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Twisted moments of automorphic L-functions

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ABSTRACT

We study the moments of the symmetric power L-functions of primitive forms at the edge of the critical strip twisted by the square of the value of the standard L-function at the center of the critical strip. We give a precise expansion of the moments as the order goes to infinity.

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1. Introduction

Let ρ be a representation on SU(2). For any $g \in \mathrm{SU}(2)$ we define the polynomial

$$D(X, \rho, g) = \det\bigl(I - X\rho(g)\bigr)^{-1}. \tag{1}$$

Endowing SU(2) with its Haar measure, Cogdell & Michel remarked that

$$\int_{\mathrm{SU}(2)} D(X, \rho, g)^z \,\mathrm{d}g = 1 + \left[\frac{z^2}{2}\,\mathrm{FrSc}(\rho)^2 + \frac{z}{2}\,\mathrm{FrSc}(\rho)\right]X^2 + O_z\bigl(X^3\bigr)$$

[CM04, (2.26)] for any complex number z, where $\mathrm{FrSc}\,\rho$ is the Frobenius–Schur indicator of ρ. The coefficient of X^2 is then

$$\begin{cases} 0 & \text{if } \rho \text{ is not self-dual,} \\ \dfrac{z(z-1)}{2} & \text{if id appears once in the irreducible decomposition of } \mathrm{Sym}^2\rho, \\ \dfrac{z(z+1)}{2} & \text{if id appears once in the irreducible decomposition of } \wedge^2\rho. \end{cases}$$

For $\rho = \mathrm{St}$ (the standard representation of SU(2)), this coefficient is $\frac{z(z-1)}{2}$. In particular, the Euler product (indexed over the set $\mathscr{P}$ of all prime numbers)

$$\prod_{p\in\mathscr{P}} \int_{\mathrm{SU}(2)} D\bigl(p^{-1/2}, \mathrm{St}, g\bigr)^z \,\mathrm{d}g$$

converges only for $z \in \{0, 1\}$.

Let $k \geqslant 2$ be a (fixed) even integer. For any squarefree integer N such that the set of primitive forms of weight k over $\Gamma_0(N)$ is not empty, we denote by $\mathrm{H}_k^*(N)$ this set. To any $f \in \mathrm{H}_k^*(N)$ we associate an L-function defined by the Euler product

$$L(s, f) = \prod_{p\in\mathscr{P}} \det\bigl(I - X\,\mathrm{St}\bigl(g_f(p)\bigr)p^{-s}\bigr)^{-1}$$

where for any prime number p, the matrix

$$g_f(p) = \begin{pmatrix} \alpha_f(p) & 0 \\ 0 & \beta_f(p) \end{pmatrix}$$

is made up of the local parameters in p associated to f. For any prime p not dividing N, this matrix belongs to SU(2) and for the $\omega(N)$ prime numbers dividing N we have $\alpha_f(p) = \pm p^{-1/2}$ and $\beta_f(p) = 0$. Hence it tempts naturally to model the moments of L-functions for the primitive forms in $\mathrm{H}_k^*(N)$ (over the discrete harmonic measure) with Euler product of polynomial of type (1) with g in SU(2) endowed with its Haar measure.

As in [CM04], denote by $\sum^{\mathrm{h}}$ the harmonic average. It is apparent that

$$\lim_{N\to+\infty}\sum_{f\in\mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac{1}{2},f\right)^0=\prod_{p\in\mathscr{P}}\int_{\mathrm{SU}(2)} D\big(p^{-1/2},\mathrm{St},g\big)^0\,\mathrm{d}g$$

and it follows from [RW07, Theorem A and Proposition B] that

$$\lim_{N\to+\infty}\sum_{f\in\mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac{1}{2},f\right)^1=\prod_{p\in\mathscr{P}}\int_{\mathrm{SU}(2)} D\big(p^{-1/2},\mathrm{St},g\big)^1\,\mathrm{d}g.$$

The generalization to high power moments sounds problematic, and in fact, there is a convergence problem on the right side. For $z=2$, the lack of convergence of the product in the representation side comes from the term $1/p$ so a natural remedy is natural to consider the normalized form

$$\prod_{p\in\mathscr{P}}\int_{\mathrm{SU}(2)}\left(1-\frac{1}{p}\right)D\big(p^{-1/2},\mathrm{St},g\big)^2\,\mathrm{d}g.$$

To fix ideas, we assume temporarily N to be prime. It turns out that the remedy is appropriate; in fact,

$$\begin{aligned}\sum_{f\in\mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac{1}{2},f\right)^2 &\sim\left[\prod_{p\in\mathscr{P}}\left(1-\frac{1}{p}\right)\int_{\mathrm{SU}(2)} D\big(p^{-1/2},\mathrm{St},g\big)^2\,\mathrm{d}g\right]\log N\quad(N\to+\infty)\\ &\sim \mathrm{e}^{-\gamma}\prod_{p\leqslant N}\int_{\mathrm{SU}(2)} D\big(p^{-1/2},\mathrm{St},g\big)^2\,\mathrm{d}g\quad(N\to+\infty),\end{aligned}$$

where γ is the Euler constant. In other words, we may model $L(1/2,f)^2$ by the product over prime numbers $p\leqslant N$ of the random variables $g\mapsto D(p^{-1/2},\mathrm{St},g)^2$ with a correction factor $\mathrm{e}^{-\gamma}$.

Our result is actually more precise and we compute all the complex moments of $L(1,\mathrm{Sym}^m f)$ twisted by $L(1/2,f)^2$ without too heavy restriction on the level N. To give our results, we need a few notation.

For any integer $m\geqslant 1$, the mth symmetric power L-function of $f\in\mathrm{H}_k^*(N)$ is

$$L\big(s,\mathrm{Sym}^m f\big)=\prod_{p\in\mathscr{P}}\det\big(I-\mathrm{Sym}^m\rho\big(g_f(p)\big)p^{-s}\big)^{-1}.$$

If $m\in\{1,2,4\}$ it is known to have all the required properties to be an L-function in the sense of [IK04, §5.1] and to have no Landau–Siegel zero [GJ78,Kim03,KS02]. For other values of m, we impose two standard hypothesis – Hypothesis $\mathrm{Sym}^k f$ and $\mathrm{LSZ}^k f$ in [CM04]. Therefore, our results are unconditional for $m\in\{1,2,4\}$ and rest on the standard conjectures for all other cases. We write, γ_∞ for the gamma factor of $L(s,f)$ which depends only on the weight of f. Explicitly it is given by

$$\gamma_\infty(s)=\pi^{-s}\Gamma\left(\frac{s}{2}+\frac{k-1}{4}\right)\Gamma\left(\frac{s}{2}+\frac{k+1}{4}\right).$$

Let

$$F^z(w,s;X)=\left(1-X^{1+2w}\right)\int\limits_{\mathrm{SU}(2)} D\left(X^{1/2+w},\mathrm{St},g\right)^2 D\left(X^{1+s},\mathrm{Sym}^m,g\right)^z\,\mathrm{d}g$$

and

$$C^z(w,s;X)=\begin{cases}(1+X^{2+2w})(1-X^{2+2w})^{-2}(1-X^{1+m/2+s})^{-z} & \text{if } 2\mid m,\\ \frac{(1+X^{1+w})^{-2}(1-X^{1+m/2+s})^{-z}+(1-X^{1+w})^{-2}(1+X^{1+m/2+s})^{-z}}{2} & \text{if } 2\nmid m.\end{cases}$$

The function $C^z(w,s;X)$ will be used as a correction factor to $F^z(w,s;X)$. Moreover we define

$$A^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^m;N\right)=\prod_{\substack{p\in\mathscr{P}\\ p\nmid N}}F^z\left(0,0;\frac{1}{p}\right)\prod_{\substack{p\in\mathscr{P}\\ p\mid N}}C^z\left(0,0;\frac{1}{p}\right)$$

and

$$B^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^m;N\right)=\frac{\mathrm{d}}{\mathrm{d}w}_{|w=0}\left(\prod_{\substack{p\in\mathscr{P}\\ p\nmid N}}F^z\left(w,0;\frac{1}{p}\right)\prod_{\substack{p\in\mathscr{P}\\ p\mid N}}C^z\left(w,0;\frac{1}{p}\right)\right).$$

Finally denote by $\varphi(n)$ (resp. $\mu(n)$) the Euler function (resp. Möbius) and by $\log_k$ the k-fold iterated logarithm.

Below are our main results.

