leftharpoonup h)) \leftharpoonup f^{-1}\ell(b)^{-1} \\ &= (b \leftharpoonup hr(a)f^{-1}\ell(b)^{-1})(a \leftharpoonup \ell(b)^{-1}) \\ &\quad - (d^{-1}\ell(b)d \rightharpoonup a \leftharpoonup \ell(b)^{-1})(b \leftharpoonup hf^{-1}\ell(b)^{-1}) \\ &= b(a \leftharpoonup \ell(b)^{-1}) - (r(b) \rightharpoonup a \leftharpoonup \ell(b)^{-1})(b \leftharpoonup hr(b)^{-1}f^{-1}) \\ &= b(a \leftharpoonup \ell(b)^{-1}) - c_a(b)a(b \leftharpoonup \ell(a)^{-1}) \\ &= b(a \leftharpoonup \ell(b)^{-1}) - a(r(a)^{-1} \rightharpoonup b) \\ &= -q(a,b). \end{aligned}$$

Therefore $S(q(a,b) \leftharpoonup g) = g^{-1} \rightharpoonup S(q(a,b)) = -g^{-1}d \rightharpoonup q(a,b) \leftharpoonup \ell(b)f = -q(g^{-1}d \rightharpoonup a, g^{-1}d \rightharpoonup b) \leftharpoonup \ell(b)f \in I_q$ for all $g \in G$, so that $S(I_q) \subseteq I_q$ as required. □

Remark V.4. Note that once we know that I_q is a Hopf ideal, then using *(ii)* we have $q(b,a) = -c_b(a)q(a,b)$.

b) The ideal I_p

Definition V.5. *For every arrow a in $\Gamma_G(W)$, choose an integer $m_a \geqslant 2$, in such a way that $m_a = m_{g \rightharpoonup a}$ for all $g \in G$.*

➢ *If a is not a loop, set* $p(a) = a(r(a)^{-1} \rightharpoonup a) \cdots (r(a)^{-m_a+1} \rightharpoonup a) = \prod_{i=0}^{m_a - 1} (r(a)^{-i} \rightharpoonup a).$

➢ *If a is a loop, set $p(a) = a^{m_a}$.*

The ideal I_p in $k\Gamma_G(W)$ is the ideal generated by the $p(a) \leftharpoonup g$ where a is an arrow and $g \in G$.

Remark V.6. Note that $a(g \rightharpoonup a)$ is a non-zero path if, and only if, $g = r(a)^{-1}$. Indeed, if the arrow a goes from v_d to v_f, then $g \rightharpoonup a$ starts at $v_{dg^{-1}}$, so that we require $dg^{-1} = f$, that is, $g^{-1} = d^{-1}f = r(a)$.

Note also that if a is a loop then $r(a) = 1$.

Therefore $p(a)$ is the non-zero path of length m_a starting with a which is the product of successive arrows in the $\rightharpoonup$-orbit of a.

In particular, any product of m_a arrows in the orbit of a is either 0 or an element of I_p.

Let $T_s(n)$ denote the set of all subsets of $\{0, 1, \ldots, n-1\}$ consisting of s elements.

Lemma V.7. *[GS, Lemma 5.3] Assume that $a \leftharpoonup \ell(a) = c_a(a)r(a) \rightharpoonup a$ for some $c_a(a) \in k^\times$ and all arrows a in $\Gamma_G(W)$. Then I_p is a Hopf ideal in $k\Gamma_G(W)$ if, and only if,*

(i) for all arrows a in $\Gamma_G(W)$ that are not loops, and for any $s \in \{1, 2, \ldots, n_a - 1\}$ where n_a is the order of $\ell(a)$ in G, we have

$$\sum_{\sigma \in T_s(m_a)} \prod_{i \notin \sigma} c_a(a)^i = 0,$$

(ii) for all loops a in $\Gamma_G(W)$ and all $i \in \{1, 2, \ldots, m_a - 1\}$, the number $\binom{m_a}{i}$ is zero in k.

Proof. Clearly, $\varepsilon(I_p) \subseteq \varepsilon(\mathfrak{r}) = 0$.

Let $a \in {}_dM_f$ be an arrow that is not a loop. Note that for any integer i, we have $r(a)^{-i} \rightharpoonup a = c_a(a)^i a \leftharpoonup \ell(a)^{-i}$ by assumption. We then have

$$p(a) = \prod_{i=0}^{m_a-1} (r(a)^{-i} \rightharpoonup a) = \prod_{i=0}^{m_a-1} c_a(a)^i (a \leftharpoonup \ell(a)^{-i})$$

so that

$$\begin{aligned}
S(p(a)) &= \prod_{i=0}^{m_a-1} c_a(a)^i S(a \leftharpoonup \ell(a)^{-i}) \\
&= \prod_{i=0}^{m_a-1} c_a(a)^i \ell(a)^i \rightharpoonup S(a) \\
&= (-1)^{m_a} c_a(a)^{-m_a(m_a-1)/2} \prod_{i=0}^{m_a-1} \ell(a)^i \rightharpoonup (d \rightharpoonup a \leftharpoonup f) \\
&= (-1)^{m_a} c_a(a)^{-m_a(m_a-1)/2} \prod_{i=0}^{m_a-1} \ell(a)^{i-m_a+1} \ell(a)^{m_a-1} d \rightharpoonup a \leftharpoonup f \\
&= (-1)^{m_a} c_a(a)^{-m_a(m_a-1)/2} \prod_{m_a-1}^{j=0} \ell(a)^{-j} \rightharpoonup (\ell(a)^{m_a-1} d \rightharpoonup a) \leftharpoonup f \\
&= (-1)^{m_a} c_a(a)^{-m_a(m_a-1)/2} (\prod_{m_a-1}^{j=0} r(a')^{-j} \rightharpoonup a') \leftharpoonup f \\
&= (-1)^{m_a} c_a(a)^{-m_a(m_a-1)/2} p(a') \leftharpoonup f \in I_p
\end{aligned}$$

where $a' = \ell(a)^{m_a-1} d \rightharpoonup a$. If a is a loop at v_d, then

$$S(p(a)) = S(a)^{m_a} = (-1)^{m_a}(d \rightharpoonup a \leftharpoonup d)^{m_a} = (-1)^{m_a}(d \rightharpoonup a)^{m_a} \leftharpoonup d = (-1)^{m_a} p(d \rightharpoonup a) \leftharpoonup d \in I_p.$$

Moreover, for $g \in G$, $S(x \leftharpoonup g) = g^{-1} \rightharpoonup S(x)$ and $g^{-1} \rightharpoonup p(b) = p(g^{-1} \rightharpoonup b)$ for any arrow b, we have $S(I_p) \subseteq I_p$.

We now consider $\Delta(I_p)$. Let $a \in {}_dM_f$ be an arrow that is not a loop. Then

$$\begin{aligned}
\Delta(p(a)) &= \prod_{i=0}^{m_a-1} r(a)^{-i} \rightharpoonup \Delta(a) \\
&= \prod_{i=0}^{m_a-1} r(a)^{-i} \rightharpoonup \sum_{g_i \in G} ((g_i \rightharpoonup a) \otimes v_{g_i} + v_{g_i} \otimes (a \leftharpoonup g_i)) \\
&= \prod_{i=0}^{m_a-1} \sum_{g_i \in G} ((g_i \rightharpoonup a) \otimes (r(a)^{-i} \rightharpoonup v_{g_i}) + v_{g_i} \otimes (r(a)^{-i} \rightharpoonup (a \leftharpoonup g_i))) \\
&= \prod_{i=0}^{m_a-1} \sum_{g_i \in G} ((g_i \rightharpoonup a) \otimes (v_{g_i r(a)^i}) + v_{g_i} \otimes (r(a)^{-i} \rightharpoonup a \leftharpoonup g_i))
\end{aligned}$$

The product above can be written as a sum of elements of the form $x_0x_1\cdots x_{m_a-1}\otimes y_0y_1\cdots y_{m_a-1}$, where $x_i\otimes y_i = g_i \rightharpoonup a\otimes v_{g_ir(a)^i}$ (type (I)) or $x_i\otimes y_i = v_{g_i}\otimes(r(a)^{-i}\rightharpoonup a)\leftharpoonup g_i$ (type (II)) for all $i=0,1,\dots,m_a-1$.

Now given an element $x_i\otimes y_i$ of type (I), then

- ➢ if $x_{i+1}\otimes y_{i+1}$ is also of type (I) we have $v_{g^{i+1}r(a)^{i+1}} = y_{i+1} = y_i = v_{g_ir(a)^i}$ so that $g_{i+1}=g_ir(a)^{-1}$ is uniquely determined; note that $\mathfrak{s}(x_{i+1}) = v_{dg_{i+1}^{-1}} = v_{dr(a)g_i^{-1}} = v_{fg_i^{-1}} = \mathfrak{t}(x_i)$ so that the product $(x_i\otimes y_i)(x_{i+1}\otimes y_{i+1})$ is well defined.
- ➢ if $x_{i+1}\otimes y_{i+1}$ is of type (II) we have $x_{i+1}=v_{g_{i+1}}=\mathfrak{t}(x_i)=v_{fg_i^{-1}}$ so that $g_{i+1}=fg_i^{-1}$; note that $\mathfrak{s}(y_{i+1}) = v_{g_{i+1}^{-1}dr(a)^{i+1}} = v_{g_ir(a)^i} = y_i$ so that the product $(x_i\otimes y_i)(x_{i+1}\otimes y_{i+1})$ is well defined.

