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SIGN OF FOURIER COEFFICIENTS OF MODULAR FORMS OF HALF-INTEGRAL WEIGHT

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Abstract. We establish lower bounds for (i) the numbers of positive and negative terms and (ii) the number of sign changes in the sequence of Fourier coefficients at squarefree integers of a half-integral weight modular Hecke eigenform.

§1. *Introduction.*

1.1. *Results.* Let $\ell \geqslant 4$ be a positive integer. Denote by $\mathfrak{S}_{\ell+1/2}$ the vector space of all cusp forms of weight $\ell + 1/2$ for the congruence subgroup $\Gamma_0(4)$. The Fourier expansion of $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ at ∞ can be written as

$$\mathfrak{f}(z) = \sum_{n=1}^{\infty} \lambda_{\mathfrak{f}}(n) n^{\ell/2-1/4} \mathrm{e}(nz) \quad (z \in \mathscr{H}), \tag{1}$$

where $\mathrm{e}(z) = \mathrm{e}^{2\pi \mathrm{i} z}$ and $\mathscr{H}$ is the Poincaré upper half-plane. For any squarefree integer t, Waldspurger [18] proved the following elegant formula:

$$\lambda_{\mathfrak{f}}(t)^2 = C_{\mathfrak{f}} L(\tfrac{1}{2}, \mathrm{Sh}_t \mathfrak{f}, \chi_t), \tag{2}$$

where $\mathrm{Sh}_t \mathfrak{f}$ is the Shimura lift of $\mathfrak{f}$ associated to t (this is a cusp form of weight 2ℓ and of level two), $\chi_t(n)$ is a real character modulo t (defined in §2) and $C_{\mathfrak{f}}$ is a constant depending on $\mathfrak{f}$ only. In the following, the letter t will always be a squarefree integer and $\sum^{\flat}$ a sum over squarefree integers.

In view of (2), Kohnen [9] posed the following question. In the case where $\lambda_{\mathfrak{f}}(t)$ is a real number, what is its sign? Very recently, Hulse, Kiral, Kuan and Lim made significant progress towards answering this question by proving that $\lambda_{\mathfrak{f}}(t)$ changes sign infinitely often if $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ is an eigenform of all the Hecke operators (see [4, Theorem 1.1]).

In order to describe the order of magnitude of $\lambda_{\mathfrak{f}}(t)$, we choose α to be a non-negative real number such that the inequality

$$\lambda_{\mathfrak{f}}(t) \ll_{\mathfrak{f},\alpha} t^{\alpha} \tag{3}$$

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holds for all squarefree integers t. The implied constant depends on $\mathfrak{f}$ and α only. It is conjectured that one can take

$$\alpha = \varepsilon$$

for any $\varepsilon > 0$. This could be regarded as an analogue of the Ramanujan conjecture on cusp forms of integral weight. Conrey and Iwaniec [3, Corollary 1.3] proved that one can take

$$\alpha = \tfrac{1}{6} + \varepsilon$$

for any $\varepsilon > 0$.

The main aim of this paper is to establish a quantitative version of the result of Hulse, Kiral, Kuan and Lim [4]. Define

$$\mathcal{T}_{\mathfrak{f}}^{+}(x) = \#\{t \leqslant x, t \text{ squarefree} \colon \lambda_{\mathfrak{f}}(t) > 0\}$$

and

$$\mathcal{T}_{\mathfrak{f}}^{-}(x) = \#\{t \leqslant x, t \text{ squarefree} \colon \lambda_{\mathfrak{f}}(t) < 0\}.$$

We establish the following results.

THEOREM 1. *Let $\ell \geqslant 4$ be a positive integer and let $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ be an eigenform of all the Hecke operators such that the $\lambda_{\mathfrak{f}}(n)$ are real for all $n \geqslant 1$. Then, for any $\varepsilon > 0$, we have*

$$\mathcal{T}_{\mathfrak{f}}^{+}(x) \geqslant x^{1-2\alpha-\varepsilon}, \qquad \mathcal{T}_{\mathfrak{f}}^{-}(x) \geqslant x^{1-2\alpha-\varepsilon}$$

for all $x \geqslant x_0(\mathfrak{f}, \varepsilon)$, where α is given by (3) and $x_0(\mathfrak{f}, \varepsilon)$ is a positive real number depending only on $\mathfrak{f}$ and ε.

Remark 2. In particular, the Conrey and Iwaniec bound leads to

$$\mathcal{T}_{\mathfrak{f}}^{+}(x) \geqslant x^{2/3-\varepsilon}, \qquad \mathcal{T}_{\mathfrak{f}}^{-}(x) \geqslant x^{2/3-\varepsilon}$$

for all $x \geqslant x_0(\mathfrak{f}, \varepsilon)$.

Remark 3. The sign equidistribution of the sequence $(\lambda_{\mathfrak{f}}(tn^2))_{n\in\mathbb{N}}$ was investigated in [2, 5, 6, 9, 10]. In particular, Inam and Wiese proved in [5] that, if t is a *fixed* squarefree integer, then

$$\lim_{x\to+\infty} \frac{\#\{p \text{ prime} \colon p \leqslant x,\ \lambda_{\mathfrak{f}}(tp^2) > 0\}}{\#\{p \text{ prime} \colon p \leqslant x\}} = \frac{1}{2}$$

and

$$\lim_{x\to+\infty} \frac{\#\{p \text{ prime} \colon p \leqslant x,\ \lambda_{\mathfrak{f}}(tp^2) < 0\}}{\#\{p \text{ prime} \colon p \leqslant x\}} = \frac{1}{2}.$$

Let us state precisely what we call the *number of squarefree sign changes* of the sequence $\lambda_{\mathfrak{f}} = (\lambda_{\mathfrak{f}}(t))_{t\geqslant 0}$ (where $\lambda_{\mathfrak{f}}(0) = 0$). This sequence is restricted to squarefree indices t. From this sequence of Fourier coefficients, we build a sequence of pairs of squarefree integers (t_n^+, t_n^-) that may be finite, or even void, in the following way: for any integer n, we have

$$\lambda_{\mathfrak{f}}(t_n^+) > 0, \qquad \lambda_{\mathfrak{f}}(t_n^-) < 0,$$

and

$$\max(t_n^+, t_n^-) < \min(t_{n+1}^+, t_{n+1}^-),$$

and $\lambda_{\mathfrak{f}}(t) = 0$ for all squarefree integers t between t_n^+ and t_n^-. The number of squarefree sign changes of $\lambda_{\mathfrak{f}}$ is the function defined by

$$\mathcal{C}_{\mathfrak{f}}(x) = \#\{n \geqslant 1\colon\ \max(t_n^+, t_n^-) \leqslant x\}.$$

THEOREM 4. *Let $\ell \geqslant 4$ be a positive integer and let $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ be an eigenform of all the Hecke operators such that the $\lambda_{\mathfrak{f}}(n)$ are real for all $n \geqslant 1$. For any $\varepsilon > 0$, the number of squarefree sign changes of $\lambda_{\mathfrak{f}}$ satisfies*

$$\mathcal{C}_{\mathfrak{f}}(x) \gg_{\mathfrak{f},\varepsilon} x^{(1-4\alpha)/5-\varepsilon}$$

for all $x \geqslant x_0(\mathfrak{f}, \varepsilon)$, where the constant $x_0(\mathfrak{f}, \varepsilon)$ and the implied constant depends on $\mathfrak{f}$ and ε.