Theorem A. *Let $m\in\{1,2,4\}$. There exists two positive real numbers c_m and δ_m such that for any sufficiently large squarefree N,*

$$\begin{aligned}&\frac{N}{\varphi(N)}\sum_{f\in\mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac{1}{2},f\right)^2 L\left(1,\mathrm{Sym}^m f\right)^z\\ &=A^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^m;N\right)\log N\\ &\quad+2A^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^m;N\right)\left(\gamma+\frac{\gamma_\infty'}{\gamma_\infty}\left(\frac{1}{2}\right)+\sum_{p\mid N}\frac{\log p}{p-1}\right)\\ &\quad+B^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^m;N\right)+O_m\left(\exp\left(-\delta_m\frac{\log N}{\log_2 N}\right)\right)\end{aligned}$$

uniformly in

$$|z|\leqslant c_m\frac{\log N}{\log_2 N\log_3 N}.$$

This theorem is proved in Section 3.1. The dependence on the level can be easily depicted when N has no small prime factors. Consider the set of numbers

$$\mathscr{N}(h) = \left\{N \in \mathbb{Z}_{>0}\colon\ \mu(N)^2 = 1 \text{ and } P^-(N) \geqslant h(N)\right\}$$

for some function h where $P^-(N)$ is the smallest prime factor of N with the convention $P^-(1) = +\infty$. We write

$$\begin{aligned} A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right) &= A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m; 1\right) \\ &= \prod_{p\in\mathscr{P}}\left(1-\frac{1}{p}\right)\int_{\mathrm{SU}(2)} D\left(\frac{1}{p^{1/2}}, \mathrm{St}, g\right)^2 D\left(\frac{1}{p}, \mathrm{Sym}^m, g\right)^z \mathrm{d}g \end{aligned}$$

and

$$\begin{aligned} B^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right) &= B^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m, 1\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}w}_{|w=0} \prod_{p\in\mathscr{P}}\left(1-\frac{1}{p^{1+2w}}\right)\int_{\mathrm{SU}(2)} D\left(\frac{1}{p^{1/2+w}}, \mathrm{St}, g\right)^2 \\ &\quad \times D\left(\frac{1}{p}, \mathrm{Sym}^m, g\right)^z \mathrm{d}g. \end{aligned}$$

Corollary B. *Let $m \in \{1,2,4\}$. There exists a positive real number c_m such that for any sufficiently large squarefree $N \in \mathscr{N}(\log^2)$,*

$$\sum_{f\in \mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac{1}{2}, f\right)^2 L\big(1, \mathrm{Sym}^m f\big)^z = \big(1+o_m(1)\big) A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right)\log N$$

uniformly in

$$|z| \leqslant c_m \frac{\log N}{\log_2 N \log_3 N}.$$

This is shown in Section 3.2.

It is interesting to evaluate the asymptotic behavior of the main term

$$A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right)$$

and the constant term $B^{2,z}(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m)$ as the exponent $z \to +\infty$ in real numbers.

Let X_m be the Chebyshev polynomial of second kind whose restriction on $[-2,2]$ is defined by

$$X_m(2\cos\theta) = \frac{\sin((m+1)\theta)}{\sin\theta}.$$

They come up naturally in theory of modular forms since, if $\{\chi_{\mathrm{Sym}^m}\colon m \in \mathbb{Z}_{\geqslant 0}\}$ is the set of irreducible characters of SU(2), then

$$\chi_{\mathrm{Sym}^m}(g) = X_m(\operatorname{tr} g).$$

Let us introduce some auxiliary functions.

$$g_m(t) := \log \int_{\mathrm{SU}(2)} \mathrm{e}^{t\chi_m(\operatorname{tr} g)}\,\mathrm{d}g = \log\left(\frac{2}{\pi}\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta\right) \quad (t \geqslant 0), \tag{2}$$

$$\widetilde{g}_m(t) := \begin{cases} g_m(t) & \text{if } 0 \leqslant t < 1, \\ g_m(t) - (m+1)t & \text{if } t \geqslant 1, \end{cases} \tag{3}$$

and

$$h_m(t) := \frac{\int_{\mathrm{SU}(2)} \mathrm{e}^{t\chi_m(\operatorname{tr} g)} \operatorname{tr} g\,\mathrm{d}g}{2\int_{\mathrm{SU}(2)} \mathrm{e}^{t\chi_m(\operatorname{tr} g)}\,\mathrm{d}g} = \frac{\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)}\cos\theta\sin^2\theta\,\mathrm{d}\theta}{\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta} \quad (t \geqslant 0), \tag{4}$$

$$\widetilde{h}_m(t) := \begin{cases} h_m(t) & \text{if } 0 \leqslant t < 1, \\ h_m(t) - 1 & \text{if } t \geqslant 1. \end{cases} \tag{5}$$

Theorem C. *Let $J \geqslant 1$ and $m \geqslant 1$ be two fixed integers. Then we have*

$$\log A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right) = z\left\{(m+1)\log_2 z + (m+1)\gamma + \sum_{j=1}^{J}\frac{a_j}{(\log z)^j} + O\left(\frac{1}{(\log z)^{J+1}}\right)\right\}$$

uniformly for $z \geqslant 3$, where γ is the Euler constant and

$$a_j := \int_0^{+\infty} \frac{\widetilde{g}_m(t)}{t^2}(\log t)^{j-1}\,\mathrm{d}t.$$

The implied constant depends on J and m only.

Theorem C is proved in Section 4.1.

Theorem D. *We have*

$$B^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right) \ll A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right)\log z$$

uniformly for $z \geqslant 3$ if m is even; *and*

$$B^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right) = A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right)\left\{b_m + O\left(\mathrm{e}^{-\sqrt{\log z}}\right)\right\}\sqrt{z}$$

uniformly for $z \geqslant 3$ if m is odd, where

$$b_m := -4\left(2 + \int_0^{+\infty} \frac{\widetilde{h}_m(t)}{t^{3/2}}\,\mathrm{d}t\right) \neq 0.$$

The implied constants depend on m only.

Section 4.2 is devoted to its proof.

It is surprising that the asymptotic behavior of $\log B^{2,z}(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m)$ changes dramatically according as the parity of m.

2. Preliminary results

For every $g \in \mathrm{SU}(2)$, define $\lambda^{z,\nu}_{\mathrm{Sym}^m}(g)$ by the expansion

$$D\big(X, \mathrm{Sym}^m, g\big)^z = \sum_{\nu=0}^{+\infty} \lambda^{z,\nu}_{\mathrm{Sym}^m}(g) X^\nu.$$

We have from [RW07, (46) and (36)],

$$\lambda^{z,\nu}_{\mathrm{Sym}^m}(g) = \sum_{u=0}^{m\nu} \mu^{z,\nu}_{\mathrm{Sym}^m,\mathrm{Sym}^u}\,\chi_{\mathrm{Sym}^u}(g)$$

with

$$\mu^{z,\nu}_{\mathrm{Sym}^m,\mathrm{Sym}^u} = \int_{\mathrm{SU}(2)} \lambda^{z,\nu}_{\mathrm{Sym}^m}(g)\chi_{\mathrm{Sym}^u}(g)\,\mathrm{d}g. \tag{6}$$

One should remark $\mu^{z,\nu}_{\mathrm{Sym}^m,\mathrm{Sym}^u} = 0$ for $n > m\nu$. Recall that $\{\chi_{\mathrm{Sym}^m}\colon\ m \in \mathbb{Z}_{\geqslant 0}\}$ is explicitly defined by the generating series

$$\sum_{m\geqslant 0} \chi_{\mathrm{Sym}^m}(g)T^m = \frac{1}{(1-\alpha T)(1-\overline{\alpha}T)} = D(T, \mathrm{St}, g) \tag{7}$$

where α and $\overline{\alpha}$ are the eigenvalues of g. It follows from the study of Cogdell & Michel [CM04] (see also [RW07, Eqs. (38), (39) and (52)]) that

$$\mu^{z,0}_{\mathrm{Sym}^m,\mathrm{Sym}^u} = \delta(u,0), \tag{8}$$

$$\mu^{z,1}_{\mathrm{Sym}^m,\mathrm{Sym}^u} = z\delta(u,m), \tag{9}$$

$$\big|\mu^{z,\nu}_{\mathrm{Sym}^m,\mathrm{Sym}^u}\big| \leqslant \binom{(m+1)|z| + \nu - 1}{\nu}. \tag{10}$$

2.1. Combinatorial results

The aim of this short section is to prove the two following useful equalities:

$$\sum_{u\geqslant 0}\frac{\tau(p^u)}{p^{(1+w)u}}\sum_{\substack{v\geqslant 0\\ u\equiv mv \ (\mathrm{mod}\ 2)}}\frac{\tau_z(p^v)}{p^{(1+m/2+s)v}}=C^z\left(w,s;\frac{1}{p}\right), \tag{11}$$

$$\sum_{u\geqslant 0}\frac{\tau(p^u)}{p^{(1/2+w)u}}\sum_{v\geqslant 0}\frac{\mu^{z,v}_{\mathrm{Sym}^m,\mathrm{Sym}^u}}{p^{(1+s)v}}=F^z\left(w,s;\frac{1}{p}\right). \tag{12}$$

Thanks to (10) and the binomial theorem, the series in (12) is absolutely convergent for $\Re s>-1/2$ and $\Re w>-1/2$.

Equality (11) follows directly from the following expressions:

$$\sum_{\substack{u\geqslant 0\\ u\ \mathrm{odd}}}(u+1)X^u=\frac{2X}{(1-X^2)^2},\qquad \sum_{\substack{u\geqslant 0\\ u\ \mathrm{even}}}(u+1)X^u=\frac{1+X^2}{(1-X^2)^2}$$

and

$$\sum_{\substack{v\geqslant 0\\ v\equiv r\ (\mathrm{mod}\ 2)}}\binom{v+z-1}{v}X^v=\frac{(1-X)^{-z}+(-1)^r(1+X)^{-z}}{2}$$

for any $r\in\{0,1\}$.