Given an element $x_i\otimes y_i$ of type (II), then

- ➢ if $x_{i+1}\otimes y_{i+1}$ is also of type (II) we have $x_{i+1}=v_{g_{i+1}}=x_i=v_{g_i}$ so that $g_{i+1}=g_i$; note that $\mathfrak{s}(y_{i+1}) = v_{g_{i+1}^{-1}dr(a)^{i+1}} = v_{g_i^{-1}fr(a)^i} = \mathfrak{t}(y_i)$ so that the product $(x_i\otimes y_i)(x_{i+1}\otimes y_{i+1})$ is well defined.
- ➢ if $x_{i+1}\otimes y_{i+1}$ is of type (I) we have $\mathfrak{s}(x_{i+1}) = v_{dg_{i+1}^{-}1} = x_i = v_{g_i}$ so that $g_{i+1}=g_i^{-1}f$; note that $y_{i+1}=v_{g_{i+1}r(a)^{i+1}} = v_{g_i^{-1}fr(a)^i} = \mathfrak{t}(y_i)$ so that the product $(x_i\otimes y_i)(x_{i+1}\otimes y_{i+1})$ is well defined.

Assume that $y_0y_1\cdots y_{m_a-1}$ starts at v_g. Then $\mathfrak{s}(y_0)=v_g$. If $x_0\otimes y_0$ is of type (I), then $y_0=v_{g_0}$ so that $g_0=g$ and $x_0=g_0\rightharpoonup a = g\rightharpoonup a$ starts at $v_{dg^{-1}}$. If $x_0\otimes y_0$ is of type (II), then $y_0=a\leftharpoonup g_0$ starts at $v_{g_0^{-1}d}$ so that $g_0=dg^{-1}$ and $x_0=v_{g_0}=v_{dg^{-1}}$ starts at $v_{dg^{-1}}$. In both cases, the source of $x_0x_1\cdots x_{m_a-1}$ is $v_{dg^{-1}}$.

Therefore, given a subset σ of $\{0,1,\cdots,m_a-1\}$ and an element g in G, there is a uniquely determined element in the product above, namely $x_0x_1\cdots x_{m_a-1}\otimes y_0y_1\cdots y_{m_a-1}$, where the path $x_0x_1\cdots x_{m_a-1}$ starts in vertex $v_{dg^{-1}}$, $y_0y_1\cdots y_{m_a-1}$ starts in vertex v_g, $x_i\otimes y_i$ is of type (I) for $i\in\sigma$, and $x_i\otimes y_i$ is of type (II) for $i\notin\sigma$.

Moreover, if $i_0=0$ then $y_0=v_{g_0}=v_g$ so that $g_{i_0}=g_0=g$, and if $i_0>0$, the $x_i\otimes y_i$ with $i<i_0$ are of type (II) so that $v_{dg^{-1}}=x_0=x_{i_0-1}=\mathfrak{s}(x_{i_0})=v_{dg_{i_0}^{-1}}$ and $g_{i_0}=g$. Next, the $x_i\otimes y_i$ with $i_0<i<i_1$ are of type (II) so that $v_{fg^{-1}}=\mathfrak{t}(x_{i_0})=x_{i_0+1}=x_{i_1-1}=\mathfrak{s}(x_{i_1})=v_{dg_{i_1}^{-1}}$ and $g_1=gr(a)^{-1}$. Inductively, we have $g_{i_j}=gr(a)^{-j}$ for $j=0,\dots,s-1$.

Similarly, for each $t=0,\dots,m_a-s-1$ we have $g_{j_t}=\ell(a)^{j_t-t}dg^{-1}$.

Therefore,

$$\begin{aligned}\Delta(p(a)) &= \sum_{g\in G}\sum_{s=0}^{m_a-1}\sum_{\sigma\in T_s(m_a)}\left(\prod_{j=0}^{s-1}(gr(a)^{-j}\rightharpoonup a)\otimes\prod_{t=0}^{m_a-s-1}(r(a)^{-j_t}\rightharpoonup a\leftharpoonup \ell(a)^{j_t-t}dg^{-1})\right)\\ &= \sum_{s=0}^{m_a-1}\sum_{g\in G}\sum_{\sigma\in T_s(m_a)}\left(\prod_{u\notin\sigma}c_a(a)^u\right)\left(\prod_{t=0}^{s-1}(gr(a)^{-t}\rightharpoonup a)\otimes\prod_{t=0}^{m_a-s-1}(a\leftharpoonup \ell(a)^{-t}dg^{-1})\right)\end{aligned}$$

since $r(a)^{-j}\rightharpoonup a = c_a(a)^j a\leftharpoonup \ell(a)^{-j}$.

If a is a loop, we have $d=f$ and $r(a)=1=\ell(a)$, and a similar argument shows that

$$\begin{aligned}\Delta(p(a))=\Delta(a^{m_a}) &= \sum_{g\in G}\sum_{s=0}^{m_a-1}\sum_{\sigma\in T_s(m_a)}\left(\prod_{j=0}^{s-1}(g\rightharpoonup a)\otimes\prod_{t=0}^{m_a-s-1}(a\leftharpoonup dg^{-1})\right)\\ &= \sum_{s=0}^{m_a-1}\sum_{g\in G}\binom{m_a}{s}\left((g\rightharpoonup a)^s\otimes(a\leftharpoonup dg^{-1})^{m_a-s}\right).\end{aligned}$$

For each $s=0,\dots,m_a$, the term $X_{s,g}:=\prod_{t=0}^{s-1}(gr(a)^{-t}\rightharpoonup a)$, or $X_{s,g}:=(g\rightharpoonup a)^s$ in the case of a loop, is a sub-path of length s of $p(g\rightharpoonup a)$ starting at $v_{dg^{-1}}$, and the term $Y_{s,g}:=\prod_{t=0}^{m_a-s-1}(a\leftharpoonup \ell(a)^{-t}dg^{-1})$, or $Y_{s,g}:=(a\leftharpoonup dg^{-1})^{m_a-s}$ in the case of a loop, is a (non-zero scalar multiple of a) sub-path of length m_a-s of $p(a\leftharpoonup dg^{-1})$ starting at v_g.

Multiplying by $v_{dg^{-1}}\otimes v_g$ shows that $\Delta(p(a))\in I_p\otimes k\Gamma_G(W)+k\Gamma_G(W)\otimes I_p$ if, and only if, for any $g\in G$ the term $\sum_{s=1}^{m_a-1}\sum_{\sigma\in T_s(m_a)}\left(\prod_{u\notin\sigma}c_a(a)^u\right)X_{s,g}\otimes Y_{s,g}$ is in $I_p\otimes k\Gamma_G(W)+k\Gamma_G(W)\otimes I_p$.

Now for $s=0$ and $s=m_a$ and for all $g\in G$, we have $\sum_{\sigma\in T_s(m_a)}\left(\prod_{u\notin\sigma}c_a(a)^u\right)X_{s,g}\otimes Y_{s,g}\in I_p\otimes k\Gamma_G(W)+k\Gamma_G(W)\otimes I_p$. Recall that $k\Gamma_G(W)\otimes k\Gamma_G(W)=\bigoplus_{t,u}k(\Gamma_G(W))_t\otimes k(\Gamma_G(W))_u$, therefore $\sum_{s=1}^{m_a-1}\sum_{g\in G}\sum_{\sigma\in T_s(m_a)}\left(\prod_{u\notin\sigma}c_a(a)^u\right)X_{s,g}\otimes Y_{s,g}$ is not in $I_p\otimes k\Gamma_G(W)+k\Gamma_G(W)\otimes I_p$ unless it is zero.

Each $X_{s,g} \otimes Y_{s,g}$ is in $k(\Gamma_G(W))_s \otimes k(gw)_{m_a-s}$ so that $\sum_{s=1}^{m_a-1} \sum_{g\in G} \sum_{\sigma\in T_s(m_a)} \left(\prod_{u\notin\sigma} c_a(a)^u\right) X_{s,g} \otimes Y_{s,g}$ vanishes if and only if for each $1 \leqslant s \leqslant m_a - 1$ we have $\sum_{\sigma\in T_s(m_a)} \left(\prod_{u\notin\sigma} c_a(a)^u\right) X_{s,g} \otimes Y_{s,g} = 0$, that is,

$$\begin{cases} \prod_{u\notin\sigma} c_a(a)^u = 0 & \text{if } a \text{ is not a loop,} \\ \binom{m_a}{s} = 0 & \text{if } a \text{ is a loop.} \end{cases}$$ □

c) The quotient $k\Gamma_G(W)/(I_q, I_p)$

Theorem V.8. *[GS, Theorem 5.6(b)] Let G be a finite group and let $W = \{w_1, \dots, w_n\}$ be a non-empty weight sequence generating an abelian subgroup of G. Let I_q and I_p be the ideals defined above for some choices of integers m_a associated to the arrows a in $\Gamma_G(W)$. Assume that the allowable kG-bimodule structure on $k\Gamma_G(W)$ is given by group homomorphisms $\Theta : G^{op} \to \mathfrak{S}_n$ and $f_i = f_{\Theta(g)(i)} : G \to k^\times$ for $i = 1, \dots, n$ as in Example IV.9. Assume moreover that*

- *$f_i(w_j) = f_j(w_i)^{-1}$ for all i and j with $i \neq j$,*
- *for any arrow a that is not a loop and for all $s = 1, \dots, m_a - 1$, $\sum_{\sigma\in T_s(m_a)} \prod_{j\notin\sigma} f_i(w_i)^j = 0$,*
- *if there is a loop in $\Gamma_G(W)$, then $\operatorname{char}(k) = p > 0$ and for any loop a and any $s = 1, \dots, m_a - 1$, p divides $\binom{m_a}{s}$.*

Then the algebra $k\Gamma_G(W)/(I_p, I_q)$ is a finite dimensional Hopf algebra.