Remark 5. In particular, the Conrey and Iwaniec bound leads to

$$\mathcal{C}_{\mathfrak{f}}(x) \gg_{\mathfrak{f},\varepsilon} x^{1/15-\varepsilon}$$

for all $x \geqslant x_0(\mathfrak{f}, \varepsilon)$.

1.2. *Methods.* To prove Theorem 1, we detect signs with

$$\frac{|\lambda_{\mathfrak{f}}(t)| + \lambda_{\mathfrak{f}}(t)}{2} = \begin{cases} \lambda_{\mathfrak{f}}(t) & \text{if } \lambda_{\mathfrak{f}}(t) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Bounding the Fourier coefficients with (3), we plainly obtain

$$\sum_{t\leqslant x}^{\flat} (|\lambda_{\mathfrak{f}}(t)| + \lambda_{\mathfrak{f}}(t)) \log\left(\frac{x}{t}\right) \ll_{\mathfrak{f},\alpha} \mathcal{T}_{\mathfrak{f}}^{+}(x) x^{\alpha} \log x$$

(recall that the letter t is for squarefree integers and hence the sum is restricted to squarefree integers). Then we use the analytic properties of the Dirichlet series

$$M(\mathfrak{f}, s) = \sum_{t\leqslant x}^{\flat} \lambda_{\mathfrak{f}}(t) t^{-s} \qquad \text{and} \qquad D(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s) = \sum_{n\geqslant 1} \lambda_{\mathfrak{f}}(n)^2 n^{-s}$$

in Lemma 8 and Proposition 7 of §2.2 to make an auxiliary tool – Lemma 9. (Note that Lemma 8 is due to [4].) More precisely, we utilize the fact that the Dirichlet series defining $M(\mathfrak{f}, s)$ and $D(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s)$ are absolutely convergent for $\operatorname{Re} s > 1$. The function $M(\mathfrak{f}, s)$ has an analytic continuation to $\operatorname{Re} s > 3/4$, whereas the function $D(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s)$ has a meromorphic continuation to $\operatorname{Re} s > 1/2$ with a unique pole; this pole is at 1 and it is simple. Thus we can easily derive Lemma 9 and then the lower bound

$$\sum_{t \leqslant x}^{\flat} (|\lambda_{\mathfrak{f}}(t)| + \lambda_{\mathfrak{f}}(t)) \log\left(\frac{x}{t}\right) \gg x^{1-\alpha}.$$

Theorem 1 follows readily.

Theorem 4 rests on the following delicate device of Soundararajan [16]: let $c > 0$ and $\delta > 0$, then

$$\begin{aligned} &\frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(\mathrm{e}^{\delta s} - 1)^2}{s^2} \xi^s \, ds \\ &\quad = \begin{cases} \min(\log(\mathrm{e}^{2\delta}\xi), \log(1/\xi)) & \text{if } \mathrm{e}^{-2\delta} \leqslant \xi \leqslant 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4}$$

(Thanks to the referee for suggesting this device.) Using it with the analytic properties of $M(\mathfrak{f}, s)$ and $D(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s)$, some weighted first and second moments on short intervals are evaluated. We use these moments to detect the sign changes via the positivity of

$$\sum_{m \leqslant A} \sum_{x/m^2 < t < (x+h)/m^2}^{\flat} (|\lambda_{\mathfrak{f}}(t)| + \varepsilon_m \lambda_{\mathfrak{f}}(t)) \min\left(\log\left(\frac{x+h}{tm^2}\right), \log\left(\frac{tm^2}{x}\right)\right)$$

for all $(\varepsilon_1, \ldots, \varepsilon_A) \in \{-1, 1\}^A$.

The paper is organized as follows. Section 2 is devoted to the background on half-integral weight modular forms (§2.1) and the establishment of the analytic properties for the Dirichlet series that we need (§2.2). Theorem 1 is proved in §3. Theorem 4 is proved in §4.

§2. *Background.*

2.1. *Modular forms of half-integral weight.* In this section, we want to recall the basic facts that we need on modular forms of half-integral weight on the congruence subgroup $\Gamma_0(4)$. All the content of this section is classical and is to be found in the main references [14] and [15]. However, it contains very few that the non-specialist reader will need.

The theta function is defined on the upper half-plane $\mathscr{H}$ by

$$\theta(z) = 1 + 2 \sum_{n=1}^{+\infty} \mathrm{e}(n^2 z)$$

for any $z \in \mathscr{H}$. Since the θ function does not vanish on $\mathscr{H}$, we can define the theta multiplier: for any $\gamma \in \Gamma_0(4)$ and $z \in \mathscr{H}$, let

$$j(\gamma, z) = \frac{\theta(\gamma z)}{\theta(z)}.$$

If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, it can be shown that $j(\gamma, z)^2 = cz + d$. For any complex number ξ, let $\xi^{1/2}$ denote $|\xi|^{1/2}\mathrm{e}^{\mathrm{i}\arg(\xi)/2}$, where $-\pi < \arg(\xi) \leqslant \pi$. The coefficient $j(\gamma, z)/(cz + d)^{1/2}$ is called the theta multiplier. It does not depend on z and can be explicitly described in terms of c and d (see, for example, [**7**, §2.8]).

Let ℓ be a non-negative integer. A modular form of weight $\ell + 1/2$ is a holomorphic function $\mathfrak{f}$ on $\mathscr{H}$ satisfying

$$\mathfrak{f}(\gamma z) = j(\gamma, z)^{2\ell+1}\mathfrak{f}(z)$$

for all $\gamma \in \Gamma_0(4)$ and $z \in \mathscr{H}$, and it is holomorphic at the cusps of $\Gamma_0(4)$. If, moreover, $\mathfrak{f}$ vanishes at the cusps of $\Gamma_0(4)$, then $\mathfrak{f}$ is called a cusp form of weight $\ell + 1/2$. The congruence subgroup has three cusps: 0, $-1/2$ and ∞. The corresponding scaling matrices are, respectively,

$$\sigma_0 = \begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix}, \qquad \sigma_{-1/2} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \qquad \text{and} \qquad \sigma_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, if $\mathfrak{f}$ is a cusp form of weight $\ell + 1/2$, the following functions have a Fourier expansion vanishing at ∞:

$$\mathfrak{f}|_{\sigma_0}(z) = (2z)^{-\ell-1/2}\mathfrak{f}\left(-\frac{1}{4z}\right)$$

and

$$\mathfrak{f}|_{\sigma_{-1/2}}(z) = (-2z+1)^{-\ell-1/2}\mathfrak{f}\left(-\frac{1}{2z-1}\right).$$

We shall write

$$\mathfrak{f}(z) = \sum_{n=1}^{+\infty}\widehat{\mathfrak{f}}(n)\mathrm{e}(nz) \tag{5}$$

for the Fourier expansion of $\mathfrak{f}$. The set $\mathfrak{S}_{\ell+1/2}$ of modular forms of weight $\ell+1/2$ is a finite-dimensional vector space over $\mathbb{C}$. If $\ell \leqslant 3$, then $\mathfrak{S}_{\ell+1/2} = \{0\}$. In the following, we shall assume that $\ell \geqslant 4$.