From (6) we deduce

$$\sum_{u\geqslant 0}\frac{\tau(p^u)}{p^{(1/2+w)u}}\sum_{v\geqslant 0}\frac{\mu^{z,v}_{\mathrm{Sym}^m,\mathrm{Sym}^u}}{p^{(1+s)v}}=\int_{\mathrm{SU}(2)}D\left(\frac{1}{p^{1+s}},\mathrm{Sym}^m,g\right)^z\sum_{u\geqslant 1}\frac{(u+1)\chi_{\mathrm{Sym}^u}(g)}{p^{(1/2+w)u}}\,\mathrm{d}g.$$

Let $g\in\mathrm{SU}(2)$ and let $\alpha,\bar{\alpha}$ be its eigenvalues. We use (7) to get

$$\sum_{u\geqslant 1}(u+1)\chi_{\mathrm{Sym}^u}(g)T^u=\frac{\mathrm{d}}{\mathrm{d}T}\frac{T}{(1-\alpha T)(1-\bar{\alpha}T)}=\left(1-T^2\right)D(T,\mathrm{St},g)^2.$$

This gives (12).

2.2. Analytical results

Lemma 2.1. *Let $m\geqslant 1$ and $z_m=(m+1)\min\{n\in\mathbb{Z}_{\geqslant 0}\colon n\geqslant |z|\}$.*

(a) *For $\sigma\geqslant 3/4$ and $r\geqslant 1/3$, we have*

$$\prod_{p\mid N}\sum_{u\geqslant 0}\frac{\tau(p^u)}{p^{ru}}\sum_{\substack{v\geqslant 0\\ u\equiv mv\ (\mathrm{mod}\ 2)}}\frac{\tau_{|z|}(p^v)}{p^{(\sigma+m/2)v}}\leqslant \mathrm{e}^{c[|z|+S_r(N)]}$$

where

$$S_r(N) = \begin{cases} 1 & \text{if } r > 1/2, \\ \log_3(N) & \text{if } r = 1/2, \\ (\log N)^{1-2r}/\log_2 N & \text{if } r < 1/2, \end{cases}$$

and the constant $c > 0$ does not depend on σ.

(b) *For $\sigma > 1$ and $r \geqslant 1/3$ we have*

$$\prod_{p \nmid N} \sum_{u \geqslant 0} \frac{\tau(p^u)}{p^{ru}} \sum_{v \geqslant 0} \frac{|\mu^{z,v}_{\mathrm{Sym}^m,\mathrm{Sym}^u}|}{p^{\sigma v}} \leqslant \exp\bigl(c_\sigma(z_m+3)\bigr),$$

where $c_\sigma > 0$ is a constant depending on σ.

(c) *For $\sigma \in [3/4, 1]$ and $r \in [1/3, 1]$ we have*

$$\prod_{p \nmid N} \sum_{u \geqslant 0} \frac{\tau(p^u)}{p^{ru}} \sum_{v \geqslant 0} \frac{|\mu^{z,v}_{\mathrm{Sym}^m,\mathrm{Sym}^u}|}{p^{\sigma v}} \leqslant \exp\left(c(z_m+3)\left[\frac{(z_m+3)^{-1+1/\sigma}-1}{(1-\sigma)\log(z_m+3)} + \log_2(z_m+3)\right]\right)$$

where $c > 0$ is a constant not depending on σ.

Proof. (a) Let

$$A_m(p) = \sum_{u \geqslant 0} \frac{\tau(p^u)}{p^{ur}} \sum_{\substack{v \geqslant 0 \\ u \equiv mv \ (\mathrm{mod}\ 2)}} \frac{\tau_{|z|}(p^v)}{p^{(\sigma+m/2)v}}.$$

If m is even then by (13),

$$A_m(p) = \sum_{u \text{ even}} \sum_{v} = \left(1+\frac{1}{p^{2r}}\right)\left(1-\frac{1}{p^{2r}}\right)^{-2}\left(1-\frac{1}{p^{\sigma+m/2}}\right)^{-|z|}. \tag{13}$$

If m is odd, then we get

$$A_m(p) = \sum_{u \text{ even}} \sum_{v \text{ even}} + \sum_{u \text{ odd}} \sum_{v \text{ odd}} \leqslant \sum_{u \text{ even}} \sum_{v \text{ even}} + \sum_{u \text{ even}} \sum_{v \text{ odd}} \leqslant \sum_{u \text{ even}} \sum_{v}.$$

In both cases, we are led to the bound in the right side of (13). Since $\sigma + m/2 \geqslant 5/4$ and $r \geqslant 1/3$, this yields

$$\prod_{p \mid N} A_m(p) \ll \exp\left(c\left(|z| + \sum_{p \mid N} \frac{1}{p^{2r}}\right)\right) \leqslant \exp\bigl(c\bigl[|z| + S_r(N)\bigr]\bigr).$$

(b) The proof is similar to [RW07, p. 728]. We separate the product into two parts according to $p^\sigma \leqslant z_m + 3$ or $p^\sigma > z_m + 3$. Using (9) and (10), we have

$$\prod_{p^\sigma > z_m+3} \leqslant \exp\left(\sum_{p^\sigma > z_m+3} \left(\frac{z_m}{p^{\sigma+rm}} + \sum_{v \geqslant 2} \frac{1}{p^{\sigma v}} \sum_{u \geqslant 0} \frac{\tau(p^u)}{p^{ru}} \bigl|\mu^{z,v}_{\mathrm{Sym}^m,\mathrm{Sym}^u}\bigr|\right)\right) \tag{14}$$

and

$$\sum_{v\geqslant 2}\leqslant\sum_{u\geqslant 0}\frac{u+1}{p^{ru}}\sum_{v\geqslant 2}\binom{z_m+v-1}{v}\frac{1}{p^{\sigma v}}$$

with

$$\sum_{v\geqslant 2}\binom{z_m+v-1}{v}\frac{1}{p^{\sigma v}}\leqslant\frac{z_m(z_m+1)}{p^{2\sigma}}\sum_{v\geqslant 2}\binom{z_m+v-1}{v-2}\frac{1}{p^{\sigma(v-2)}}$$

so that

$$\sum_{v\geqslant 2}\leqslant\left(1-\frac{1}{p^r}\right)^{-2}\left(\frac{z_m+1}{p^\sigma}\right)^2\left(1-\frac{1}{p^\sigma}\right)^{-z_m-2}\leqslant 4\left(1-\frac{1}{2^{1/3}}\right)^{-2}\left(\frac{z_m+1}{p^\sigma}\right)^2 \tag{15}$$

since $p^\sigma > z_m+3$. Reporting (15) in (14) leads to

$$\prod_{p^\sigma>z_m+3}\leqslant\exp\bigl(c(z_m+3)^{1/\sigma}\bigr).$$

Now we deal with $p^\sigma < z_m+3$. Using (8)–(10), we have

$$\sum_{u\geqslant 0}\frac{\tau(p^u)}{p^{ru}}\sum_{v\geqslant 0}\frac{|\mu^{z,v}_{\mathrm{Sym}^m,\mathrm{Sym}^u}|}{p^{\sigma v}}\leqslant 1+\frac{z_m}{p^\sigma}+\sum_{v\geqslant 2}\frac{1}{p^{\sigma v}}\sum_{u\geqslant 0}\frac{\tau(p^u)}{p^{ru}}\binom{z_m+v-1}{v}.$$

The right-hand side, denoted by R, satisfies

$$\begin{aligned}
R &= 1+\frac{z_m}{p^\sigma}+\sum_{v\geqslant 2}\frac{1}{p^{\sigma v}}\binom{z_m+v-1}{v}+\sum_{v\geqslant 2}\frac{1}{p^{\sigma v}}\sum_{u\geqslant 1}\frac{\tau(p^u)}{p^{ru}}\binom{z_m+v-1}{v}\\
&= \left(1-\frac{1}{p^\sigma}\right)^{-z_m}+\frac{1}{p^{\sigma+r}}\sum_{u\geqslant 0}\frac{u+2}{p^{ru}}\sum_{v\geqslant 1}\binom{z_m+v}{v+1}\frac{1}{p^{\sigma v}}\\
&\leqslant \left(1-\frac{1}{p^\sigma}\right)^{-z_m}+\frac{2z_m}{p^{\sigma+r}}\left(1-\frac{1}{p^r}\right)^{-2}\sum_{v\geqslant 1}\binom{z_m+v}{v}\frac{1}{p^{\sigma v}}\\
&\leqslant \left(1-\frac{1}{p^\sigma}\right)^{-z_m}+\frac{2z_m}{p^{\sigma+r}}\left(1-\frac{1}{p^r}\right)^{-2}\left(1-\frac{1}{p^\sigma}\right)^{-z_m-1}\\
&\leqslant \left(1-\frac{1}{p^\sigma}\right)^{-z_m-1}\left(1+c\frac{z_m}{p^{\sigma+r}}\right)
\end{aligned} \tag{16}$$

for some absolute constant $c>0$. Since σ and $\sigma+r$ are greater than 1 it follows that

$$\prod_{p^\sigma<z_m+3}\leqslant\exp\bigl(c_\sigma(z_m+1)\bigr).$$

(c) As for establishing (15) we have an absolute constant c such that

$$\prod_{p^\sigma > z_m+3} \leqslant \exp\left(\sum_{p^\sigma > z_m+3} \frac{z_m}{p^{\sigma+rm}} + c\frac{(z_m+1)^2}{p^{2\sigma}}\right) \leqslant \exp\left(c\frac{(z_m+3)^{1/\sigma}}{\log(z_m+3)}\right). \tag{17}$$

From (16) we have

$$\prod_{p^\sigma < z_m+3} \leqslant \exp\left(c(z_m+1)\sum_{p^\sigma < z_m+3} \frac{1}{p^\sigma} + \frac{1}{p^{\sigma+r}}\right)$$

and using

$$\sum_{p \leqslant y} \frac{1}{p^\sigma} \ll \log_2 y + \frac{y^{1-\sigma}-1}{(1-\sigma)\log y}$$

valid uniformly for $1/2 \leqslant \sigma \leqslant 1$ and $y \geqslant \mathrm{e}^2$ [TW03, Lemma 3.2] we get

$$\prod_{p^\sigma < z_m+3} \leqslant \exp\left(c(z_m+3)\left[\frac{(z_m+3)^{(1-\sigma)/\sigma}-1}{(1-\sigma)\log(z_m+3)} + \log_2(z_m+3)\right]\right). \tag{18}$$

The result is a consequence of (17) and (18). □

3. Evaluation of the moments

3.1. Moments in the all level case

We fix G any function which is holomorphic and bounded in some sufficiently wide vertical strip $|\Re s| \ll 1$, even and normalized by $G(0) = 1$. (Note $G'(0) = 0$.)