Proof.
- We first show that I_q is a Hopf ideal using Lemma V.2. Since W generates an abelian subgroup of G by assumption, we need only show that condition *(ii)* is satisfied.

  Let (a_i, h) and (a_j, h) be two arrows with the same source $v_{h^{-1}}$. Then

$$r((a_j, h)) \rightharpoonup (a_i, h) = hw_jh^{-1} \rightharpoonup (a_i, h) = (a_i, hw_j)$$
$$(a_i, h) \leftharpoonup \ell((a_j, h)) = (a_i, h) \leftharpoonup w_j = f_i(w_j)(a_{\Theta(w_j)(i)}, hw_j) = f_i(w_j)(a_i, hw_j)$$

  since $w_{\Theta(w_j)(i)} = w_j^{-1} w_i w_j = w_i$ because the elements of W commute. Therefore $c_{(a_j,h)}((a_i, h)) = f_i(w_j)$ and by assumption, if $i \neq j$, we have $c_{(a_i,h)}((a_j, h)) = c_{(a_j,h)}((a_i, h))^{-1}$. Therefore *(ii)* in Lemma V.2 is satisfied.

- From the above, we have $c_{(a_i,h)}((a_i, h)) = f_i(w_i)$ and conditions *(i)* and *(ii)* in Lemma V.7 are satisfied by assumption. Therefore I_p is also a Hopf ideal, and so is (I_p, I_q). Hence $H := k\Gamma_G(W)/(I_p, I_q)$ is a Hopf algebra.

- It remains to be shown that H is finite dimensional.

  Let a and b be arrows such that $\mathfrak{t}(a) = \mathfrak{s}(b)$. Then a and $r(a) \rightharpoonup b$ are arrows with the same source. Assume that they are different, that is, that $b \neq r(a)^{-1} \rightharpoonup b$. Then, in H, we have

$$\begin{aligned} 0 = q(a, r(a) \rightharpoonup b) &= a(r(a)^{-1} \rightharpoonup (r(a) \rightharpoonup b)) - (r(a) \rightharpoonup b)(a \leftharpoonup \ell(r(a) \rightharpoonup b)^{-1}) \\ &= ab - (r(a) \rightharpoonup b)(a \leftharpoonup \ell(b)^{-1}) \\ &= ab - c_b(a)(r(a) \rightharpoonup b)(r(b)^{-1} \rightharpoonup a) \end{aligned}$$

  so that, in H, we have $ab = cb'a'$ where $c \in k^\times$, a' is an arrow in the left G-orbit of a and b' is an arrow in the left G-orbit of b.

  Note that there are n left G-orbits in $(k\Gamma_G(W))_1$, one for each $w_i \in W$ (the orbits of the $(a_i, 1)$).

  Set $N = \max\{m_a; a \in k(\Gamma_G(W))_1\} = \max\left\{m_{(a_i,1)}; i = 1, \dots, n\right\}$. We prove that any path of length at least nN vanishes in H.

  Let $z = b_1 b_2 \cdots b_t$ be a path of length $t \geqslant nN$. Then at least N of the arrows b_i are in the same left G-orbit. By the first part of the proof that H is finite dimensional, there is a scalar c such that $z = b_1 \cdots b_r b'_{r+1} \cdots b'_{r+N} b'_{r+N+1} \cdots b'_t + z'$ with $b'_{r+1}, \dots, b'_{r+N}$ in the same G-orbit and $z' \in I_q$. Therefore $b'_{r+1} \cdots b'_{r+N}$ is in I_p by Remark V.6 so that $z \in (I_p, I_q)$ as required. □

Remark V.9. Green and Solberg also give, in [GS, Theorem 5.6], the order of the antipode ($2 \cdot \operatorname{lcm}\{|f_i(w_i)|; i = 1, \dots, n\}$) as well as necessary and sufficient conditions for $k\Gamma_G(W)/(I_q, I_p)$ to be commutative ($w_i = 1$ for all i) or cocommutative (G abelian and $f_i \equiv 1$ for all i).

Moreover, in Corollary 5.4, they do the case of a general allowable kG-bimodule structure on $k\Gamma_G(W)$.

In Examples V.10, V.12 and V.13, we check separately that I_q and I_p are Hopf ideals using Lemmas V.2 and V.7, although we do not need to in order to apply Theorem V.8.

Example V.10. Let $G = \mathbb{Z}/n\mathbb{Z}$ be the cyclic group of order n, generated by γ. The subset $W = \{\gamma\}$ is a weight sequence (G is abelian).

The quiver $\Gamma_G(W)$ is then an oriented cycle with n vertices (and arrows): the arrows are the (a, g) from $v_{g^{-1}}$ to $v_{\gamma g^{-1}}$ for all $g \in G$ that is, $\alpha_t := (a, \gamma^t)$ is the arrow from $v_{\gamma^{-t}}$ to $v_{\gamma^{-t+1}}$. Set $e_t = v_{\gamma^{-t}}$.

Take $\Theta \equiv \mathrm{id}$ and let $f : G \to k^\times$ be defined by $f(\gamma) = \zeta$ with $\zeta^n = 1$. These determine an allowable kG-bimodule structure on $k\Gamma_G(W)$ as in Example IV.9. The corresponding Hopf algebra structure on $k\Gamma_G(W)$ is determined by:

$$\begin{aligned}
\varepsilon(e_t) &= \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{if } t \neq 0, \end{cases} \qquad \varepsilon(\alpha_t) = 0, \\
\Delta(e_t) &= \sum_{s=0}^{n-1} e_s \otimes e_{t-s}, \qquad S(e_t) = e_{-t}, \\
\Delta(\alpha_t) &= \sum_{s=0}^{n-1} (\gamma^s \rightharpoonup \alpha_t \otimes v_{\gamma^s} + v_{\gamma^s} \otimes \alpha_t \leftharpoonup \gamma^s) \\
&= \sum_{s=0}^{n-1} (\alpha_{t+s} \otimes e_{-s} + \zeta^s e_{-s} \otimes \alpha_{t+s}) \\
&= \sum_{s+u=t} (\alpha_s \otimes e_u + \zeta^{-u} e_u \otimes \alpha_s), \\
S(\alpha_t) &= -\gamma^{-t} \rightharpoonup \alpha_t \leftharpoonup \gamma^{1-t} \\
&= -\zeta^{1-t} \alpha_{1-t}
\end{aligned}$$

where the indices are taken modulo n.

Finally let $d \geqslant 2$ be the order of ζ and set $m_a = d$ for all arrows $a \in (\Gamma_G(W))_1$. Note that d divides n. We now determine the quotient $k\Gamma_G(W)/(I_p, I_q)$. Clearly, $I_q = 0$ since no two arrows have the same source.

Now consider I_p. For all t we have $\alpha_t \leftharpoonup \ell(\alpha_t) = \alpha_t \leftharpoonup \gamma = \zeta\gamma \rightharpoonup \alpha_t = \zeta r(\alpha_t) \rightharpoonup \alpha_t$ so that $c_{\alpha_t}(\alpha_t) = \zeta$ for all α_t. Since there are no loops in the quiver, we need only check that for all $s = 1, \ldots, d-1$, we have $\sum_{\sigma \in T_s(d)} \prod_{i \notin \sigma} \zeta^i = 0$. This follows immediately from Lemma V.11 below applied to the cyclic group G, using that $f(\gamma^s) = \zeta^s \neq 1$.

Since $\gamma \rightharpoonup \alpha_t = \alpha_{t+1}$, the path $p(\alpha_t)$ is the unique path of length d starting at α_t. Note that $p(\alpha_t) \leftharpoonup \gamma^s = \zeta^{ds} p(\alpha_{t+s}) = p(\alpha_{t+s})$. Hence I_p is the ideal generated by all paths of length d.

These algebras are called the generalised Taft algebras. They were studied in detail by Cibils in [C] and also in [CHYZ]. They are neither commutative nor cocommutative.