Shimura established a correspondence between half-integral cusp forms and integral weight cusp forms on a congruence subgroup. Niwa [**13**] gave a more direct proof of this correspondence and lowered the level of the congruence group involved. Fix a squarefree integer t. We write χ_0 for the principal character of modulus two and define a character χ_t by

$$\chi_t(n) = \chi_0(n)\left(\frac{-1}{n}\right)^{\ell}\left(\frac{t}{n}\right).$$

Let $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$. Then, the Dirichlet series defined by the product

$$L(\chi_t, s-\ell+1)\sum_{n=1}^{+\infty}\frac{\widehat{\mathfrak{f}}(tn^2)}{n^s}$$

is the Dirichlet series of a cusp form of integral weight 2ℓ over the congruence subgroup $\Gamma_0(2)$. We denote this cusp form by $\mathrm{Sh}_t\mathfrak{f}$ and the vector space of cusp forms of weight 2ℓ over $\Gamma_0(2)$ by $S_{2\ell}$. At this point, the dependence in t of $\mathrm{Sh}_t\mathfrak{f}$ is not really clear. It will become clearer after we introduce the Hecke operators.

The Hecke operator of half-integral weight $\ell + 1/2$ and order p^2 is the linear endomorphism $\mathfrak{T}_{p^2}$ on $\mathfrak{S}_{\ell+1/2}$ that sends any cusp form with Fourier coefficients $(\widehat{\mathfrak{f}}(n))_{n\geqslant 1}$ to the cusp form with Fourier coefficients defined by

$$\widehat{\mathfrak{T}_{p^2}(\mathfrak{f})}(n) = \widehat{\mathfrak{f}}(p^2 n) + \chi_0(p)\left(\frac{(-1)^\ell n}{p}\right)p^{\ell-1}\widehat{f}(n) + \chi_0(p)p^{2\ell-1}\widehat{\mathfrak{f}}\left(\frac{n}{p^2}\right).$$

If n/p^2 is not an integer, then $\widehat{\mathfrak{f}}(n/p^2)$ is considered to be zero. Hecke operators and the Shimura correspondence commute, meaning that if T_p is the Hecke operator of order p over $S_{2\ell}$, then

$$\mathrm{Sh}_t(\mathfrak{T}_{p^2}\mathfrak{f}) = T_p(\mathrm{Sh}_t\mathfrak{f})$$

for any $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$. In particular, if $\mathfrak{f}$ is an eigenform of $\mathfrak{T}_{p^2}$, then $\mathrm{Sh}_t\mathfrak{f}$ is an eigenform of T_p with the same eigenvalue. Let $\mathfrak{f}$ be an eigenform of all the Hecke operators $\mathfrak{T}_{p^2}$; denote by w_p the corresponding eigenvalue. One has

$$L(\chi_t, s-\ell+1)\sum_{n=1}^{+\infty}\frac{\widehat{\mathfrak{f}}(tn^2)}{n^s} = \widehat{\mathfrak{f}}(t)\prod_p\left(1 - \frac{\omega_p}{p^s} + \frac{\chi_0(p)}{p^{2s-2\ell+1}}\right)^{-1}, \qquad (6)$$

the product being over all prime numbers. This product on primes is the L-function of a cusp form in $S_{2\ell}$. We denote this cusp form by $\mathrm{Sh}\,\mathfrak{f}$. Note that it does not depend on t and that $\mathrm{Sh}_t\mathfrak{f} = \widehat{\mathfrak{f}}(t)\,\mathrm{Sh}\,\mathfrak{f}$.

Let ψ be the arithmetic function defined by

$$\psi(n) = \prod_{p|n}(1 + p^{-1/2}),$$

the product being on prime numbers. We write τ for the divisor function and, clearly, $\psi(n) \leqslant \tau(n)$ for every $n \in \mathbb{N}^*$. The next Lemma slightly improves Lemma 4.1 in [4].

LEMMA 6. *Let $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ be an eigenform of all the Hecke operators $\mathfrak{T}_{p^2}$. There exists a constant $C > 0$ such that, for any squarefree integer t and any integer n,*

$$|\widehat{\mathfrak{f}}(tn^2)| \leqslant C|\widehat{\mathfrak{f}}(t)|n^{\ell-1/2}\tau(n)\psi(n).$$

Proof. From (6), we get

$$\widehat{\mathfrak{f}}(tn^2) = \widehat{\mathfrak{f}}(t) \sum_{d|n} \chi_t\left(\frac{n}{d}\right)\mu\left(\frac{n}{d}\right)\left(\frac{n}{d}\right)^{\ell-1} \widehat{\mathrm{Sh}\,\mathfrak{f}}(d). \tag{7}$$

By the Deligne estimate, there exists $C > 0$ such that

$$|\widehat{\mathrm{Sh}\,\mathfrak{f}}(d)| \leqslant Cd^{(2\ell-1)/2}\tau(d) \tag{8}$$

for any d. It follows, from (7) and (8), that

$$|\widehat{\mathfrak{f}}(tn^2)| \leqslant C|\widehat{\mathfrak{f}}(t)|n^{\ell-1} \sum_{d|n} \left|\mu\left(\frac{n}{d}\right)\right| d^{1/2}\tau(d) \leqslant C|\widehat{\mathfrak{f}}(t)|n^{\ell-1/2}\tau(n)\psi(n). \quad \square$$

The size of the Fourier coefficients of a half-integral weight modular form is therefore controlled by the size of its Fourier coefficients at squarefree integers. Deligne's bound for integral weight modular forms does not apply, although it conjecturally does. Let α be a positive real number such that, if $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$, then

$$|\widehat{\mathfrak{f}}(t)| \leqslant Ct^{(\ell+1/2-1)/2+\alpha}$$

for any squarefree integer t (and C is a real number depending only on $\mathfrak{f}$ and α). The Ramanujan–Petersson conjecture asserts that α can be taken arbitrarily small. The best proven result is due to Conrey and Iwaniec [**3**] (see also the appendix by Mao in [**1**] for a uniform value of C). Their result implies that we can take $\alpha = 1/6 + \varepsilon$ with any real positive ε. If $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ is an eigenform of all the Hecke operators, then, by comparison of (1) and (5),

$$\lambda_{\mathfrak{f}}(n) = \frac{\widehat{\mathfrak{f}}(n)}{n^{(\ell+1/2-1)/2}}.$$

For any squarefree integer t and integer n, we have

$$|\lambda_{\mathfrak{f}}(tn^2)| \leqslant C_1|\lambda_{\mathfrak{f}}(t)|\tau(n)\psi(n) \leqslant C_2 t^{\alpha}\tau(n)\psi(n) \tag{9}$$

with the admissible choice $\alpha = 1/6 + \varepsilon$, where C_1 and C_2 are positive real numbers that do not depend on t or n.

2.2. *Some associated Dirichlet series.* Let $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$, and assume that it is an eigenform of all the Hecke operators. We define

$$D(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s) = \sum_{n=1}^{+\infty} \lambda_{\mathfrak{f}}(n)^2 n^{-s}. \tag{10}$$

Write $\sigma = \operatorname{Re} s$ and $\tau = \operatorname{Im} s$†. According to (9), we know that it is absolutely convergent as soon as $\sigma > 1 + 2\alpha$. We state analytical information on this function. The proof is quite standard but, since we have not found a handy proof in the literature for this case, we provide the details for completeness.