Let $z \in \mathbb{C}$ and $x \geqslant 1$. Define

$$\omega^z_{\mathrm{Sym}^m f}(x) = \sum_{n=1}^{+\infty} \frac{\lambda^z_{\mathrm{Sym}^m f}(n)}{n}\mathrm{e}^{-n/x} \tag{19}$$

for all $f \in \mathrm{H}_k^*(N)$. We prove the following lemma.

Lemma 3.1. *For all x, z and N we have*

$$\sum_{f \in \mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac{1}{2}, f\right)^2 \omega^z_{\mathrm{Sym}^m f}(x) = 2\sum_{q \geqslant 1} \frac{\tau(q)}{\sqrt{q}} V_N\left(\frac{q}{N}\right) \sum_{n \geqslant 1} \frac{\mathrm{e}^{-n/x}}{n}\tau_z(n_N)$$
$$\times \left(\prod_{p \mid n^{(N)}} \mu^{z,\nu_p(n)}_{\mathrm{Sym}^m,\mathrm{Sym}^{\nu_p(q)}}\right)\delta\left(q^{(N)} \middle| n^{(N)m}\right)\frac{\square(n_N^m q_N)}{\sqrt{n_N^m q_N}} + O(\mathrm{Err})$$

where

$$V_N(y) = \frac{1}{2\pi \mathrm{i}}\int_{(2)} \zeta^{(N)}(1+2w)\left(\frac{\gamma_\infty(1/2+w)}{\gamma_\infty(1/2)}\right)^2 \frac{G(w)}{w} y^{-w}\,\mathrm{d}w \tag{20}$$

and

$$\mathrm{Err} = \frac{\tau(N)^2 \log N \log_2 N}{N^{1/4}} x^{m/4} (\log x)^{z_m+1} (z_m + m + 1)!.$$

Proof. Let $L(s, f \boxplus f) = L(s, f)^2$. This is an L-function in the sense of [IK04, §5.1]. In particular the gamma factor is $\gamma_\infty(s)^2$, the sign of the functional equation is 1, the conductor is N^2 and the n-th Dirichlet coefficient is

$$\lambda_{f \boxplus f}(n) = \sum_{\substack{(q,r)\in\mathbb{Z}^2_{\geqslant 0} \\ qr^2 = n}} \mathbf{1}^{(N)}(r) \lambda_f(q) \tau(q),$$

where $\mathbf{1}^{(N)}(r)$ is the characteristic function of integers coprime with N. Therefore we can apply [IK04, Theorem 5.3] to obtain

$$L\left(\frac{1}{2}, f\right)^2 = 2 \sum_{q \geqslant 1} \frac{\lambda_f(q)\tau(q)}{\sqrt{q}} V_N\left(\frac{q}{N}\right) \tag{21}$$

where

$$\begin{aligned} V_N(y) &= \sum_r \frac{\mathbf{1}^{(N)}(r)}{r} \int_{(3)} \left(yr^2\right)^{-u} G(u) \left(\frac{\gamma_\infty(1/2+u)}{\gamma_\infty(1/2)}\right)^2 \frac{\mathrm{d}u}{u} \\ &= \int_{(3)} y^{-u} \zeta^{(N)}(1+2u) G(u) \left(\frac{\gamma_\infty(1/2+u)}{\gamma_\infty(1/2)}\right)^2 \frac{\mathrm{d}u}{u}. \end{aligned}$$

We have to evaluate

$$T = \sum_{f \in \mathrm{H}_k^*(N)}^{\mathrm{h}} \lambda_f(q) \lambda^z_{\mathrm{Sym}^m f}(n).$$

Similarly to [RW07, Lemma 12] we have

$$\begin{aligned} T &= \frac{\tau_z(n_N)}{\sqrt{n_N^m q_N}} \square\left(n_N^m q_N\right) \delta\left(q^{(N)} \middle| n^{(N)m}\right) \prod_{p \mid q^{(N)}} \mu^{z, \nu_p(n)}_{\mathrm{Sym}^m, \mathrm{Sym}^{\nu_p(q)}} \\ &\quad + O\left(\frac{\tau(N)^2 \log_2 N}{N} n^{m/4} q^{1/4} \tau(q) \log(Nnq) \tau_{(m+1)|z|}(n)\right). \end{aligned} \tag{22}$$

From (19), (21) and (22) we deduce

$$\sum_{f \in \mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac{1}{2}, f\right)^2 \omega^z_{\mathrm{Sym}^m f}(x) = P + E$$

where P is the announced principal term and

$$E = \frac{\tau(N)^2 \log_2 N}{N} \sum_q \frac{\tau(q)^2}{q^{1/4}} \log(Nq) V_N\left(\frac{q}{N}\right) \sum_n \frac{\tau_{(m+1)|z|}(n) \log n}{n^{1-m/4}} \mathrm{e}^{-n/x}. \tag{23}$$

We proved in [RW07, Proof of Lemma 16] that the summation over n is

$$\sum_n \ll x^{m/4} (\log x)^{z_m+1} (z_m + m + 1)!. \tag{24}$$

Moreover, by (20) and since

$$\sum_q \frac{\tau(q)^2 \log(Nq)}{q^s} = \left[\log(N) - \frac{\mathrm{d}}{\mathrm{d}s}\right] \frac{\zeta^4(s)}{\zeta(2s)}$$

we get, after having moved the integration line in V_N from (2) to (7/10) and crossed a pole at $w = 3/4$ the majoration

$$\sum_q \frac{\tau(q)^2 \log(Nq)}{q^{1/4}} V_N\left(\frac{q}{N}\right) \ll N^{3/4} \log N. \tag{25}$$

The announced error term is a consequence of (23) with (24) and (25). □

We study the principal term exhibited in Lemma 3.1 in the following lemma.

Lemma 3.2. *For any squarefree integer N, any $z \in \mathbb{C}$ and any $x \in \mathbb{R}$ such that*

$$\frac{1}{100m} \log N \leqslant \log x \leqslant \frac{1}{12} \log N$$

we have

$$\begin{aligned}
&\sum_{q \geqslant 1} \frac{\tau(q)}{\sqrt{q}} V_N\left(\frac{q}{N}\right) \sum_{n \geqslant 1} \frac{\mathrm{e}^{-n/x}}{n} \tau_z(n_N) \left(\prod_{p \mid n^{(N)}} \mu^{z, \nu_p(n)}_{\mathrm{Sym}^m, \mathrm{Sym}^{\nu_p(q)}}\right) \delta\left(q^{(N)} \middle| n^{(N)m}\right) \frac{\square(n_N^m q_N)}{\sqrt{n_N^m q_N}} \\
&\quad = \frac{\varphi(N)}{N} A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m; N\right) \left(\frac{1}{2} \log N + \gamma + \frac{\gamma'_\infty}{\gamma_\infty}\left(\frac{1}{2}\right) + \sum_{p \mid N} \frac{\log p}{p-1}\right) \\
&\qquad + \frac{1}{2} B^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m; N\right) + O(\mathrm{Err})
\end{aligned}$$

where

$$\mathrm{Err} = \exp\left(c\left[\log_2 N - \frac{\log N}{\log(z_m+3)} + (z_m+3)\log(z_m+3)\right]\right).$$

Proof. We write Σ for the sum to be evaluated:

$$\Sigma = \frac{1}{(2\pi \mathrm{i})^2} \int_{(1)} \int_{(1)} N^w \zeta^{(N)}(1+2w) \left(\frac{\gamma_\infty(1/2+w)}{\gamma_\infty(1/2)}\right)^2 H_N^z(w,s) G(w) \frac{\mathrm{d}w}{w} \Gamma(s) x^s \,\mathrm{d}s \tag{26}$$

with

$$H_N^z(w,s)=\sum_q \frac{\tau(q)}{q^{w+1/2}q_N^{1/2}}\sum_n \frac{\tau_z(n_N)}{n^{s+1}n_N^{m/2}}\delta\big(q^{(N)}\big|n^{(N)m}\big)\,\square\big(n_N^m q_N\big)\prod_{p\mid q^{(N)}}\mu^{z,\nu_p(n)}_{\mathrm{Sym}^m,\mathrm{Sym}^{\nu_p(q)}}.$$