Lemma V.11. *[GS, Lemma 5.5] Let G be a finite group of order n and let k be a field. Suppose that $f : G \to k^\times$ is a group morphism. Let s be an integer with $1 \leqslant s < n$. Assume that there exists an element $g \in G$ such that $f(g^s) \neq 1$. Let $T_s(G)$ be the set of all subsets of G consisting of s elements. Then*

$$\sum_{\sigma \in T_s(G)} \prod_{g \notin \sigma} f(g) = 0$$

Proof. For $\sigma \in T_s(G)$, set $f(\sigma) = \prod_{g \notin \sigma} f(g)$ For $\sigma \in T_s(G)$ and $g \in G$, set $g\sigma := \{gh; h \in \sigma\}$. Then $\tau_g : T_s(G) \to T_s(G)$ defined by $\tau_g(\sigma) = g\sigma$ is a bijection, with inverse $\tau_{g^{-1}}$. Moreover,

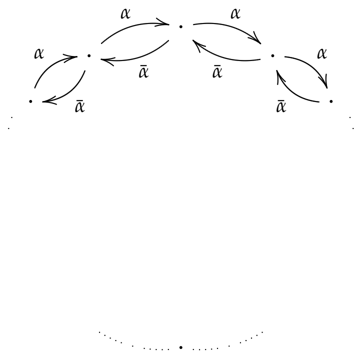
$$f(g\sigma) = \prod_{h \notin \sigma} f(gh) = f(g)^s \prod_{h \notin \sigma} f(h) = f(g)^s f(\sigma).$$

Therefore

$$\sum_{\sigma \in T_s(G)} f(\sigma) = \sum_{\sigma \in T_s(G)} f(g\sigma) = f(g^s) \sum_{\sigma \in T_s(G)} f(\sigma).$$

Since $f(g^s) \neq 1$, we have $\sum_{\sigma \in T_s(G)} f(\sigma) = 0$. □

Example V.12. Let $G = \mathbb{Z}/n\mathbb{Z}$ be the cyclic group of order n, generated by γ. The subset $W = \{w_1 = \gamma, w_2 = \gamma^{-1}\}$ is a weight sequence (G is abelian). The quiver $\Gamma_G(W)$ is then of the form



with n vertices and $2n$ arrows: if we set $e_t = v_{\gamma^{-t}}$ for $0 \leqslant t < n$, the $\alpha_t := (a_1, \gamma^t)$ go from $e_t = v_{\gamma^{-t}}$ to $v_{\gamma\gamma^{-t}} = e_{t-1}$ and the $\bar{\alpha}_t := (a_2, \gamma^t)$ go from e_t to $v_{\gamma^{-1}\gamma^{-t}} = e_{t+1}$ for all t considered modulo n.

Take $\Theta \equiv \mathrm{id}$ and let $f_i : G \to k^\times$ for $i = 1, 2$ be defined by $f_i(\gamma) = \zeta$ with $\zeta^n = 1$. These determine an allowable kG-bimodule structure on $k\Gamma_G(W)$ as in Example IV.9. The corresponding Hopf algebra structure on $k\Gamma_G(W)$ is determined by the formulas in the previous example for the e_t and the α_t and by:

$$\begin{aligned}
\varepsilon(\bar{\alpha}_t) &= 0, \\
\Delta(\bar{\alpha}_t) &= \sum_{s=0}^{n-1} (\gamma^s \rightharpoonup \bar{\alpha}_t \otimes v_{\gamma^s} + v_{\gamma^s} \otimes \bar{\alpha}_t \leftharpoonup \gamma^s) \\
&= \sum_{s=0}^{n-1} (\bar{\alpha}_{t+s} \otimes e_{-s} + \zeta^s e_{-s} \otimes \bar{\alpha}_{t+s}) \\
&= \sum_{s+u=t} (\bar{\alpha}_s \otimes e_u + \zeta^{-u} e_u \otimes \bar{\alpha}_s), \\
S(\bar{\alpha}_t) &= -\gamma^{-t} \rightharpoonup \bar{\alpha}_t \leftharpoonup \gamma^{-t-1} \\
&= -\zeta^{-1-t} \bar{\alpha}_{-t+1}
\end{aligned}$$

where the indices are taken modulo n.

Finally let $d \geqslant 2$ be the order of ζ and set $m_a = d$ for all arrows $a \in k(\Gamma_G(W))_1$. Note that d divides n. We now determine the quotient $k\Gamma_G(W)/(I_p, I_q)$.

The arrows α_t and $\bar{\alpha}_t$ are distinct and have the same source e_t. We have

$$q(\alpha_t, \bar{\alpha}_t) = \alpha_t(r(\alpha_t)^{-1} \rightharpoonup \bar{\alpha}_t) - \bar{\alpha}_t(\alpha_t \leftharpoonup \ell(\bar{\alpha}_t)^{-1}) = \alpha_t(\gamma^{-1} \rightharpoonup \bar{\alpha}_t) - \bar{\alpha}_t(\alpha_t \leftharpoonup \gamma) = \alpha_t\bar{\alpha}_{t-1} - \zeta\bar{\alpha}_t\alpha_{t+1}.$$

Moreover, $q(\alpha_t, \bar{\alpha}_t) \leftharpoonup \gamma^s = \zeta^{2s} q(\alpha_{t+s}, \bar{\alpha}_{t+s})$.

The subgroup generated by W is G which is abelian, and

$$\begin{aligned}
\alpha_t \leftharpoonup \ell(\bar{\alpha}_t) &= \alpha_t \leftharpoonup \gamma^{-1} = \zeta^{-1}\alpha_{t-1} & \qquad r(\bar{\alpha}_t) \rightharpoonup \alpha_t &= \gamma^{-1} \rightharpoonup \alpha_t = \alpha_{t-1} \\
\bar{\alpha}_t \leftharpoonup \ell(\alpha_t) &= \bar{\alpha}_t \leftharpoonup \gamma = \zeta\bar{\alpha}_{t+1} & r(\alpha_t) \rightharpoonup \bar{\alpha}_t &= \gamma \rightharpoonup \bar{\alpha}_t = \bar{\alpha}_{t+1}
\end{aligned}$$

so that $c_{\alpha_t}(\bar{\alpha}_t) = \zeta = c_{\bar{\alpha}_t}(\alpha_t)^{-1}$ and the conditions in Lemma V.2 are satisfied. Therefore I_q is a Hopf ideal, generated by all elements of the form $\alpha_t\bar{\alpha}_{t-1} - \zeta\bar{\alpha}_t\alpha_{t+1}$ for $0 \leqslant t < n$ considered mod n.

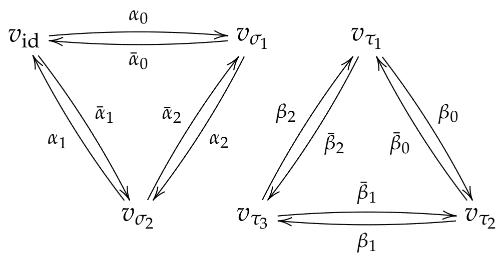
Now consider I_p. As in the previous example, we have $c_{\alpha_t}(\alpha_t) = \zeta$ for all t. We also have $c_{\bar{\alpha}_t}(\bar{\alpha}_t) = \zeta^{-1}$. Moreover, $p(\alpha_t) = \alpha_t\alpha_{t-1}\cdots\alpha_{t-d+1}$ and $p(\bar{\alpha}_t) = \bar{\alpha}_t\bar{\alpha}_{t+1}\cdots\bar{\alpha}_{t+d-1}$, and we have $p(\alpha_t) \leftharpoonup \gamma^s = \zeta^{sd} p(\alpha_{t+s}) = p(\alpha_{t+s})$ and $p(\bar{\alpha}_t) \leftharpoonup \gamma^s = p(\bar{\alpha}_{t+s})$. Hence I_p is the ideal generated by all paths of length d going in the same direction around the circular quiver.

These Hopf algebras are neither commutative nor cocommutative.

In the case where $d = 2$, that is, $\zeta = -1$ (and n even), the Hopf algebras $k\Gamma_G(W)/(I_q, I_p)$ are isomorphic as algebras to some algebras Λ that occur in the study of the representation theory of the Drinfeld doubles of the generalised Taft algebras, see [EGST]. These algebra isomorphisms allow us to define Hopf algebra structures on the algebras Λ. However, unless $\mathrm{char}(k) = 2$,they are not Hopf algebras of the form $k\Gamma_G(W)/(I_p, I_q)$.

Example V.13. Let $G = \mathfrak{S}_3$ be the symmetric group of order 6; we denote its elements by id, $\sigma_1 = (1 \;\; 2 \;\; 3)$, $\sigma_2 = \sigma_1^2$, and τ_i the transposition that fixes i for $i = 1, 2, 3$. The subset $W = \{w_1 = \sigma_1, w_2 = \sigma_2\}$ is a weight sequence (conjugation by $g \in G$ either fixes both σ_i or exchanges them).

The quiver $\Gamma_G(W)$ is then



where $\alpha_i = (a_1, \sigma_1^i)$, $\beta_i = (a_1, \tau_{i+1})$, $\bar{\alpha}_i = (a_2, \sigma_1^{i-1})$ and $\bar{\beta}_i = (a_2, \tau_{i-1})$ for $i = 0, 1, 2$ (indices considered mod 3 where necessary).