PROPOSITION 7. *Let $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$, and assume that it is an eigenform of all the Hecke operators. The Dirichlet series (10) converges absolutely as soon as* $\operatorname{Re} s > 1$. *It can be continued analytically to a meromorphic function in the*

† No confusion will arise with the divisor function $\tau(n)$ from the context.

half-plane $\operatorname{Re} s > \frac{1}{2}$ *with the only pole at* $s = 1$*. This pole is simple. Further, for any* $\varepsilon > 0$*, we have*

$$D(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s) \ll_{\mathfrak{f},\varepsilon} |\tau|^{2\max(1-\sigma,0)+\varepsilon} \quad (\tfrac{1}{2} + \varepsilon \leqslant \sigma \leqslant 3, |\tau| \geqslant 1).$$

The implied constant depends on $\mathfrak{f}$ *and* ε *only.*

Proof. Let $\mathfrak{a}$ be a cusp of $\Gamma = \Gamma_0(4)$. We denote its stability group by $\Gamma_{\mathfrak{a}}$ and its scaling matrix by $\sigma_{\mathfrak{a}}$ (see [7, §2.3]). The Eisenstein series associated to $\mathfrak{a}$ is

$$\begin{aligned} E_{\mathfrak{a}}(z, s) &= \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma z)^s \\ &= \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} \operatorname{Im}(\gamma \sigma_{\mathfrak{a}}^{-1} z)^s = E_{\infty}(\sigma_{\mathfrak{a}}^{-1} z, s). \end{aligned}$$

We take $\{0, -1/2, \infty\}$ as a representative set of cusps and obtain

$$E_0(z, s) = E_{\infty}\left(-\frac{1}{4z}, s\right) \qquad \text{and} \qquad E_{-1/2}(z, s) = E_{\infty}\left(-\frac{z}{2z-1}, s\right).$$

These series converge absolutely for $\operatorname{Re} s > 1$ (see, for example, [12, Theorem 2.1.1]). Moreover, $\mathfrak{f}_{|\sigma_{\mathfrak{a}}}$ admits a Fourier expansion

$$\mathfrak{f}_{|\sigma_{\mathfrak{a}}}(z) = \sum_{n=1}^{+\infty} n^{(\ell+1/2-1)/2} \lambda_{\mathfrak{f},\mathfrak{a}}(n) \mathrm{e}((n - r_{\mathfrak{a}})z),$$

where $r_{\mathfrak{a}} = 0$ if $\mathfrak{a} \in \{0, \infty\}$ and $r_{\mathfrak{a}} = \frac{1}{4}(2 + (-1)^{\ell})$ if $\mathfrak{a} = -\frac{1}{2}$. Note that $\mathfrak{f}_{|\sigma_{\infty}} = \mathfrak{f}$. Let

$$D(\mathfrak{f}_{\mathfrak{a}} \otimes \overline{\mathfrak{f}_{\mathfrak{a}}}, s) = \sum_{n=1}^{+\infty} |\lambda_{\mathfrak{f},\mathfrak{a}}(n)|^2 \left(\frac{n}{n - r_{\mathfrak{a}}}\right)^{\ell-1/2} (n - r_{\mathfrak{a}})^{-s}. \tag{11}$$

Classically (see, for example, [7, §13.2]), we have

$$(4\pi)^{-(s+\ell-1/2)} \Gamma\left(s + \ell - \frac{1}{2}\right) D(\mathfrak{f}_{\mathfrak{a}} \otimes \overline{\mathfrak{f}_{\mathfrak{a}}}, s) = \int_{\Gamma/\mathscr{H}} y^{\ell+1/2} |\mathfrak{f}(z)|^2 E_{\mathfrak{a}}(z, s) \frac{dx\,dy}{y^2}$$

for $\operatorname{Re} s$ large enough. The right-hand side provides an analytic continuation in the region $\operatorname{Re} s > 1$. By the Landau lemma, this implies that the Dirichlet series (11) is absolutely convergent for $\operatorname{Re} s > 1$. The general theory implies that $s \mapsto E_{\mathfrak{a}}(z, s)$ has a meromorphic continuation to the whole complex plane and satisfies the functional equation

$$\vec{E}(z, s) = \Phi(s) \vec{E}(z, 1 - s),$$

where $\vec{E}$ is the transpose of $(E_{\infty}, E_0, E_{-1/2})$ and $\Phi = (\varphi_{\mathfrak{a},\mathfrak{b}})_{(\mathfrak{a},\mathfrak{b}) \in \{\infty, 0, -1/2\}^2}$ is the scattering matrix. Indeed,

$$\varphi_{\mathfrak{a},\mathfrak{b}}(s) = \pi^{1/2} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \sum_{c>0} \mathcal{N}(c) c^{-2s},$$

where $\mathcal{N}(c)$ is the number of d, incongruent modulo c, such that there exist a and b satisfying

$$\sigma_{\mathfrak{a}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\sigma_{\mathfrak{b}}^{-1} \in \Gamma_0(4).$$

This leads to

$$\begin{aligned}\Phi(s) &= \frac{\Lambda(2s-1)}{\Lambda(2s)}\frac{2^{1-2s}}{2^{2s}-1}\begin{pmatrix} 1 & 2^{2s-1}-1 & 2^{2s-1}-1 \\ 2^{2s-1}-1 & 1 & 2^{2s-1}-1 \\ 2^{2s-1}-1 & 2^{2s-1}-1 & 1 \end{pmatrix} \\ &= \frac{\Lambda(2s-1)}{\Lambda(2s)}\Psi(s), \text{ say,}\end{aligned}$$

where $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. On the half-plane $\operatorname{Re} s \geqslant 1/2$, $E_{\mathfrak{a}}$ and $\varphi_{\mathfrak{a},\mathfrak{a}}$ have the same poles of the same orders [**12**, Theorems 4.4.2, 4.3.4, 4.3.5]. The only pole on $\operatorname{Re} s \geqslant 1/2$ is then $s = 1$ and it is simple. Note that this follows also from the general theory since we are working on a congruence subgroup [**8**, Theorem 11.3].

Let $\vec{L}(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s)$ be the transpose of

$$(D(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s), D(\mathfrak{f}_0 \otimes \overline{\mathfrak{f}_0}, s), D(\mathfrak{f}_{-1/2} \otimes \overline{\mathfrak{f}_{-1/2}}, s))$$

and

$$\vec{\Lambda}(\mathfrak{f}, s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s+\ell-1/2)\zeta(2s)\vec{L}(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s).$$

We proved that

- $\vec{\Lambda}(\mathfrak{f}, s) = \Psi(s)\vec{\Lambda}(\mathfrak{f}, 1-s)$ and
- in the half-plane $\operatorname{Re} s \geqslant 1/2$, the function $D(\mathfrak{f}_{\mathfrak{a}} \otimes \overline{\mathfrak{f}_{\mathfrak{a}}}, s)$ has only a simple pole at $s = 1$.