Writing $a=n^{(N)}$, $b=n_N$, $c=q^{(N)}$ and $d=q_N$ we have $H_N^z(w,s)=AB$ where

$$\begin{aligned}A&=\sum_{b\mid N^\infty}\frac{\tau_z(b)}{b^{1+m/2+s}}\sum_{d\mid N^\infty}\frac{\tau(d)}{d^{w+1}}\,\square\big(db^m\big)\\&=\prod_{p\mid N}\sum_{u\geqslant 0}\frac{\tau(p^u)}{p^{u(w+1)}}\sum_{\substack{v\geqslant 0\\ u\equiv mv \pmod 2}}\frac{\tau_z(p^v)}{p^{(s+1+m/2)v}}=C^z\left(w,s;\frac{1}{p}\right)\end{aligned}$$

by (11) and

$$\begin{aligned}B&=\sum_{(a,N)=1}\frac{1}{a^{s+1}}\sum_{c\mid a^m}\frac{\tau(c)}{c^{w+1/2}}\prod_{p\mid c}\mu^{z,\nu_p(a)}_{\mathrm{Sym}^m,\mathrm{Sym}^{\nu_p(c)}}\\&=\prod_{p\nmid N}\sum_{u\geqslant 0}\frac{\tau(p^u)}{p^{(1/2+w)u}}\sum_{v\geqslant 0}\frac{\mu^{z,v}_{\mathrm{Sym}^m,\mathrm{Sym}^u}}{p^{(1+s)v}}\\&=F^z\left(w,s;\frac{1}{p}\right)\end{aligned}$$

by (12). (Recall that $\mu^{z,v}_{\mathrm{Sym}^m,\mathrm{Sym}^u}$ vanishes when $u>mv$.)

In (26) we shift the w-contour to $\Re w=-1/6$ encountering a simple pole at 0 and obtain

$$\Sigma=P+\frac{1}{2\pi\mathrm{i}}\int_{(1)}\Sigma^-(s)\Gamma(s)x^s\,\mathrm{d}s \tag{27}$$

with

$$\begin{aligned}P=\frac{\varphi(N)}{N}\frac{1}{2\pi\mathrm{i}}\int_{(1)}\bigg[&\left(\frac{1}{2}\log N+\gamma+\sum_{p\mid N}\frac{\log p}{p-1}+\frac{\gamma_\infty'(1/2)}{\gamma_\infty(1/2)}\right)H_N^z(0,s)\\&+\frac{1}{2}\frac{\partial}{\partial w}_{\big|(0,s)}H_N^z(w,s)\bigg]\Gamma(s)x^s\,\mathrm{d}s.\end{aligned}$$

We bound $|\Sigma^-|$ as follows. We use Lemma 2.1 choosing $\sigma=2$ and $r=5/6$ in (a), $r=1/3$ in (b) to get

$$\big|\Sigma^-(s)\big|\ll N^{-1/6}\exp\left[c\left(\frac{(\log N)^{1/3}}{\log_2 N}+z_m\right)\right]$$

hence

$$\Sigma=P+O\left\{xN^{-1/6}\exp\left[c\left(\frac{(\log N)^{1/3}}{\log_2 N}+z_m\right)\right]\right\}.$$

We now treat the integral in the defining expression for P. For this, we replace the segment $[1 - i\log^2 x, 1 + i\log^2 x]$ by the union of the three segments $[1 - i\log^2 x, -\sigma - i\log^2 x]$, $[-\sigma - i\log^2 x, -\sigma + i\log^2 x]$, $[-\sigma + i\log^2 x, 1 + i\log^2 x]$ with $\sigma = 1/\log(|z| + 3)$. We shall show that the residue Res of the pole of Γ at 0 provides the main contribution whereas the integral on the new contour enters the error term.

We write

$$P - \mathrm{Res} = A_0 + A_1 + A_2 + B_0 + B_1 + B_2 \tag{28}$$

where

$$\mathrm{Res} = \frac{\varphi(N)}{N}\left(\frac{\log N}{2} + \gamma + \sum_{p|N}\frac{\log p}{p-1} + \frac{\gamma'_\infty}{\gamma_\infty}\left(\frac{1}{2}\right)\right)H_N^z(0,0) + \frac{\varphi(N)}{2N}\frac{\partial}{\partial w}_{|(0,0)}H_N^z(w,s),$$

$$A_0 = \frac{\varphi(N)}{N}\left(\frac{\log N}{2} + \gamma + \sum_{p|N}\frac{\log p}{p-1} + \frac{\gamma'_\infty}{\gamma_\infty}\left(\frac{1}{2}\right)\right)\frac{1}{2\pi \mathrm{i}}\int_{1\pm i\log^2 x}^{1\pm i\infty} H_N^z(0,s)\Gamma(s)x^s\,\mathrm{d}s,$$

$$B_0 = \frac{\varphi(N)}{2N}\frac{1}{2\pi \mathrm{i}}\int_{1\pm i\log^2 x}^{1\pm i\infty}\frac{\partial}{\partial w}_{|(0,s)}H_N^z(w,s)\Gamma(s)x^s\,\mathrm{d}s,$$

and A_1 (resp. B_1) has the same integrand as A_0 (resp. B_0) but the contour is $[1 - i\log^2 x, -\sigma - i\log^2 x]$ and A_2 (resp. B_2) has the same integrand as A_0 (resp. B_0) but the contour is $[-\sigma - i\log^2 x, -\sigma + i\log^2 x]$.

From Lemma 2.1(a) and (b) and the Stirling formula [IK04, (5.113)] we have

$$A_0 \ll \frac{\varphi(N)\log N}{N}\mathrm{e}^{-\log^2 x + c(z_m+3)}. \tag{29}$$

From Lemma 2.1(a) and (c) and the Stirling formula we have

$$A_1 \ll \frac{\varphi(N)\log N}{N}\mathrm{e}^{-\log^2 x + c(z_m+3)\log_2(z_m+3)} \tag{30}$$

and

$$A_2 \ll \frac{\varphi(N)\log N}{N}\exp\left(-\frac{\log x}{\log(z_m+3)}\right)\mathrm{e}^{c(z_m+3)\log_2(z_m+3)}. \tag{31}$$

The contribution of B_0, B_1 and B_2 are easily seen to be dominated by the ones of A_0, A_1 and A_2 thanks to Cauchy integral formula. Reporting (29)–(31) in (28) and the result in (27) we obtain that Σ is the announced principal term (the residue Res) up to an error term

$$\ll \exp\left(c\left(-\frac{\log N}{\log(z_m+3)} + (z_m+3)\log(z_m+3) + \log_2 N\right)\right).$$

This completes the proof. □

We have now the ingredients to prove Theorem A. As in [RW07, p. 743] we have

$$\sum_{f\in \mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac{1}{2}, f\right)^2 L\left(1, \mathrm{Sym}^m f\right)^z = \sum_{f\in \mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac{1}{2}, f\right)^2 \omega^z_{\mathrm{Sym}^m f}(x) + O(\mathrm{Err}) \tag{32}$$

where

$$\begin{aligned}\mathrm{Err} &= x^{-1/\log_2 N} \mathrm{e}^{D|z|\log_3 N} \log^4 N + \mathrm{e}^{D|z|\log_2 N - \frac{1}{2}\log^2 N} + N^{-1/4}\log^{D|z|} N \\ &\ll \exp\left(D|z|\log_2 N - \alpha \frac{\log N}{\log_2 N}\right)\end{aligned}$$

by setting $x = N^\alpha$. We have also used

$$\sum_{f\in \mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac{1}{2}, f\right)^2 \ll \log N$$

which follows from (21) and Petersson trace formula [ILS00, Corollary 2.10] or [RW07, Lemma 10].

Reporting Lemmas 3.2, 3.1 in (32) and assuming

$$|z| \leqslant \varepsilon \frac{\log N}{\log_2 N \log_3 N}$$

for $\varepsilon > 0$ small enough (regarding to α) we obtain the theorem.

3.2. *Moments for levels without small prime factors*

Corollary B is a consequence of the following lemma.