Take $\Theta : G = \mathfrak{S}_3 \to \mathfrak{S}_2 \cong \mathbb{Z}/2\mathbb{Z} = \langle \gamma \rangle$ be defined by $\Theta(\sigma_i) = \mathrm{id}$ and $\Theta(\tau_i) = \gamma$ and let $f_i : G \to k^\times$ for $i = 1, 2$ be identically 1. Clearly, $\Theta(g)$ fixes each of the w_i for any $g \in G$. These determine an allowable kG-bimodule structure on $k\Gamma_G(W)$ as in Example IV.9. The left and right actions on arrows are given in the following table.

$\rightharpoonup$	α_0	α_1	α_2	β_0	β_1	β_2	$\bar{\alpha}_0$	$\bar{\alpha}_1$	$\bar{\alpha}_2$	$\bar{\beta}_0$	$\bar{\beta}_1$	$\bar{\beta}_2$
σ_1	α_1	α_2	α_0	β_1	β_2	β_0	$\bar{\alpha}_1$	$\bar{\alpha}_2$	$\bar{\alpha}_0$	$\bar{\beta}_1$	$\bar{\beta}_2$	$\bar{\beta}_0$
σ_2	α_2	α_0	α_1	β_2	β_0	β_1	$\bar{\alpha}_2$	$\bar{\alpha}_0$	$\bar{\alpha}_1$	$\bar{\beta}_2$	$\bar{\beta}_0$	$\bar{\beta}_1$
τ_1	β_0	β_2	β_1	α_0	α_2	α_1	$\bar{\beta}_0$	$\bar{\beta}_2$	$\bar{\beta}_1$	$\bar{\alpha}_0$	$\bar{\alpha}_2$	$\bar{\alpha}_1$
τ_2	β_1	β_0	β_2	α_1	α_0	α_2	$\bar{\beta}_1$	$\bar{\beta}_0$	$\bar{\beta}_2$	$\bar{\alpha}_1$	$\bar{\alpha}_0$	$\bar{\alpha}_2$
τ_3	β_2	β_1	β_0	α_2	α_1	α_0	$\bar{\beta}_2$	$\bar{\beta}_1$	$\bar{\beta}_0$	$\bar{\alpha}_2$	$\bar{\alpha}_1$	$\bar{\alpha}_0$

$\leftharpoonup$	α_0	α_1	α_2	β_0	β_1	β_2	$\bar{\alpha}_0$	$\bar{\alpha}_1$	$\bar{\alpha}_2$	$\bar{\beta}_0$	$\bar{\beta}_1$	$\bar{\beta}_2$
σ_1	α_1	α_2	α_0	β_2	β_0	β_1	$\bar{\alpha}_1$	$\bar{\alpha}_2$	$\bar{\alpha}_0$	$\bar{\beta}_2$	$\bar{\beta}_0$	$\bar{\beta}_1$
σ_2	α_2	α_0	α_1	β_1	β_2	β_0	$\bar{\alpha}_2$	$\bar{\alpha}_0$	$\bar{\alpha}_1$	$\bar{\beta}_1$	$\bar{\beta}_2$	$\bar{\beta}_0$
τ_1	$\bar{\beta}_2$	$\bar{\beta}_0$	$\bar{\beta}_1$	$\bar{\alpha}_1$	$\bar{\alpha}_2$	$\bar{\alpha}_0$	β_2	β_0	β_1	α_1	α_2	α_0
τ_2	$\bar{\beta}_0$	$\bar{\beta}_1$	$\bar{\beta}_2$	$\bar{\alpha}_0$	$\bar{\alpha}_1$	$\bar{\alpha}_2$	β_0	β_1	β_2	α_0	α_1	α_2
τ_3	$\bar{\beta}_1$	$\bar{\beta}_2$	$\bar{\beta}_0$	$\bar{\alpha}_2$	$\bar{\alpha}_0$	$\bar{\alpha}_1$	β_1	β_2	β_0	α_2	α_0	α_1

(note that $(a_1, g) \leftharpoonup \tau_i = (a_2, g\tau_i)$ and $(a_2, g) \leftharpoonup \tau_i = (a_1, g\tau_i)$.

The corresponding Hopf algebra structure on $k\Gamma_G(W)$ is determined as before by the actions above. We get

$$S(\beta_i) = \bar{\beta}_i \qquad S(\alpha) = -\alpha \text{ if } \alpha \in \{\alpha_2, \bar{\alpha}_2\}$$
$$S(\bar{\beta}_i) = \beta_i \qquad S(\alpha) = -\alpha' \text{ if } \{\alpha, \alpha'\} = \{\alpha_0, \alpha_1\} \text{ or } \{\alpha, \alpha'\} = \{\bar{\alpha}_0, \bar{\alpha}_1\}$$

and, for instance,

$$\begin{aligned}
\Delta(\alpha_0) &= \sum_{g \in G} (g \rightharpoonup \alpha_0 \otimes v_g + v_g \otimes \alpha_0 \leftharpoonup g) \\
&= \alpha_0 \otimes v_{\mathrm{id}} + \alpha_1 \otimes v_{\sigma_1} + \alpha_2 \otimes v_{\sigma_2} + \beta_0 \otimes v_{\tau_1} + \beta_1 \otimes v_{\tau_2} + \beta_2 \otimes v_{\tau_3} \\
&\quad + v_{\mathrm{id}} \otimes \alpha_0 + v_{\sigma_1} \otimes \alpha_1 + v_{\sigma_2} \otimes \alpha_2 + v_{\tau_1} \otimes \bar{\beta}_2 + v_{\tau_2} \otimes \bar{\beta}_0 + v_{\tau_2} \otimes \bar{\beta}_1 \\
\Delta(\bar{\beta}_1) &= \bar{\beta}_1 \otimes v_{\mathrm{id}} + \bar{\beta}_2 \otimes v_{\sigma_1} + \bar{\beta}_0 \otimes v_{\sigma_2} + \bar{\alpha}_2 \otimes v_{\tau_1} + \bar{\alpha}_0 \otimes v_{\tau_2} + \bar{\alpha}_1 \otimes v_{\tau_3} \\
&\quad + v_{\mathrm{id}} \otimes \bar{\beta}_1 + v_{\sigma_1} \otimes \bar{\beta}_0 + v_{\sigma_2} \otimes \bar{\beta}_2 + v_{\tau_1} \otimes \alpha_2 + v_{\tau_2} \otimes \alpha_1 + v_{\tau_2} \otimes \alpha_0.
\end{aligned}$$

Note that $\ell(\alpha_i) = r(\alpha_i) = \ell(\beta_i) = r(\bar{\beta}_i) = \sigma_1$ and that $\ell(\bar{\alpha}_i) = r(\bar{\alpha}_i) = r(\beta_i) = \ell(\bar{\beta}_i) = \sigma_2$. It is then easy, using the table above, to check that Condition *(ii)* in Lemma V.2 is satisfied. Since W generates an abelian subgroup of G, I_q is a Hopf ideal. It is the ideal generated by

$$\left\{ \alpha_i \bar{\alpha}_i - \bar{\alpha}_{i+1} \alpha_{i+1}, \beta_i \bar{\beta}_i - \bar{\beta}_{i-1} \beta_{i-1}; i = 0, 1, 2 \pmod 3 \right\}.$$

Set $m_a = 3$ for all arrows $a \in (\Gamma_G(W))_1$. Using the table above, it is easy to see that $c_a(a) = 1$ for every arrow a. Assume that $\mathrm{char}(k) = 3$. Then Condition *(i)* in Lemma V.7 is satisfied. Since there are no loops in $\Gamma_G(W)$, I_p is a Hopf ideal. The left action of σ_i on the set of arrows for $i = 1, 2$ has four orbits, $\{\alpha_i; i = 0, 1, 2\}$, $\{\beta_i; i = 0, 1, 2\}$, $\{\bar{\alpha}_i; i = 0, 1, 2\}$, $\{\bar{\beta}_i; i = 0, 1, 2\}$. Therefore $p(\alpha_i)$ is the path of length 3 starting at α_i and going in one direction, and similarly for the other arrows. The right action of elements of G permutes these paths. Therefore I_p is generated by all paths of length 3 going in one direction:

$$\left\{ \alpha_i \alpha_{i-1} \alpha_{i-2}, \beta_i \beta_{i+1} \beta_{i+2}, \bar{\alpha}_i \bar{\alpha}_{i+1} \bar{\alpha}_{i+2}, \bar{\beta}_i \bar{\beta}_{i-1} \bar{\beta}_{i-2}; i = 0, 1, 2 \pmod 3 \right\}.$$

This Hopf algebra is neither commutative nor cocommutative. It is also clear that the antipode has order 2.

We conclude with another example which shows that W need not be a subset of G.

Example V.14. Let $G = \mathbb{Z}/n\mathbb{Z}$ be the cyclic group of order n, generated by γ. The subset $W = \{w_1 = 1, w_2 = 1\}$ is a weight sequence (G is abelian). The quiver $\Gamma_G(W)$ is then of the form

with n vertices and $2n$ arrows: if we set $e_t = v_{\gamma^{-t}}$ for $0 \leqslant t < n$, the $\alpha_t := (a_1, \gamma^t)$ go from $e_t = v_{\gamma^{-t}}$ to $v_{1\gamma^{-t}} = e_t$ and the $\beta_t := (a_2, \gamma^t)$ go from e_t to $v_{1\gamma^{-t}} = e_t$ for all $t = 0, 1, \ldots, n-1$.