Now let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^3$. Using $\|D(\mathfrak{f}_{\mathfrak{a}} \otimes \overline{\mathfrak{f}_{\mathfrak{a}}}, 1+\varepsilon+i\tau)\| \ll_{\mathfrak{f},\varepsilon} 1$ for any $\tau \in \mathbb{R}$ and any fixed $\varepsilon > 0$, we deduce that

$$|\zeta(-2\varepsilon+2i\tau)| \cdot \|\vec{L}(\mathfrak{f} \otimes \bar{\mathfrak{f}}, -\varepsilon+i\tau)\| \ll_{\mathfrak{f},\varepsilon} (1+|\tau|)^{2+\varepsilon},$$

from the functional equation, and obtain the estimate

$$|\zeta(2s)| \cdot \|\vec{L}(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s)\| \ll_{\mathfrak{f},\varepsilon} (1+|\tau|)^{2(1-\sigma)+\varepsilon} \quad (s = \sigma+i\tau, \sigma \in [0,1], |\tau| \geqslant 1),$$

by the standard argument with the convexity principle‡. This leads to the desired result. □

Remark. Kohnen and Zagier [**11**, pp. 189–191] proved the functional equation for $D(\mathfrak{f} \otimes \mathfrak{f}, s)$ when $\mathfrak{f}$ is in the plus case. The authors thank Professor Kohnen for pointing this out.

‡ One needs the estimate $|\zeta(2s)| \cdot \|\vec{L}(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s)\| \ll e^{e^{\eta|\tau|}}$ for some $\eta > 0$ in the strip so as to apply the convexity principle. This can be easily verified by the Fourier expansion of $E_{\mathfrak{a}}(z, s)$ and [**12**, (2.2.6)–(2.2.11)].

Another useful Dirichlet series is

$$M(\mathfrak{f}, s) = \sum_{t \geqslant 1}^{\flat} \lambda_{\mathfrak{f}}(t) t^{-s}. \tag{12}$$

The series $M(\mathfrak{f}, s)$ is absolutely convergent for $\operatorname{Re} s > 1$, by the Cauchy–Schwarz inequality and Proposition 7. The next lemma is due to Hulse, Kiral, Kuan and Lim [**4**, Proposition 4.4].

LEMMA 8. *Let $\ell \geqslant 4$ be a positive integer and let $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ be an eigenform of all the Hecke operators. The series $M(\mathfrak{f}, s)$, given by (12), converges for* $\operatorname{Re} s > \frac{3}{4}$. *Further, for any $\varepsilon > 0$, we have*

$$M(\mathfrak{f}, \sigma + \mathrm{i}\tau) \ll_{\mathfrak{f},\varepsilon} (|\tau| + 1)^{\max(1-\sigma,0)+2\varepsilon} \quad (\tfrac{3}{4} + \varepsilon \leqslant \sigma \leqslant 3, |\tau| \geqslant 1),$$

where the implied constant depends on $\mathfrak{f}$ and ε only.

Proof. We only sketch the proof since it is nearly the same as in [**4**, Proposition 4.4]. By the relation

$$\mu(m)^2 = \sum_{r^2 | m} \mu(r),$$

we have

$$M(\mathfrak{f}, s) = \sum_{r=1}^{+\infty} \mu(r) D_r(s), \tag{13}$$

where

$$D_r(s) = \sum_{\substack{m=1 \\ m \equiv 0 \ (\mathrm{mod}\ r^2)}}^{+\infty} \lambda_{\mathfrak{f}}(m) m^{-s}.$$

This series is absolutely convergent for $\operatorname{Re} s > 1$, by the Cauchy–Schwarz inequality and Proposition 7. Then, by introducing additive characters to remove the congruence condition and applying the Mellin transform, we obtain

$$D_r(s) = \frac{(2\pi)^{s+(\ell+1/2-1)/2}}{\Gamma(s + (\ell + 1/2 - 1)/2)} \cdot \frac{1}{r^2} \sum_{d | r^2} \sum_{\substack{u \ (\mathrm{mod}\ \mathrm{d}) \\ (u,d)=1}} \Lambda\left(\mathfrak{f}, \frac{u}{d}, s\right)$$

with

$$\Lambda(\mathfrak{f}, q, s) = \int_0^{+\infty} \mathfrak{f}(\mathrm{i}y + q) y^{s+(\ell-1/2)/2} \frac{dy}{y}$$

for any rational number q. Using the functional equation for $\Lambda(\mathfrak{f}, q, s)$ (see [**4**, Lemma 4.3]), we obtain

$$D_r(-\varepsilon + \mathrm{i}\tau) \ll_{\varepsilon,\mathfrak{f}} (1 + |\tau|)^{1+2\varepsilon} r^{2+5\varepsilon}.$$

From (9), also

$$D_r(1 + \varepsilon + \mathrm{i}\tau) \ll_{\varepsilon,\mathfrak{f}} \frac{1}{r^2}.$$

Finally, by the Phrägmen–Lindelöf principle, we deduce that

$$D_r(\sigma + \mathrm{i}\tau) \ll_{\varepsilon,\mathfrak{f}} (1+|\tau|)^{1-\sigma+\varepsilon} r^{2-4\sigma+\varepsilon}.$$

Reinserting this bound into (13) leads to the result. □

§3. *Proof of Theorem 1.* We begin by establishing mean value results for the Fourier coefficients at squarefree integers.

LEMMA 9. *Let* $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ *and assume that it is an eigenform of all the Hecke operators. Let* $\varepsilon > 0$. *There exist positive real numbers* C_1, C_2 *and* C_3 *such that, for any* $x \geqslant 1$,

$$\sum_{t\leqslant x}^{\flat} \lambda_{\mathfrak{f}}(t) \log\left(\frac{x}{t}\right) \leqslant C_1 x^{3/4+\varepsilon}$$

and

$$C_2 x \leqslant \sum_{t\leqslant x}^{\flat} \lambda_{\mathfrak{f}}(t)^2 \leqslant C_3 x$$

for any $x \geqslant x_0(\mathfrak{f})$.

Proof. Using the Perron formula [17, Theorem II.2.3], we write

$$\sum_{t\leqslant x}^{\flat} \lambda_{\mathfrak{f}}(t) \log\left(\frac{x}{t}\right) = \frac{1}{2\pi\mathrm{i}} \int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} M(\mathfrak{f}, s) x^s \frac{ds}{s^2}.$$

We move the line of integration to $\operatorname{Re} s = 3/4 + \varepsilon$ and use Lemma 8 to obtain

$$\sum_{t\leqslant x}^{\flat} \lambda_{\mathfrak{f}}(t) \log\left(\frac{x}{t}\right) \leqslant C_1 x^{3/4+\varepsilon}.$$