Lemma 3.3. *We have*

$$\frac{\varphi(N)}{N} A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m, N\right) = A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right)\left[1 + o_m(1)\right]$$

and

$$\begin{aligned}\frac{\varphi(N)}{N} B^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m, N\right) &= B^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right)\left[1 + o_m(1)\right] \\ &\quad + A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right) o_m(1)\end{aligned}$$

uniformly for

$$\begin{cases} N \in \mathscr{N}(\log^2), \\ |z| \ll_m \dfrac{\log N}{\log_2 N \log_3 N}. \end{cases} \tag{33}$$

Proof. To prove the first equality, we write

$$\frac{\varphi(N)}{N}A^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^m,N\right)=A^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^m\right)\frac{E_1(N)}{E_2(N)}\tag{34}$$

with

$$E_1(N)=\prod_{p|N}C^z\left(0,0;\frac{1}{p}\right),$$

$$E_2(N)=\prod_{p|N}\int_{\mathrm{SU}(2)}D\left(p^{-1/2},\mathrm{St},g\right)^2 D\left(p^{-1},\mathrm{Sym}^m,g\right)^z\,\mathrm{d}g.$$

First, we deal with $E_1(N)$. For m even we have

$$\begin{aligned}E_1(N)&=\left(1+O\left(\frac{\omega(N)}{P^-(N)^2}\right)\right)\left(1+O\left(\frac{(|z|+1)\omega(N)}{P^-(N)^{1+m/2}}\right)\right)\\&=1+O\left(\frac{(|z|+1)\omega(N)}{P^-(N)^{\min(2,1+m/2)}}\right)\end{aligned}$$

as soon as the function inside the error term is bounded. If m is odd then

$$\begin{aligned}C^z\left(0,0;\frac{1}{p}\right)&=\frac{1}{2}\left(1+\frac{2}{p}+O\left(\frac{1}{p^2}\right)\right)\left(1+\frac{z}{p^{1+m/2}}+O\left(\frac{(|z|+1)^2}{p^{2+m}}\right)\right)\\&\quad+\frac{1}{2}\left(1-\frac{2}{p}+O\left(\frac{1}{p^2}\right)\right)\left(1-\frac{z}{p^{1+m/2}}+O\left(\frac{(|z|+1)^2}{p^{2+m}}\right)\right)\\&=1+O\left(\frac{(|z|+1)^2}{p^{2+m/2}}\right)\end{aligned}$$

so that

$$E_1(N)=1+O\left(\frac{(|z|+1)^2\omega(N)}{P^-(N)^{2+m/2}}\right).\tag{35}$$

From (35) we deduce that

$$E_1(N)=1+o_m(1)\tag{36}$$

if N and z satisfy (33).

To study $E_2(N)$ we define

$$\begin{aligned}e(z,p)&=\int_{\mathrm{SU}(2)}D\left(p^{-1/2},\mathrm{St},g\right)^2 D\left(p^{-1},\mathrm{Sym}^m,g\right)^z\mathrm{d}g\\&=\sum_{\nu_1=0}^{+\infty}p^{-\nu_1}\sum_{\nu_2=0}^{+\infty}p^{-\nu_2/2}\sum_{u=0}^{\min(m\nu_1,\nu_2)}\mu^{z,\nu_1}_{\mathrm{Sym}^m,\mathrm{Sym}^u}\mu^{2,\nu_2}_{\mathrm{St},\mathrm{Sym}^u}\end{aligned}\tag{37}$$

by orthogonality. Using (8) and (9) we compute the contribution of $\nu_1=1$ and $\nu_2=2$ to (37) and with (10) we obtain

$$\begin{aligned}\left|e(z,p)-1\right| &\leqslant \sum_{\nu_2=2}^{+\infty}\binom{3+\nu_2}{\nu_2}\frac{1}{p^{\nu_2/2}}+\frac{|z|}{p}\sum_{\nu_2=m}^{+\infty}\binom{3+\nu_2}{\nu_2}\frac{1}{p^{\nu_2/2}}\\ &\quad+\sum_{\nu_1=2}^{+\infty}\binom{(m+1)|z|+\nu_1-1}{\nu_1}\frac{1}{p^{\nu_1}}\sum_{\nu_2=0}^{+\infty}\binom{3+\nu_2}{\nu_2}\frac{1}{p^{\nu_2/2}}\\ &\ll_m \frac{1}{p}+\frac{|z|}{p^{1+m/2}}+\frac{|z|(|z|+1)}{p^2}.\end{aligned}$$

It follows that

$$E_2(N)=1+O\left(\frac{\omega(N)}{P^-(N)}\left(1+\frac{|z|}{P^-(N)^{m/2}}+\frac{(|z|+1)^2}{P^-(N)}\right)\right)=1+o_m(1) \tag{38}$$

if N and z satisfy (33). The first result of the lemma follows from (34), (36) and (38).

We consider now $B^{2,z}(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^m,N)$. We begin in considering

$$F_N^z(w,0)=\prod_{\substack{p\in\mathscr{P}\\ p\nmid N}}F^z\left(w,0;\frac{1}{p}\right)\prod_{\substack{p\in\mathscr{P}\\ p\mid N}}C^z\left(w,0;\frac{1}{p}\right)$$

with enough uniformity in some fixed neighborhood of w to be authorized to apply Cauchy integral formula. We write $F_N^z(w,0)=F_1^z(w,0)Q_N(w)$ with

$$Q_N(w)=Q_N^{(1)}(w)/Q_N^{(2)}(w)$$

and

$$Q_N^{(1)}(w)=\prod_{p\mid N}C^z\left(w,0,\frac{1}{p}\right),\qquad Q_N^{(2)}(w)=\prod_{p\mid N}F^z\left(w,0,\frac{1}{p}\right).$$

As for $E_1(N)$ and $E_2(N)$ we compute

$$Q_N^{(1)}(w)=1+O_\varepsilon\left(\frac{\omega(N)}{P^-(N)^{1-\varepsilon}}\left(1+\frac{|z|}{P^-(N)^{m/2+\varepsilon}}\right)\right) \tag{39}$$

and

$$\frac{N}{\varphi(N)}Q_N^{(2)}(w)=1+O_\varepsilon\left(\frac{\omega(N)}{P^-(N)^{1-2\varepsilon}}\left(1+\frac{|z|}{P^-(N)^{1/2+\varepsilon}}+\frac{(|z|+1)^2}{P^-(N)^{1+2\varepsilon}}\right)\right) \tag{40}$$

the constant implied by the error term being independent of w such that $\Re w>-\varepsilon$. It follows in particular that

$$Q_N(0)=1+o_m(1) \tag{41}$$

if N and z satisfy (33). Denote $C(0,\varepsilon)$ the circle of center 0 and radius ε. We have

$$\frac{\mathrm{d}}{\mathrm{d}w}_{|w=0} F_N^z(w,0) = \frac{\mathrm{d}}{\mathrm{d}w}_{|w=0} F_1^z(w,0) Q_N(0) + F_1^z(0,0)\cdot \frac{1}{2\pi \mathrm{i}} \int_{C(0,\varepsilon)} Q_N(w)\frac{\mathrm{d}w}{w^2} \tag{42}$$

and from the uniformity in w in (39) and (40) we deduce

$$\frac{1}{2\pi \mathrm{i}} \int_{C(0,\varepsilon)} Q_N(w)\frac{\mathrm{d}w}{w^2} = o(1). \tag{43}$$

Reporting (41) and (43) in (42) we obtain the second result of the lemma. □

4. Behavior for the asymptotic real moments

4.1. Behavior of the main term

The aim of this section is to prove Theorem C. In fact we shall establish a more general result (see Proposition 4.1 below). Write

$$D_m(\theta,t) := D\big(t,\mathrm{Sym}^m,g\big) = \prod_{j=0}^{m}\big(1-\mathrm{e}^{\mathrm{i}(m-2j)\theta}t\big)^{-1}, \tag{44}$$

and

$$F_m^{\ell,z}(w,s;t) := \big(1-t^{1+2w}\big)^{\frac{\ell(\ell-1)}{2}} \frac{2}{\pi}\int_0^{\pi} D_1\big(\theta,t^{1/2+w}\big)^{\ell} D_m\big(\theta,t^{1+s}\big)^{z} \sin^2\theta \,\mathrm{d}\theta$$

so that

$$F^z(w,s;t) = F_m^{2,z}(w,s;t).$$

Proposition 4.1. *Let $J \geqslant 1$, $\ell \geqslant 0$ and $m \geqslant 1$ be three fixed integers. Then we have*

$$\sum_{p\leqslant y} \log F_m^{\ell,z}\left(0,0;\frac{1}{p}\right) = z\left\{(m+1)\log_2 z + (m+1)\gamma + \sum_{j=1}^{J}\frac{a_j}{(\log z)^j} + O\left(\frac{1}{(\log z)^{J+1}}\right)\right\}$$

uniformly for $y \geqslant z^{3/2} \geqslant 10$, where γ is the Euler constant and a_j is defined as in Theorem C.

Since

$$A^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^m\right) = \prod_{p\in\mathscr{P}} F_m^{2,z}\left(0,0;\frac{1}{p}\right),$$

Theorem C is an immediate consequence of Proposition 4.1 by taking $\ell = 2$ and making $y \to +\infty$.

In order to prove this proposition, we first establish some preliminary lemmas.