Take $\Theta \equiv \mathrm{id}$ and let $f_i : G \to k^\times$ for $i = 1, 2$ be defined by $f_i(\gamma) = \zeta_i$ with $\zeta_i^n = 1$ for $i = 1, 2$. These determine an allowable kG-bimodule structure on $k\Gamma_G(W)$ as in Example IV.9. Since $\gamma \rightharpoonup \alpha_t = \alpha_{t+1}$, $\gamma \rightharpoonup \beta_t = \beta_{t+1}$ and

$$\alpha_t \leftharpoonup \gamma = (a_1, \gamma^t) \leftharpoonup \gamma = f_1(\gamma)(a_{\Theta(\gamma)(1)}, \gamma^t\gamma) = \zeta_1(a_1, \gamma^{t+1}) = \zeta_1\alpha_{t+1}$$
$$\beta_t \leftharpoonup \gamma = (a_2, \gamma^t) \leftharpoonup \gamma = f_2(\gamma)(a_{\Theta(\gamma)(2)}, \gamma^t\gamma) = \zeta_2(a_2, \gamma^{t+1}) = \zeta_2\beta_{t+1},$$

the corresponding Hopf algebra structure on $k\Gamma_G(W)$ is determined by the formulas in the first two examples for the e_t and by:

$$\begin{aligned}
\varepsilon(\alpha_t) &= 0,\\
\varepsilon(\beta_t) &= 0,\\
\Delta(\alpha_t) &= \sum_{s+u=t} (\alpha_s \otimes e_u + \zeta_1^{-u} e_u \otimes \alpha_s),\\
\Delta(\beta_t) &= \sum_{s+u=t} (\beta_s \otimes e_u + \zeta_2^{-u} e_u \otimes \beta_s),\\
S(\alpha_t) &= -\gamma^{-t} \rightharpoonup \alpha_t \leftharpoonup \gamma^{-t} = -\zeta_1^{-t}\alpha_{-t}\\
S(\beta_t) &= -\gamma^{-t} \rightharpoonup \beta_t \leftharpoonup \gamma^{-t} = -\zeta_2^{-t}\beta_{-t}
\end{aligned}$$

where the indices are taken modulo n.

Finally let $\mathrm{char}(k) = p > 0$ and set $m_a = p$ for all arrows $a \in k(\Gamma_G(W))_1$. Fix $\zeta_1 = \zeta_2 = 1$. We now determine the quotient $k\Gamma_G(W)/(I_p, I_q)$.

The arrows α_t and β_t are distinct and have the same source e_t. We have

$$q(\alpha_t, \beta_t) = \alpha_t(r(\alpha_t)^{-1} \rightharpoonup \beta_t) - \alpha_t(\alpha_t \leftharpoonup \ell(\beta_t)^{-1}) = \alpha_t\beta_t - \beta_t\alpha_t.$$

Moreover, $q(\alpha_t, \beta_t) \leftharpoonup \gamma = q(\alpha_{t+1}, \beta_{t+1})$. Hence I_q is the ideal generated by $\{\alpha_t\beta_t - \beta_t\alpha_t; 0 \leqslant t \leqslant n-1\}$.

Now consider I_p. Since all arrows are loops, we have $p(\alpha_t) = \alpha_t^d$ and $p(\beta_t) = \beta_t^d$. Moreover, $p(\alpha_t) \leftharpoonup \gamma = p(\alpha_{t+1})$ and $p(\beta_t) = p(\beta_{t+1})$ so that I_p is the ideal generated by $\left\{\alpha_t^d, \beta_t^d; 0 \leqslant t \leqslant n-1\right\}$.

Since the subgroup generated by W is $\{1\}$ which is abelian, all the conditions in Theorem V.8 are satisfied and therefore $k\Gamma_G(W)/(\alpha_t\beta_t - \beta_t\alpha_t, \alpha_t^p, \beta_t^p; 0 \leqslant t \leqslant n-1)$ is a finite dimensional Hopf algebra. It is commutative and cocommutative.

Abstract

This appendix gives some extra details for some of the proofs in [CR] (when k is a field). Moreover, the definition of a Cayley graph has been changed for compatibility with [GS] (Proposition IV.14), with (trivial) consequences on the statement and proof of Proposition 3.3 below. The section titles and the numbered results are those in [CR].

3. Bimodules de Hopf d'un groupe

Lemme 3.2. If H is a finite dimensional Hopf algebra, then the category $b_k(H)$ of finite dimensional Hopf bimodules over H is anti-equivalent to $b_k(H^*)$.

Proof. Recall that H^* is a Hopf algebra whose structure maps are given in Propositions I.12 and I.28.

Let M be a Hopf bimodule over H, with structure maps $\mu_\ell, \mu_r, \rho_\ell$ and ρ_r. Then M^* is a Hopf bimodule over H^* with structure maps defined similarly to those of H^* in Proposition I.12

$$\rho_\ell^* : H^* \otimes M^* \to M^* \qquad \rho_r^* : M^* \otimes H^* \to M^*$$
$$\mu_\ell^* : M^* \to H^* \otimes M^* \qquad \mu_r^* : M^* \to M^* \otimes H^*$$

where in each case $V^* \otimes W^* \cong (V \otimes W)^*$ as in Remark I.11. With this same convention, for k-linear maps $f : U_1 \to U_2$ and $g : V_1 \to V_2$ we may identify $(f \otimes g)^*$ and $f^* \otimes g^*$ via the following diagram

$$\begin{array}{ccc} (U_2 \otimes V_2)^* & \xrightarrow{(f\otimes g)^*} & (U_1 \otimes V_1)^* \\ \cong \downarrow & & \downarrow \cong \\ U_2^* \otimes V_2^* & \xrightarrow[f^*\otimes g^*]{} & U_1^* \otimes V_1^*. \end{array}$$

We then have for instance

$$\rho_\ell^*(\mathrm{id} \otimes \rho_\ell^*) = [(\mathrm{id} \otimes \rho_\ell)\rho_\ell]^* = [(\Delta \otimes \mathrm{id})\rho_\ell] = \rho_\ell^*(\Delta^* \otimes \mathrm{id})$$
$$\rho_\ell^*(\varepsilon^* \otimes \mathrm{id}) = [(\varepsilon \otimes \mathrm{id})\rho_\ell]^* = \mathrm{id}^* = \mathrm{id}$$

so that M^* is a left H^*-module.

The other properties that need to be checked are similar.

Moreover, it is easy to check that if $f : M \to N$ is a morphism of Hopf bimodules, then $f^* : N^* \to M^*$ is a morphism of Hopf bimodules.

Since all spaces are finite dimensional, dualising again gives a Hopf bimodule over H^{**} canonically isomorphic to the original Hopf bimodule over H. □

Lemma. Let M be a right comodule over kG. Then $M = \bigoplus_{g \in G} M^g$ where $M^g = \{m \in M; \rho(m) = m \otimes g\}$. Similarly, if M is a left comodule over kG then $M = \bigoplus_{g\in G} {}^gM$. Consequently, if M is a bicomodule over kG then $M = \bigoplus_{g\in G, h \in G} {}^gM^h$.

Proof. For $m \in M$ we can write $\rho(m) = \sum_{g\in G} m_g \otimes g \in M \otimes kG$. We have $(\rho \otimes \mathrm{id})(\rho(m)) = (\mathrm{id} \otimes \Delta)(\rho(m)) = \sum_{g \in G} m_g \otimes g \otimes g$ and $(\rho \otimes \mathrm{id})(\rho(m)) = \sum_{g\in G} \rho(m_g) \otimes g$. Since $M \otimes kG \otimes kG = \bigoplus_{g\in G} M \otimes kG \otimes g$, we have $\rho(m_g) = m_g \otimes g$ so that $m_g \in M^g$. Moreover, $m = (\mathrm{id} \otimes \varepsilon)(\rho(m)) = \sum_{g\in G} m_g \in \bigoplus_{g\in G} M^g$.

When M is a bicomodule, each M^g is a left subcomodule of M, therefore $M^g = \bigoplus_{h\in G} {}^hM^g$. Finally $M = \bigoplus_{g\in G, h\in G} {}^gM^h$. □

Notation. Let $\mathscr{C}$ be the set of conjugacy classes in G and for each conjugacy class $C \in \mathscr{C}$ choose an element $u(C)$. Let $Z_{u(C)}$ denote the centraliser of $u(C)$. Moreover, if $g \in G$, let $\Omega(g)$ be the conjugacy class of g.

Proposition 3.3. The category $\mathscr{B}(kG)$ of all Hopf bimodules over kG is equivalent to the cartesian product $\underset{C \in \mathscr{C}}{\times}$ Mod-$kZ_{u(C)}$.

Proof. **1) Description of the functor** $\mathcal{V} : \underset{C\in\mathscr{C}}{\times} kZ_{u(C)}\text{-Mod} \to \mathscr{B}(kG)$**.**

If $M = \{M(C)\}_{C\in\mathscr{C}}$ with $M(C) \in kZ_{u(C)}\text{-Mod}$, define $\mathcal{V}M := \bigoplus_{d,f\in G} {}^dM^f$ with ${}^dM^f = M(\Omega(fd^{-1}))$.