For the second formula, we use an effective version of the Perron formula [17, Corollary II.2.2.1]:

$$\sum_{n\leqslant x} \lambda_{\mathfrak{f}}(n)^2 = \frac{1}{2\pi\mathrm{i}} \int_{\kappa-\mathrm{i}T}^{\kappa+\mathrm{i}T} D(\mathfrak{f}\otimes\bar{\mathfrak{f}}, s) x^s \frac{ds}{s} + O\left(\frac{x^{1+2\alpha+\varepsilon}}{T}\right)$$

for any $T \leqslant x$ and $\kappa = 1 + 1/\log x$. Proposition 7 allows us to shift the line of integration to $\operatorname{Re} s = 1/2 + \varepsilon$. We get

$$\frac{1}{2\pi\mathrm{i}} \int_{\kappa-\mathrm{i}T}^{\kappa+\mathrm{i}T} D(\mathfrak{f}\otimes\bar{\mathfrak{f}}, s) x^s \frac{ds}{s} = r_{\mathfrak{f}} x + \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{L}} D(\mathfrak{f}\otimes\bar{\mathfrak{f}}, s) x^s \frac{ds}{s},$$

where $r_{\mathfrak{f}}$ is the residue at $s = 1$ of $D(\mathfrak{f}\otimes\bar{\mathfrak{f}}, s)$ and $\mathcal{L}$ is the contour made from segments that join in order the points $\kappa - \mathrm{i}T$, $1/2 + \varepsilon - \mathrm{i}T$, $1/2 + \varepsilon + \mathrm{i}T$ and $\kappa + \mathrm{i}T$. With the convexity bound in Proposition 7,

$$\int_{1/2+\varepsilon\pm\mathrm{i}T}^{\kappa\pm\mathrm{i}T} D(\mathfrak{f}\otimes\bar{\mathfrak{f}}, s) x^s \frac{ds}{s} \ll \frac{x^{1+\varepsilon}}{T}$$

if $T \leqslant x^{1/2}$ and

$$\int_{1/2+\varepsilon-\mathrm{i}T}^{1/2+\varepsilon+\mathrm{i}T} D(\mathfrak{f}\otimes\bar{\mathfrak{f}}, s)x^s\frac{ds}{s} \ll x^{1/2+\varepsilon}T.$$

We choose $T = x^{1/4+\alpha}$ and obtain

$$\sum_{n\leqslant x}\lambda_{\mathfrak{f}}(n)^2 = r_{\mathfrak{f}}x + O(x^{3/4+\alpha+\varepsilon}). \tag{14}$$

Each positive integer n may be decomposed uniquely as $n = tm^2$ with squarefree t. Using (9),

$$\begin{aligned}\sum_{n\leqslant x}\lambda_{\mathfrak{f}}(n)^2 &\ll_{\mathfrak{f}} \sideset{}{^\flat}\sum_{t\leqslant x}\lambda_{\mathfrak{f}}(t)^2\sum_{m\leqslant(x/t)^{1/2}}\tau(m)\psi(m)\\ &\ll_{\mathfrak{f}} x^{1/2}\sideset{}{^\flat}\sum_{t\leqslant x}\frac{\lambda_{\mathfrak{f}}(t)^2}{t^{1/2}}\log\left(\frac{x}{t}\right).\end{aligned}$$

Combining this with (14), we find that

$$\sideset{}{^\flat}\sum_{t\leqslant x}\frac{\lambda_{\mathfrak{f}}(t)^2}{t^{1/2}}\log\left(\frac{x}{t}\right) \geqslant c_1x^{1/2} \quad (x \geqslant x_0(\mathfrak{f})), \tag{15}$$

where the constant c_1 depends only on $\mathfrak{f}$. On the other hand, (14) leads to

$$\sideset{}{^\flat}\sum_{t\leqslant x}\frac{\lambda_{\mathfrak{f}}(t)^2}{t^{1/2}}\log\left(\frac{x}{t}\right) \leqslant \sum_{n\leqslant x}\frac{\lambda_{\mathfrak{f}}(n)^2}{n^{1/2}}\log\left(\frac{x}{n}\right) \leqslant c_2x^{1/2}, \tag{16}$$

where c_2 depends only on $\mathfrak{f}$. Let $c_3 \in]0, 1[$. From (15) and (16), it follows that

$$\begin{aligned}\frac{\log(1/c_3)}{(c_3x)^{1/2}}\sideset{}{^\flat}\sum_{c_3x<t\leqslant x}\lambda_{\mathfrak{f}}(t)^2 &\geqslant \sideset{}{^\flat}\sum_{c_3x<t\leqslant x}\frac{\lambda_{\mathfrak{f}}(t)^2}{t^{1/2}}\log\left(\frac{x}{t}\right)\\ &= \sideset{}{^\flat}\sum_{t\leqslant x}\frac{\lambda_{\mathfrak{f}}(t)^2}{t^{1/2}}\log\left(\frac{x}{t}\right) - \sideset{}{^\flat}\sum_{t\leqslant c_3x}\frac{\lambda_{\mathfrak{f}}(t)^2}{t^{1/2}}\log\left(\frac{x}{t}\right)\\ &\geqslant (c_1 - c_2c_3^{1/2})x^{1/2}.\end{aligned}$$

We deduce that

$$\sideset{}{^\flat}\sum_{c_3x<t\leqslant x}\lambda_{\mathfrak{f}}(t)^2 \geqslant \frac{c_3^{1/2}}{\log(1/c_3)}(c_1 - c_2c_3^{1/2})x.$$

Choosing $c_3 < \min(1, c_1^2/c_2^2)$, we have

$$\sideset{}{^\flat}\sum_{t\leqslant x}\lambda_{\mathfrak{f}}(t)^2 \gg \sideset{}{^\flat}\sum_{c_3x<t\leqslant x}\lambda_{\mathfrak{f}}(t)^2 \gg x.$$

Finally, (14) gives

$$\sum_{t\leqslant x}^{\flat}\lambda_{\mathfrak{f}}(t)^2 \leqslant \sum_{n\leqslant x}\lambda_{\mathfrak{f}}(n)^2 \ll x$$

and hence

$$\sum_{t\leqslant x}^{\flat}\lambda_{\mathfrak{f}}(t)^2 \asymp x. \qquad \square$$

With this lemma, we can complete the proof of Theorem 1. From (9), we derive

$$\begin{aligned}\sum_{t\leqslant x}^{\flat}|\lambda_{\mathfrak{f}}(t)|\log\left(\frac{x}{t}\right) &\gg x^{-\alpha}\sum_{t\leqslant x}^{\flat}|\lambda_{\mathfrak{f}}(t)|^2\log\left(\frac{x}{t}\right)\\ &\gg x^{-\alpha}\sum_{t\leqslant x/2}^{\flat}|\lambda_{\mathfrak{f}}(t)|^2.\end{aligned}$$

Hence, Lemma 9 implies that

$$\sum_{t\leqslant x}^{\flat}|\lambda_{\mathfrak{f}}(t)|\log\left(\frac{x}{t}\right) \gg_{\mathfrak{f},\alpha} x^{1-\alpha}. \tag{17}$$

We detect signs of Fourier coefficients with the help of

$$\frac{|\lambda_{\mathfrak{f}}(t)|+\lambda_{\mathfrak{f}}(t)}{2} = \begin{cases}\lambda_{\mathfrak{f}}(t) & \text{if } \lambda_{\mathfrak{f}}(t)>0,\\ 0 & \text{otherwise.}\end{cases}$$

Using (9),

$$\sum_{t\leqslant x}^{\flat}\big(|\lambda_{\mathfrak{f}}(t)|+\lambda_{\mathfrak{f}}(t)\big)\log\left(\frac{x}{t}\right) \ll \mathcal{T}_{\mathfrak{f}}^{+}(x)x^{\alpha}\log x. \tag{18}$$

Moreover, (17) and Lemma 9 imply that

$$\begin{aligned}\sum_{t\leqslant x}^{\flat}\big(|\lambda_{\mathfrak{f}}(t)|+\lambda_{\mathfrak{f}}(t)\big)\log\left(\frac{x}{t}\right) &= \sum_{t\leqslant x}^{\flat}|\lambda_{\mathfrak{f}}(t)|\log\left(\frac{x}{t}\right)+\sum_{t\leqslant x}^{\flat}\lambda_{\mathfrak{f}}(t)\log\left(\frac{x}{t}\right)\\ &\gg x^{1-\alpha}+O(x^{3/4+\varepsilon})\\ &\gg x^{1-\alpha}.\end{aligned} \tag{19}$$

Finally, equations (18) and (19) give

$$\mathcal{T}_{\mathfrak{f}}^{+}(x) \gg \frac{x^{1-2\alpha}}{\log x}.$$

Similarly, using

$$\frac{|\lambda_{\mathfrak{f}}(t)|-\lambda_{\mathfrak{f}}(t)}{2} = \begin{cases}-\lambda_{\mathfrak{f}}(t) & \text{if } \lambda_{\mathfrak{f}}(t)<0,\\ 0 & \text{otherwise,}\end{cases}$$

we obtain

$$\mathcal{T}_{\mathfrak{f}}^{-}(x) \gg \frac{x^{1-2\alpha}}{\log x}.$$

This finishes the proof of Theorem 1.