Lemma 4.2. *Let $g_m(t)$ and $\widetilde{g}_m(t)$ be defined as in* (2) *and* (3)*. Then*

$$\widetilde{g}_m(t) \ll \begin{cases} t^2 & \text{if } 0 \leqslant t < 1, \\ \log(2t) & \text{if } t \geqslant 1, \end{cases}$$

and

$$\widetilde{g}'_m(t) \ll \begin{cases} t & \text{if } 0 \leqslant t < 1, \\ t^{-1} & \text{if } t \geqslant 1. \end{cases}$$

Proof. When $t \geqslant 0$, we can write

$$\mathrm{e}^{tX_m(2\cos\theta)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{t\sin((m+1)\theta)}{\sin\theta}\right)^n.$$

From this we deduce, for $0 \leqslant t < 1$,

$$\begin{aligned}\widetilde{g}_m(t) &= \log\left(1 + \sum_{n=2}^{\infty} \frac{t^n}{n!}\frac{2}{\pi}\int_0^{\pi}\left(\frac{\sin((m+1)\theta)}{\sin\theta}\right)^n \sin^2\theta\,\mathrm{d}\theta\right) \\ &= \log\bigl(1 + t^2 + O\bigl(t^3\bigr)\bigr) \asymp t^2\end{aligned}$$

and

$$\widetilde{g}'_m(t) \asymp t.$$

Let C_m be the maximum of $2X'_m(x)$ in $[-2, 2]$. Then, since $X_m(2) = m+1$, we have

$$0 \leqslant m + 1 - X_m(2\cos\theta) \leqslant C_m(1 - \cos\theta)$$

for every $\theta \in [0, \pi]$. Thus for $t \geqslant 1$, we have by (3) and (5),

$$\widetilde{g}'_m(t) = -\frac{\int_0^{\pi} \mathrm{e}^{tX_m(2\cos\theta)}(m + 1 - X_m(2\cos\theta))\sin^2\theta\,\mathrm{d}\theta}{\int_0^{\pi} \mathrm{e}^{tX_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta} \ll_m \bigl|\widetilde{h}_m(t)\bigr|.$$

Now (54) of Lemma 4.6 below implies $\widetilde{g}'_m(t) \ll t^{-1}$ for $t \geqslant 1$. From this we immediately deduce $\widetilde{g}_m(t) \ll \log(2t)$ for $t \geqslant 1$. □

Lemma 4.3. *Let $m \geqslant 1$ be a fixed integer. Then we have*

$$\int_0^{\pi} \mathrm{e}^{tX_m(2\cos\theta)}\cos\theta\sin^2\theta\,\mathrm{d}\theta \ll t\int_0^{\pi} \mathrm{e}^{tX_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta \tag{45}$$

and

$$\frac{2}{\pi}\int_0^{\pi} \mathrm{e}^{tX_m(2\cos\theta)}\cos^2\theta\sin^2\theta\,\mathrm{d}\theta = \left\{\frac{1}{4} + O(t)\right\}\frac{2}{\pi}\int_0^{\pi} \mathrm{e}^{tX_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta \tag{46}$$

uniformly for $t \geqslant 0$. The implied constants depend on m only.

Proof. First we note that these estimates are trivial for $t \geqslant 1$, so we suppose that $0 \leqslant t \leqslant 1$. In view of the following relations:

$$\frac{2}{\pi}\int_0^\pi \cos^n\theta \sin^2\theta \,\mathrm{d}\theta = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n = 0, \\ 2\frac{(2r-1)!!}{(2r+2)!!} & \text{if } n = 2r \end{cases}$$

(with $n!! := n\cdot(n-2)\cdots$) and $\mathrm{e}^{tX_m(2\cos\theta)} = 1 + O(t)$, it follows that

$$\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)}\cos\theta\sin^2\theta\,\mathrm{d}\theta \ll t\int_0^\pi |\cos\theta|\sin^2\theta\,\mathrm{d}\theta \ll t\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta.$$

Similarly

$$\frac{2}{\pi}\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)}\cos^2\theta\sin^2\theta\,\mathrm{d}\theta = \frac{1}{4} + O(t)$$

which implies (46). □

Lemma 4.4. *Let $\ell \geqslant 0$ and $m \geqslant 1$ be two fixed integers. Suppose $z \geqslant 4$ is real. Then we have*

$$\log F_m^{\ell,z}\left(0,0;\frac{1}{p}\right) = -(m+1)z\log\left(1-\frac{1}{p}\right) + O(\log z) \tag{47}$$

uniformly for $2 \leqslant p \leqslant \sqrt{z}$; and

$$\log F_m^{\ell,z}\left(0,0;\frac{1}{p}\right) = g_m\left(\frac{z}{p}\right) + O\left(\frac{z}{p^{3/2}}\right) \tag{48}$$

uniformly for $p \geqslant \sqrt{z} \geqslant 2$. The implied constants depend on ℓ and m only.

Proof. We have

$$\prod_{j=0}^m\left(1-\frac{\mathrm{e}^{\mathrm{i}(m-2j)\theta}}{p}\right) = \sum_{\nu=0}^{m+1}\frac{(-1)^\nu}{p^\nu}\sum_{0\leqslant j_1<\cdots<j_\nu\leqslant m}\mathrm{e}^{\mathrm{i}(\nu m-2j_1-\cdots-2j_\nu)\theta}.$$

Since the left-hand side is real and

$$\sum_{0\leqslant j_1<\cdots<j_\nu\leqslant m}\mathrm{e}^{\mathrm{i}(\nu m-2j_1-\cdots-2j_\nu)\theta} = 1 \quad (\nu = 0, m+1),$$

it follows that, with notation $\mathbf{j}_\nu = (j_1,\ldots,j_\nu)$ and $\ell_{\mathbf{j}_\nu}^m = \nu m - 2j_1 - \cdots - 2j_\nu$,

$$\prod_{j=0}^{m}\left(1-\frac{e^{i(m-2j)\theta}}{p}\right)=\sum_{\nu=0}^{m+1}\frac{(-1)^{\nu}}{p^{\nu}}\sum_{0\leqslant j_1<\cdots<j_\nu\leqslant m}\cos\bigl(\ell^m_{\mathbf{j}_\nu}\theta\bigr)$$
$$=\left(1-\frac{1}{p}\right)^{m+1}+\sum_{\nu=1}^{m}\frac{(-1)^{\nu-1}}{p^{\nu}}\sum_{0\leqslant j_1<\cdots<j_\nu\leqslant m}\bigl\{1-\cos\bigl(\ell^m_{\mathbf{j}_\nu}\theta\bigr)\bigr\}$$
$$=\left(1-\frac{1}{p}\right)^{m+1}+\sum_{\nu=1}^{m}\frac{(-1)^{\nu-1}}{p^{\nu}}\sum_{0\leqslant j_1<\cdots<j_\nu\leqslant m}2\sin^2\bigl(\ell^m_{\mathbf{j}_\nu}\theta/2\bigr).$$

Introducing the notation

$$\widetilde{D}_m\bigl(\theta,p^{-1}\bigr):=1+\left(1-\frac{1}{p}\right)^{-(m+1)}\sum_{\nu=1}^{m}\frac{(-1)^{\nu-1}}{p^{\nu}}\sum_{0\leqslant j_1<\cdots<j_\nu\leqslant m}2\sin^2\bigl(\ell^m_{\mathbf{j}_\nu}\theta/2\bigr),$$

we can write

$$\prod_{j=0}^{m}\left(1-\frac{e^{i(m-2j)\theta}}{p}\right)=\left(1-\frac{1}{p}\right)^{m+1}\widetilde{D}_m\bigl(\theta,p^{-1}\bigr)$$

and

$$F_m^{\ell,z}\left(0,0;\frac{1}{p}\right)=\left(1-\frac{1}{p}\right)^{-(m+1)z+\ell(\ell-1)/2}\check{F}_m^{\ell,z}(p)$$

with

$$\check{F}_m^{\ell,z}(p):=\frac{2}{\pi}\int_0^{\pi}D_1\bigl(\theta,p^{-1/2}\bigr)^{\ell}\widetilde{D}_m\bigl(\theta,p^{-1}\bigr)^{-z}\sin^2\theta\,d\theta.$$

Observing the nonnegativity of the integrand, we infer that for some suitably small positive constant δ,

$$\check{F}_m^{\ell,z}(p)\geqslant\frac{2}{\pi}\left(1-\frac{1}{\sqrt{2}}\right)^{2\ell}\int_0^{\delta\sqrt{p/z}}\left(1+\frac{c_m\theta^2}{p}\right)^{-z}\theta^2d\theta$$
$$\gg\int_0^{\delta\sqrt{p/z}}\left(1+\frac{c_m\delta^2}{z}\right)^{-z}\theta^2d\theta$$
$$\gg\left(1+\frac{c_m\delta^2}{z}\right)^{-z}\left(\frac{p}{z}\right)^{3/2}$$
$$\gg_m\left(\frac{p}{z}\right)^{3/2}$$

for $p\leqslant z$. On the other hand, it is obvious that

$$\big|\widetilde{D}_m\big(\theta, p^{-1}\big)^{-1}\big| \leqslant 1 \quad \text{and} \quad \check{F}_m^{\ell,z}(p) \ll 1$$

uniformly for $p \leqslant \sqrt{z}$. By combining these estimates, we find that

$$\log F_m^{\ell,z}\left(0, 0; \frac{1}{p}\right) = \log\left(1 - \frac{1}{p}\right)^{-(m+1)z+\ell(\ell-1)/2} + \log \check{F}_m^{\ell,z}(p)$$
$$= (m+1)z \log\left(1 - \frac{1}{p}\right)^{-1} + O(\log z)$$

for $p \leqslant \sqrt{z}$.