If $\phi = \{\phi_C\}_{C\in\mathscr{C}} : M \to N$ is a morphism in $\underset{C\in\mathscr{C}}{\times} kZ_{u(C)}\text{-Mod}$, define $\mathcal{V}\phi = \bigoplus_{d,f\in G} \phi_{\Omega(fd^{-1})}$.

➢ Hopf bimodule structure on $\mathcal{V}M$.

- ✧ If $v \in {}^d\mathcal{V}M^f$, the coactions are given by $\rho_\ell(v) = d\otimes v$ and $\rho_r(v) = v\otimes f$.
- ✧ If $v \in {}^d\mathcal{V}M^f$ and $g \in G$, the right action of g on v sends v to $v \in {}^{dg}\mathcal{V}M^{fg} = M(\Omega(fd^{-1}))$ (g acts by translation of the co-isotypic components).
- ✧ If $z \in C$, there exists $t \in G$ such that $z = tu(C)t^{-1}$. Moreover, $s \in G$ also satisfies $z = su(C)s^{-1}$ if and only if $(t^{-1}s)u(C)(t^{-1}s)^{-1}$, if and only if $t^{-1}s \in Z_{u(C)}$. Hence t is well defined up to multiplication on the right by an element of $Z_{u(C)}$.
  Therefore there is a bijection

$$\begin{array}{rcl} C & \longleftrightarrow & \left\{tZ_{u(C)}; t\in G\right\} \\ z & \mapsto & E(z) = tZ_{u(C)} \text{ where } z = tu(C)t^{-1} \\ tu(C)t^{-1} & \leftarrow\!\shortmid & tZ_{u(C)}. \end{array}$$

The left action of g on $\mathcal{V}M$ may now be defined. The module $M(C)$ is a left $kZ_{u(C)}$-module by assumption and $kE(z)$ is a free right $Z_{u(C)}$-module of rank 1 so that $M(C) \cong kE(z)\otimes_{kZ_{u(C)}} M(C)$ as k-vector spaces. The left action of g on $\mathcal{V}M$ sends ${}^d\mathcal{V}M^f$ to ${}^{gd}\mathcal{V}M^{gf}$ as follows:

$${}^d\mathcal{V}M^f \cong M(\Omega(fd^{-1}))\otimes_{kZ_{u(\Omega(fd^{-1}))}} kE(fd^{-1})$$
$$\swarrow$$
$$M(\Omega(fd^{-1})) \underset{kZ_{u(\Omega(fd^{-1}))}}{\otimes} kE(gfd^{-1}g^{-1}) = M(\Omega(gfd^{-1}g^{-1})) \underset{kZ_{u(\Omega(gfd^{-1}g^{-1}))}}{\otimes} kE(gfd^{-1}g^{-1}) \cong {}^{gd}\mathcal{V}M^{gf}$$

where the middle map sends $t\otimes m$ to $gt\otimes m$.

$\mathcal{V}M$ is obviously a bicomodule and a bimodule. Moreover, these two structures are compatible. Fix $v \in {}^d\mathcal{V}M^f$ and $g \in G$. We have $gv \in {}^{gd}\mathcal{V}M^{gf}$ and $vg \in {}^{dg}\mathcal{V}M^{fg}$.

- ✧ $\rho_\ell(vg) = dg\otimes vg = \rho_\ell(v)\cdot g$ (the action is diagonal).
- ✧ $\rho_r(vg) = vg\otimes fg = \rho_r(v)\cdot g$.
- ✧ $\rho_\ell(gv) = gd\otimes gv = g\cdot(d\otimes v) = g\cdot\rho_\ell(v)$.
- ✧ $\rho_r(gv) = gv\otimes gf = g\cdot(v\otimes f) = g\cdot\rho_r(v)$.

➢ $\mathcal{V}\phi$ is clearly a morphism of bicomodules by construction. Moreover, if $v = t\otimes m \in {}^d\mathcal{V}M^f$ and $g \in G$,

- ✧ $\mathcal{V}\phi(vg) = \phi_{\Omega(fg(dg)^{-1})}(vg) = \phi_{\Omega(fd^{-1})}(v) \in {}^{dg}\mathcal{V}N^{fg}$ so that $\mathcal{V}\phi(vg) = \phi_{\Omega(fd^{-1})}(v)g = \mathcal{V}\phi(v)g$.
- ✧ $\mathcal{V}\phi(gv) = (\mathrm{id}\otimes\phi_{\Omega(gf(gd)^{-1})})(gt\otimes m) = gt\otimes\phi_{\Omega(gfd^{-1}g^{-1})}(m) = g\cdot(t\otimes\phi_{\Omega(gfd^{-1}g^{-1})}(m)) = g\mathcal{V}\phi(v)$.

Therefore $\mathcal{V}\phi$ is a morphism of Hopf bimodules.

2) Description of the functor $\mathcal{W} : \mathscr{B}(kG) \to \underset{C\in\mathscr{C}}{\times} kZ_{u(C)}\text{-Mod}$**.**

If B is a Hopf bimodule over kG, then ${}^1B^{u(C)}$ is a left $kZ_{u(C)}$-module, where $Z_{u(C)}$ acts by conjugation: if $g \in Z_{u(C)}$, then

$$g\cdot {}^1B^{u(C)} \subset {}^1B^{gu(C)g^{-1}} = {}^1B^{u(C)}$$

Define $\mathcal{W}B = \left\{{}^1B^{u(C)}\right\}_{C\in\mathscr{C}} \in \underset{C\in\mathscr{C}}{\times} kZ_{u(C)}\text{-Mod}$. Moreover, if $\phi : B \to B'$ is a morphism of Hopf bimodules, then $\phi({}^1B^{u(C)}) \subseteq {}^1B'^{u(C)}$ since ϕ is a morphism of bicomodules, so that $\mathcal{W}\phi$ can be defined by $(\mathcal{W}\phi)_C = \phi_{|{}^1B^{u(C)}}$ for $C \in \mathscr{C}$. Each $(\mathcal{W}\phi)_C$ is a morphism of $kZ_{u(C)}$-modules since ϕ is a morphism of bimodules (if $g \in Z_{u(C)}$ and $b \in {}^1B^{u(C)}$, then $(\mathcal{W}\phi)_C(g\cdot b) = \phi(gbg^{-1}) = g\phi(b)g^{-1} = g\cdot(\mathcal{W}\phi)_C(b)$).

3) $\mathcal{V}\mathcal{W} \cong \mathrm{id}$.

Recall that ${}^d\mathcal{V}\mathcal{W}B^f \cong kE(fd^{-1}) \otimes_{kZ_{u(\Omega(fd^{-1}))}} \mathcal{W}B(\Omega(fd^{-1})) \cong kE(fd^{-1}) \otimes_{kZ_{u(\Omega(fd^{-1}))}} {}^1B^{u(\Omega(fd^{-1}))}$.

Moreover, if $b \in {}^dB^f$ and $t \in G$ is such that $fd^{-1} = tu(\Omega(fd^{-1}))t^{-1}$, then $t^{-1}bd^{-1}t \in {}^{t^{-1}dd^{-1}t}B^{t^{-1}fd^{-1}t} = {}^1B^{u(\Omega(fd^{-1}))}$.

Define ${}^d\theta_B^f : {}^dB^f \to kE(fd^{-1}) \otimes_{kZ_{u(\Omega(fd^{-1}))}} {}^1B^{u(\Omega(fd^{-1}))}$ by ${}^d\theta_B^f(b) = t \otimes t^{-1}bd^{-1}t$. This is well defined, independently of t : if s is another element in G such that $fd^{-1} = su(\Omega(fd^{-1}))s^{-1}$, then $s = tz$ for some $z \in Z_{u(\Omega(fd^{-1}))}$ and we have

$$s \otimes s^{-1}bd^{-1}s = tz \otimes (tz)^{-1}bd^{-1}tz = t \otimes z \cdot (z^{-1}t^{-1}bd^{-1}tz) = t \otimes zz^{-1}t^{-1}bd^{-1}tzz^{-1} = t \otimes t^{-1}bd^{-1}t.$$

- ${}^d\theta_B^f$ is a bijection with inverse $t \otimes b \mapsto tbt^{-1}d$.
- ${}^d\theta_B^f$ is a morphism of bicomodules by construction.
- If $b \in {}^dB^f$ and $g \in G$, then $bg \in {}^{dg}B^{fg}$ and $(fg)(dg)^{-1} = fd^{-1}$ so we can choose the same t. Therefore
$$ {}^d\theta_B^f(bg) = t \otimes t^{-1}bg(dg)^{-1}t = t \otimes t^{-1}bd^{-1}t = {}^d\theta_B^f(b)g$$
since the right action is the regular action.
- If $b \in {}^dB^f$ and $g \in G$, then $gb \in {}^{gd}B^{gf}$ and $(gf)(gd)^{-1} = gfd^{-1}g^{-1}$ so we can choose gt. Therefore
$$ {}^d\theta_B^f(gb) = gt \otimes (gt)^{-1}gb(gd)^{-1}(gt) = gt \otimes t^{-1}bd^{-1}t = g\,{}^d\theta_B^f(b).$$

Therefore ${}^d\theta_B^f$ is an isomorphism of Hopf bimodules.