§4. *Proof of Theorem 4.* The basic idea of the proof is the same as for Theorem 1, although here we localize on short intervals. The device (4), with the analytic properties of $M(\mathfrak{f}, s)$, gives a nice mean value estimate for $\lambda_{\mathfrak{f}}(t)$ over the squarefree integers in a short interval (see (20)). However, our series $D(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s)$ runs over all positive (not just squarefree) integers. We cannot obtain a counterpart for $|\lambda_{\mathfrak{f}}(t)|^2$. To get around this, we consider a bundle of short intervals which leads to two moment estimates (21) and (26) in §4.1. Then we can enumerate the sign changes in §4.2.

4.1. *Computation of moments of order one and two.* Let

$$0 \leqslant \alpha < 1/4 \qquad \text{and} \qquad 1 > \eta > 3/4 + \alpha.$$

Suppose that x is sufficiently large. We set $h = x^{\eta}$ and define δ by $\mathrm{e}^{2\delta} = 1 + h/x$. We have $\delta \asymp h/x$.

For all $s \in \mathbb{C}$ such that $|\mathrm{Re}\, s| \leqslant 2$, we have $(\mathrm{e}^{\delta s} - 1)^2/s^2 \ll \min(\delta^2, 1/|s|^2)$. It follows, by Lemma 8 and (4), that

$$\begin{aligned}
&\sum_{x \leqslant t \leqslant x+h}^{\flat} \lambda_{\mathfrak{f}}(t) \min\left(\log\left(\frac{x+h}{t}\right), \log\left(\frac{t}{x}\right)\right) \\
&\quad = \frac{1}{2\pi \mathrm{i}} \int_{3/4+\varepsilon-\mathrm{i}\infty}^{3/4+\varepsilon+\mathrm{i}\infty} M(\mathfrak{f}, s) \frac{(\mathrm{e}^{\delta s} - 1)^2}{s^2} x^s \, ds \\
&\quad \ll x^{3/4+\varepsilon} \int_{-\infty}^{+\infty} (|\tau| + 1)^{1/4+\varepsilon} \min\left(\delta^2, \frac{1}{1 + |\tau|^2}\right) d\tau \\
&\quad \ll h^{3/4} x^{\varepsilon}.
\end{aligned} \tag{20}$$

For any integer constant $A > 0$, let $(\varepsilon_1, \ldots, \varepsilon_A) \in \{-1, 1\}^A$. The bound for the moment of order one follows from (20); that is

$$\sum_{m \leqslant A} \varepsilon_m \sum_{x/m^2 < t < (x+h)/m^2}^{\flat} \lambda_{\mathfrak{f}}(t) \min\left(\log\left(\frac{x+h}{tm^2}\right), \log\left(\frac{tm^2}{x}\right)\right) \ll h^{3/4} x^{\varepsilon}. \tag{21}$$

We turn to the evaluation of the moment of order two. Since $\eta > 3/4 + \alpha$, by (14) and Lemma 6, we obtain for some positive constant C,

$$Ch \leqslant C' \sum_{x < n \leqslant x+h} \lambda_{\mathfrak{f}}(n)^2 \leqslant \sum_{m \leqslant \sqrt{x+h}} \tau(m)^4 \sum_{x/m^2 \leqslant t \leqslant (x+h)/m^2}^{\flat} \lambda_{\mathfrak{f}}(t)^2.$$

Next we prove that $\sqrt{x+h}$ can be replaced by some constant A in the outer sum up to the cost of a replacement of a smaller C. Indeed, we will prove that, for any fixed $A > 0$,

$$\sum_{A<m\leqslant\sqrt{x+x^\eta}} \tau(m)^4 \sideset{}{^\flat}\sum_{x/m^2\leqslant t\leqslant(x+x^\eta)/m^2} \lambda_{\mathfrak{f}}(t)^2 \ll x^\eta A^{-1+\varepsilon}.$$

Note that

$$\begin{aligned}\sum_{\sqrt{x}\leqslant m\leqslant\sqrt{x+x^\eta}} \tau(m)^4 &\sideset{}{^\flat}\sum_{x/m^2\leqslant t\leqslant(x+x^\eta)/m^2} \lambda_{\mathfrak{f}}(t)^2 \\ &= \sum_{\sqrt{x}\leqslant m\leqslant\sqrt{x+x^\eta}} \tau(m)^4 \sideset{}{^\flat}\sum_{t\leqslant(x+x^\eta)/m^2} \lambda_{\mathfrak{f}}(t)^2 \\ &\ll x^{1/2+\varepsilon},\end{aligned} \tag{22}$$

by (14). In light of (22), (14) and (3), it suffices to evaluate

$$\sum_{A<m\leqslant\sqrt{x}} \tau(m)^4 \min\left\{\max\left[\frac{x^\eta}{m^2}, \left(\frac{x}{m^2}\right)^{3/4+\alpha+\varepsilon}\right], \left(1+\frac{x^\eta}{m^2}\right)\frac{x^{2\alpha}}{m^{4\alpha}}\right\}.$$

Write $y = x^\eta/m^2$ and $Y = x/m^2$; then $0 < y < Y$ and $Y \gg 1$. Note that $2\alpha < 3/4 + \alpha$. The term $\min\{\cdots\}$ in the preceding formula is then handled by observing that

$$\min\{\max(y, Y^{3/4+\alpha+\varepsilon}), (1+y)Y^{2\alpha}\} \ll \begin{cases} Y^{2\alpha} & \text{if } y \leqslant 1, \\ yY^{2\alpha} & \text{if } 1 < y \leqslant Y^{3/4-\alpha}, \\ Y^{3/4+\alpha+\varepsilon} & \text{if } Y^{3/4-\alpha} < y \leqslant Y^{3/4+\alpha+\varepsilon}, \\ y & \text{if } Y^{3/4+\alpha+\varepsilon} < y < Y. \end{cases}$$

We split the sum over m into four subsums with the ranges of summation dividing at the points for which $y = 1$, $y = Y^{3/4-\alpha}$ and $y = Y^{3/4+\alpha+\varepsilon}$, respectively. Write

$$\eta_3 = \frac{\eta}{2}, \qquad \eta_2 = \frac{\eta - 3/4 + \alpha}{1/2 + 2\alpha}, \qquad \eta_1 = \frac{\eta - (3/4 + \alpha + \varepsilon)}{1/2 - 2(\alpha + \varepsilon)}$$