Now if $\phi : B \to B'$, define $\theta(\phi)$ on ${}^d\mathcal{V}\mathcal{W}B^f$ by ${}^d\theta(\phi)^f = \mathrm{id} \otimes {}^1\phi^{u(\Omega(fd^{-1}))}$. Clearly $\theta(\phi)$ is a morphism of Hopf bimodules (bicomodules by construction and bimodules easy to check).

Finally, θ is natural:

$$ {}^d\theta(\phi)^d \circ {}^d\theta_B^f(b) = t \otimes {}^{u(\Omega(fd^{-1}))}\phi^1(t^{-1}bd^{-1}t) = t \otimes t^{-1}\phi(b)d^{-1}t = {}^d\theta_B^f(\phi(b))$$

so that $\theta(\phi) \circ \theta_B = \theta_{B'} \circ \phi$.

4) $\mathcal{W}\mathcal{V} \cong \mathrm{id}$.

If $M = \{M(C)\}_{C \in \mathscr{C}} \in \times_{C\in\mathscr{C}} kZ_{u(C)}$-Mod, define

$$\Psi_C : M(C) \to \mathcal{W}\mathcal{V}M(C) = {}^1\mathcal{V}M^{u(C)} = kE(u(C)) \otimes_{kZ_{u(\Omega(u(C)))}} M(\Omega(u(C))) = kZ_{u(C)} \otimes kZ_{u(C)}M(C)$$

which sends m to $m \otimes 1$. Then Ψ_C is an isomorphism of left $kZ_{u(C)}$-modules.

If $\phi : M \to N$ is a morphism, then define $(\Psi\phi)_C : \mathcal{W}\mathcal{V}M(C) \to \mathcal{W}\mathcal{V}N(C)$ by $(\Psi\phi)_C = \phi_C \otimes \mathrm{id}$, which is a morphism of left $kZ_{u(C)}$-modules.

Moreover, $\Psi : \mathrm{id} \to \mathcal{W}\mathcal{V}$ is natural: $(\Psi\phi)_C \circ \Psi_C = \Psi_C \circ \phi_C : M(C) \to \mathcal{W}\mathcal{V}M(C)$. □

Remark. The functors $\mathcal{V}$ and $\mathcal{W}$ preserve dimensions and therefore induce an equivalence between $b_k(kG)$ and $\times_{C\in\mathscr{C}} kZ_{u(C)}$-mod.

Definition. The Cayley graph of a group G with respect to a marking map $m : G \to \mathbb{N}$ is an oriented graph Γ whose vertices are indexed by the elements of the group, $\Gamma_0 = \{\delta_g; g \in G\}$, and such that the number of arrows from δ_d to δ_f is $m(fd^{-1})$.

Théorème 3.1. Let Γ be a quiver. Then $k\Gamma$ is a graded Hopf algebra if and only if Γ is the Cayley graph of a finite group G with respect to to a marking map $m : G \to \mathbb{N}$ constant on conjugacy classes.

Proof. Recall that $k\Gamma = T_{k\Gamma_0}(k\Gamma_1)$.

1) Assume that $k\Gamma$ is a graded Hopf algebra. Then its degree 0 part $k\Gamma_0$ is a Hopf subalgebra, isomorphic to a product of $\#\Gamma_0$ copies of k so that $k\Gamma_0 \cong k^G$ for some group G of order $\#\Gamma_0$. Moreover, $k\Gamma_1$ (the degree 1 part) is a Hopf bimodule over k^G so that $(k\Gamma_1)^* =: B$ is a Hopf bimodule over kG. Set ${}_d(k\Gamma_1)_f := \delta_d(k\Gamma_1)\delta_f$ where d, f are in G and δ_d, δ_f are the corresponding elements in $k^G \cong k\Gamma_0$.

By construction, $\dim {}^dB^f = \dim {}_d(k\Gamma_1)_f$ is the number of arrows from δ_d to δ_f.

We have $B = \mathcal{V}M$ for some $M = \{M(C)\}_{C\in\mathscr{C}}$ and ${}^dB^f = M(\Omega(fd^{-1}))$ so that $\dim {}^dB^f$ only depends on the conjugacy class of fd^{-1}.

Define $m : G \to \mathbb{N}$ by $m(g) = \dim M(\Omega(g))$. Then m is constant on conjugacy classes by construction and $m(fd^{-1})$ is the number of arrows from δ_d to δ_f.

Therefore Γ is the Cayley graph of G with respect to m.

2) Assume that Γ is the Cayley graph of a finite group G with respect to to a marking map $m : G \to \mathbb{N}$ constant on conjugacy classes. By definition, $\Gamma_0 = \{v_g; g \in G\}$, therefore $k\Gamma_0 \cong k^G$ so that $k\Gamma = T_{k^G}(k\Gamma_1)$.

$k\Gamma$ is therefore a Hopf algebra if and only if the k^G-bimodule $k\Gamma_1$ is a Hopf bimodule over k^G, if and only if the kG-bicomodule $B := (k\Gamma_1)^*$ is a Hopf bimodule over kG. Note that the number of arrows from v_d to v_f is $m(fd^{-1}) = \dim {}_d(k\Gamma_1)_f$.

For $C \in \mathscr{C}$, let $M(C)$ be a vector space of dimension $m(C)$, endowed with a left $kZ_{u(C)}$-module structure (*eg.* the trivial one). Then $M = \{M(C)\}_{C\in\mathscr{C}}$ is in $\underset{C\in\mathscr{C}}{\times} kZ_{u(C)}\text{-Mod}$ so that $\mathcal{V}M \in b_k(kG)$.

We have $\dim {}^d\mathcal{V}M^f = \dim M(\Omega(fd^{-1})) = m(fd^{-1}) = \dim {}^dB^f$ and ${}^d\mathcal{V}M^f$ and ${}^dB^f$ have the same bicomodule structure so that ${}^d\mathcal{V}M^f \cong {}^dB^f$ as bicomodules. Therefore B is a Hopf bimodule over kG via this isomorphism, so that $k\Gamma_1$ is a Hopf bimodule over k^G and $k\Gamma$ is a Hopf algebra. □

Remark. Different $kZ_{u(C)}$-module structures on the $M(C)$ yield different Hopf bimodule structures on B and $k\Gamma_1$ and hence different Hopf algebra structures on $k\Gamma$.

Explicit description of the comultiplication: link with [GS]. Given a Cayley graph Γ for a group G with respect to m constant on conjugacy classes, what is the comultiplication explicitly on $k\Gamma_1$?

We know that $k\Gamma_1 = B^*$ for some Hopf bimodule $B = \mathcal{V}M$ over kG where $M = \{M(C)\}_{C\in\mathscr{C}} \in \underset{C\in\mathscr{C}}{\times} kZ_{u(C)}\text{-Mod}$ with $\dim_k M(C) = m(C)$ for each $C \in \mathscr{C}$. We have $\delta_d(k\Gamma_1)\delta_f = ({}^dB^f)^* = ({}^d\mathcal{V}M^f)^*$.

Given a kG-bimodule V, the vector space V^* is also a kG-bimodule: for $\alpha \in V^*$ and $g \in G$, set

$$g \triangleright \alpha : v \mapsto \alpha(vg) \quad \text{and} \quad \alpha \triangleleft g : v \mapsto \alpha(gv).$$

Note that if $\alpha \in ({}^d\mathcal{V}M^f)^*$ then $g \triangleright \alpha \in ({}^{dg^{-1}}\mathcal{V}M^{fg^{-1}})^* = \delta_{dg^{-1}}(k\Gamma_1)\delta_{fg^{-1}}$ and $\alpha \triangleleft g \in ({}^{g^{-1}d}\mathcal{V}M^{g^{-1}f})^* = \delta_{g^{-1}d}(k\Gamma_1)\delta_{g^{-1}f}$.

The kG-bimodule structure on ${}^d\mathcal{V}M^f$ (regular on the right and obtained using the left $kZ_{u(\Omega(fd^{-1}))}$-module structure on $M(\Omega(fd^{-1}))$ on the left) gives a k^G-bicomodule structure on $\delta_d(k\Gamma_1)\delta_f$ as follows:

$$\rho_\ell(\alpha) = \sum_{g\in G} (g \triangleright \alpha) \otimes \delta_g \quad \text{and} \quad \rho_r(\alpha) = \sum_{g\in G} \delta_g \otimes (\alpha \triangleleft g) \quad \text{for } \alpha \in ({}^d\mathcal{V}M^f)^*.$$

Therefore $\Delta(\alpha) = \sum_{g\in G}((g \triangleright \alpha) \otimes \delta_g + \delta_g \otimes (\alpha \triangleleft g))$.

Note that since the right action on $\mathcal{V}M$ is regular, the left action on $k\Gamma_1$ is regular (or trivial as required/defined in [GS]) and the right action on $k\Gamma_1$ satisfies the condition for the kG-bimodule structure on $k\Gamma$ to be allowable.

Conversely, given an allowable kG-bimodule structure on $k\Gamma$, the reverse construction give an object in $\underset{C\in\mathscr{C}}{\times} kZ_{u(C)}\text{-Mod}$.

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