(note $\eta_3 > \eta_2 > \eta_1 > 0$). The four subsums are evaluated via the summations

$$\sum_{x^{\eta_3}<m\leqslant\sqrt{x}} \tau(m)^4 \frac{x^{2\alpha}}{m^{4\alpha}} \ll x^{2\alpha+(1-4\alpha)/2+\varepsilon} = x^{1/2+\varepsilon} = o(x^\eta),$$

$$\sum_{x^{\eta_2}<m\leqslant x^{\eta_3}} \tau(m)^4 \frac{x^{\eta+2\alpha}}{m^{2+4\alpha}} \ll x^{\eta+2\alpha-\eta_2(4\alpha+1)+\varepsilon} = x^{\eta-\frac{(\eta-3/4)(1+4\alpha)}{1/2+2\alpha}+\varepsilon},$$

$$\sum_{x^{\eta_1}<m\leqslant x^{\eta_2}} \tau(m)^4 \left(\frac{x}{m^2}\right)^{3/4+\alpha+\varepsilon} \ll x^{3/4+\alpha-\eta_1(2\alpha+1/2)+\varepsilon} = x^{\eta-\frac{\eta-(3/4+\alpha+\varepsilon)}{1/2-2(\alpha+\varepsilon)}+\varepsilon},$$

$$\sum_{A<m\leqslant x^{\eta_1}} \tau(m)^4 \frac{x^\eta}{m^2} \ll x^\eta A^{-1+\varepsilon}.$$

By taking a large enough constant A, we infer that

$$\sum_{m\leqslant A}\tau(m)^4\sum^{\flat}_{x/m^2<t<(x+h)/m^2}\lambda_{\mathfrak{f}}(t)^2\geqslant(C-O(A^{-1+\varepsilon}))h\gg h.$$

This equation remains true if we replace (x,h) by $(x+h/4,h/2)$, so

$$\sum_{m\leqslant A}\tau(m)^4\sum^{\flat}_{(x+h/2)/m^2<t\leqslant(x+3h/4)/m^2}\lambda_{\mathfrak{f}}(t)^2\gg h. \tag{23}$$

Moreover,

$$\begin{aligned}\sum_{m\leqslant A}\tau(m)^4&\sum^{\flat}_{x/m^2<t<(x+h)/m^2}\lambda_{\mathfrak{f}}(t)^2\min\left(\log\left(\frac{x+h}{tm^2}\right),\log\left(\frac{tm^2}{x}\right)\right)\\&\geqslant\sum_{m\leqslant A}\tau(m)^4\sum^{\flat}_{(x+h/4)/m^2<t\leqslant(x+3h/4)/m^2}\lambda_{\mathfrak{f}}(t)^2\min\left(\log\left(\frac{x+h}{tm^2}\right),\log\left(\frac{tm^2}{x}\right)\right)\end{aligned} \tag{24}$$

and, if $t\in[(x+h/4)/m^2,(x+3h/4)/m^2]$, then

$$\frac{x}{h}\min\left(\log\left(\frac{x+h}{tm^2}\right),\log\left(\frac{tm^2}{x}\right)\right)\gg 1. \tag{25}$$

We deduce, from (24), (25) and (23), that

$$\sum_{m\leqslant A}\tau(m)^4\sum^{\flat}_{x/m^2<t<(x+h)/m^2}\lambda_{\mathfrak{f}}(t)^2\min\left(\log\left(\frac{x+h}{tm^2}\right),\log\left(\frac{tm^2}{x}\right)\right)\gg\frac{h^2}{x}. \tag{26}$$

This is our moment of order two.

4.2. *Implication on the number of sign changes.* We use (21) and (9) to write

$$\begin{aligned}\sum_{m\leqslant A}\sum^{\flat}_{x/m^2<t<(x+h)/m^2}&(|\lambda_{\mathfrak{f}}(t)|+\varepsilon_m\lambda_{\mathfrak{f}}(t))\min\left(\log\left(\frac{x+h}{tm^2}\right),\log\left(\frac{tm^2}{x}\right)\right)\\&\gg\sum_{m\leqslant A}\sum^{\flat}_{x/m^2<t<(x+h)/m^2}t^{-\alpha}\lambda_{\mathfrak{f}}(t)^2\min\left(\log\left(\frac{x+h}{tm^2}\right),\log\left(\frac{tm^2}{x}\right)\right)\\&\quad+O(h^{3/4+\varepsilon})\\&\gg x^{-1-\alpha}h^2+O(h^{3/4+\varepsilon}),\end{aligned} \tag{27}$$

by (26). If $\eta>\frac{4}{5}(1+\alpha)$, we deduce that

$$\begin{aligned}\sum_{m\leqslant A}\sum^{\flat}_{x/m^2<t<(x+h)/m^2}&(|\lambda_{\mathfrak{f}}(t)|+\varepsilon_m\lambda_{\mathfrak{f}}(t))\min\left(\log\left(\frac{x+h}{tm^2}\right),\log\left(\frac{tm^2}{x}\right)\right)\\&\gg x^{2\eta-1-\alpha}.\end{aligned}$$

Assume that, for all $m \in \{1, \ldots, A\}$, there exists $\varepsilon_m \in \{-1, 1\}$ such that the sign of $\lambda_{\mathfrak{f}}(t)$ is $-\varepsilon_m$ for every squarefree $t \in]x/m^2, (x+h)/m^2[$. Then,

$$\sum_{m \leqslant A} \sideset{}{^\flat}\sum_{x/m^2 < t < (x+h)/m^2} (|\lambda_{\mathfrak{f}}(t)| + \varepsilon_m \lambda_{\mathfrak{f}}(t)) \min\left(\log\left(\frac{x+h}{tm^2}\right), \log\left(\frac{tm^2}{x}\right)\right) = 0,$$

which is in contradiction with (27). Consequently, there exists $m \in \{1, \ldots, A\}$ such that the interval $]x/m^2, (x+h)/m^2[$ contains squarefree integers t and t' satisfying

$$|\lambda_{\mathfrak{f}}(t)| = \lambda_{\mathfrak{f}}(t) \neq 0 \qquad \text{and} \qquad |\lambda_{\mathfrak{f}}(t')| = -\lambda_{\mathfrak{f}}(t') \neq 0,$$

that is, $\lambda_{\mathfrak{f}}(t)\lambda_{\mathfrak{f}}(t') < 0$.

Let X be any sufficiently large number. Write $B = (1 + 1/A)^2$, $H = (BX)^\eta$ and $J = \lfloor (B-1)X/H \rfloor$. For any $j \in \{0, \ldots, J-1\}$ and any $m \in \{1, \ldots, A\}$, let

$$I_j(m) = \left]\frac{X + jH}{m^2}, \frac{X + (j+1)H}{m^2}\right[.$$

The interval $I_J(m+1)$ is on the left-hand side of $I_0(m)$. Moreover, if $j \neq k$, then $I_j(m) \cap I_k(m) = \emptyset$. It follows that the AJ intervals $I_j(m)$ are disjoint. Since, for any j, there exists m such that $I_j(m)$ contains a sign change, we obtain at least $J \gg X^{1-\eta}$ sign changes over the interval $[1, X]$. The proof is complete after replacing η by $\eta + \varepsilon$.

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