Next we prove (48). In view of (44) and (9), it is easy to see that

$$D_m\big(\theta, p^{-1}\big)^z = \mathrm{e}^{(z/p)X_m(2\cos\theta)}\left\{1 + O\left(\frac{z}{p^2}\right)\right\} \quad (p \geqslant \sqrt{z}), \tag{49}$$

where the implied constant depends on m at most. Thus for $p \geqslant \sqrt{z}$, we can write

$$F_m^{\ell,z}\left(0, 0; \frac{1}{p}\right) = \left\{1 + O\left(\frac{z}{p^2}\right)\right\}\left(1 - \frac{1}{p}\right)^{\ell(\ell-1)/2} \widetilde{F}_m^{\ell,z}(p)$$

with

$$\widetilde{F}_m^{\ell,z}(p) := \frac{2}{\pi}\int_0^\pi D_1\big(\theta, p^{-1/2}\big)^\ell \mathrm{e}^{(z/p)X_m(2\cos\theta)} \sin^2\theta \,\mathrm{d}\theta.$$

Since

$$D_1\big(\theta, p^{-1/2}\big)^\ell = 1 + \frac{2\ell\cos\theta}{p^{1/2}} + \frac{2(\ell+1)\ell\cos^2\theta - \ell}{p} + O\left(\frac{1}{p^{3/2}}\right) \tag{50}$$

where the implied constant depends on ℓ at most, (45) and (46) of Lemma 4.3 allow us to deduce that

$$\widetilde{F}_m^{\ell,z}(p) = \left\{1 + \frac{\ell(\ell-1)}{2p} + O\left(\frac{z}{p^{3/2}}\right)\right\}\frac{2}{\pi}\int_0^\pi \mathrm{e}^{(z/p)X_m(2\cos\theta)} \sin^2\theta \,\mathrm{d}\theta.$$

Inserting it into the preceding relation, we easily obtain (48). □

Now we are ready to prove Proposition 4.1. From (47) and (48), we deduce that for $y \geqslant z^{3/2}$,

$$\sum_{p\leqslant y} \log F_m^{\ell,z}\left(0, 0; \frac{1}{p}\right) = (m+1)z \sum_{p\leqslant\sqrt{z}} \log\left(1 - \frac{1}{p}\right)^{-1} + \sum_{\sqrt{z}<p\leqslant y} g_m\left(\frac{z}{p}\right) + O\left(\frac{z^{3/4}}{\log z}\right).$$

In view of (2), (3) and the following estimate

$$\sum_{\sqrt{z}<p\leqslant z}\left\{(m+1)z\log\left[\left(1-\frac{1}{p}\right)^{-1}\right]-(m+1)\frac{z}{p}\right\}\ll\frac{\sqrt{z}}{\log z},$$

the last asymptotic formula can be written as

$$\sum_{p\leqslant y}\log F_m^{\ell,z}\left(0,0;\frac{1}{p}\right)=(m+1)z\sum_{p\leqslant z}\log\left(1-\frac{1}{p}\right)^{-1}+\sum_{\sqrt{z}<p\leqslant y}\widetilde{g}_m\left(\frac{z}{p}\right)+O\left(\frac{z^{3/4}}{\log z}\right). \tag{51}$$

By the prime number theorem, it follows that

$$\sum_{\sqrt{z}<p\leqslant y}\widetilde{g}_m\left(\frac{z}{p}\right)=\int_{\sqrt{z}}^{y}\widetilde{g}_m\left(\frac{z}{u}\right)\mathrm{d}\sum_{p\leqslant u}1=\int_{\sqrt{z}}^{y}\frac{\widetilde{g}_m(z/u)}{\log u}\,\mathrm{d}u+R_1, \tag{52}$$

where

$$R_1:=\int_{\sqrt{z}}^{y}\widetilde{g}_m\left(\frac{z}{u}\right)\mathrm{d}O\left(u\mathrm{e}^{-2\sqrt{\log u}}\right).$$

In view of Lemma 4.2, a simple partial integration gives us

$$R_1\ll z\mathrm{e}^{-\sqrt{\log z}}.$$

In order to evaluate the last integral of (52), we use the change of variables $t=z/u$ to write

$$\begin{aligned}\int_{\sqrt{z}}^{y}\frac{\widetilde{g}_m(z/u)}{\log u}\,\mathrm{d}u&=z\int_{z/y}^{\sqrt{z}}\frac{\widetilde{g}_m(t)}{t^2\log(z/t)}\,\mathrm{d}t\\&=z\int_{1/\sqrt{z}}^{\sqrt{z}}\frac{\widetilde{g}_m(t)}{t^2\log(z/t)}\,\mathrm{d}t+O(R_2)\end{aligned}$$

where

$$R_2:=z\int_{z/y}^{1/\sqrt{z}}\frac{|\widetilde{g}_m(t)|}{t^2\log(z/t)}\,\mathrm{d}t\ll\frac{z^{1/2}}{\log z}$$

by using Lemma 4.2. On the other hand, we have

$$\int_{1/\sqrt{z}}^{\sqrt{z}} \frac{\widetilde{g}_m(t)}{t^2 \log(z/t)} \, \mathrm{d}t = \frac{1}{\log z} \int_{1/\sqrt{z}}^{\sqrt{z}} \frac{\widetilde{g}_m(t)}{t^2(1 - (\log t)/\log z)} \, \mathrm{d}t$$

$$= \sum_{j=1}^{J} \frac{1}{(\log z)^j} \int_{1/\sqrt{z}}^{\sqrt{z}} \frac{\widetilde{g}_m(t)}{t^2} (\log t)^{j-1} \, \mathrm{d}t + O_J\left(\frac{1}{(\log z)^{J+1}}\right).$$

Extending the interval of integration $[1/\sqrt{z}, \sqrt{z}]$ to $(0, \infty)$ and bounding the contributions of $(0, 1/\sqrt{z}]$ and $[\sqrt{z}, \infty)$ by using Lemma 4.2, we have

$$\int_{1/\sqrt{z}}^{\sqrt{z}} \frac{\widetilde{g}_m(t)}{t^2} (\log t)^{j-1} \, \mathrm{d}t = a_j + O\left(\frac{(\log z)^j}{\sqrt{z}}\right).$$

Combining these estimates, we find that

$$\sum_{\sqrt{z} < p \leqslant y} \widetilde{g}_m\left(\frac{z}{p}\right) = z\left\{\sum_{j=1}^{J} \frac{a_j}{(\log z)^j} + O_J\left(\frac{1}{(\log z)^{J+1}}\right)\right\}. \tag{53}$$

Now the desired result follows from (51), (53) and the prime number theorem in the form

$$\sum_{p \leqslant z} \log\left(1 - \frac{1}{p}\right)^{-1} = \log_2 z + \gamma + O\left(\mathrm{e}^{-2\sqrt{\log z}}\right).$$

This completes the proof. □

4.2. Behavior of the constant term

The aim of this section is to prove Theorem D. We shall prove a slightly more general result, i.e. Proposition 4.5. Clearly Theorem D is its simple consequence with the choice of $\ell = 2$.

Let $\ell \geqslant 0$ and $m \geqslant 1$ be two fixed integers. Define

$$B_m(w) = B_m(w, z, p) := \frac{2}{\pi} \int_0^{\pi} D_1\left(\theta, p^{-(1/2+w)}\right)^{\ell} D_m\left(\theta, p^{-1}\right)^{z} \sin^2\theta \, \mathrm{d}\theta$$

so that

$$F_m^{\ell,z}\left(w, 0; p^{-1}\right) = \left(1 - p^{-(1+2w)}\right)^{\ell(\ell-1)/2} B_m(w).$$

Proposition 4.5. *Let $\ell \geqslant 0$. We have*

$$\sum_{p \leqslant y} \frac{\mathrm{d}}{\mathrm{d}w}_{|w=0} \log F_m^{\ell,z}\left(w, 0; \frac{1}{p}\right) \ll \log z$$

uniformly for $y \geqslant z \geqslant 10$ if m is even; and

$$\sum_{p\leqslant y} \frac{\mathrm{d}}{\mathrm{d}w}_{|w=0} \log F_m^{\ell,z}\left(w,0;\frac{1}{p}\right) = \sqrt{z}\left\{b_{\ell,m} + O\left(\mathrm{e}^{-\sqrt{\log z}}\right)\right\}$$

uniformly for $y \geqslant z\mathrm{e}^{2\sqrt{\log z}} \geqslant 10$ *if* m *is odd, where*

$$b_{\ell,m} := -2\ell\left(2 + \int_0^{+\infty} \frac{\widetilde{h}_m(t)}{t^{3/2}}\,\mathrm{d}t\right).$$

The implied constant depends on ℓ *and* m *only.*

We need preliminary lemmas.

Lemma 4.6. *Let* $h_m(t)$ *and* $\widetilde{h}_m(t)$ *be defined as in* (4) *and* (5)*. Then*

$$\widetilde{h}_m(t) \ll \begin{cases} t & \text{if } 0 \leqslant t < 1,\\ t^{-1} & \text{if } t \geqslant 1,\end{cases} \qquad \widetilde{h}'_m(t) \ll \begin{cases} 1 & \text{if } 0 \leqslant t < 1,\\ t^{-1} & \text{if } t \geqslant 1.\end{cases} \tag{54}$$

Further if m *is even, then*

$$h_m(t) = 0 \quad (t \geqslant 0). \tag{55}$$

Proof. Eq. (55) follows from

$$h_m(t) = \int_{-\pi/2}^{\pi/2} \mathrm{e}^{tX_m(2\cos\theta)} \cos\theta \sin^2\theta \,\mathrm{d}\theta \quad (m \text{ even})$$

by parity. The estimates of (54) with $0 \leqslant t \leqslant 1$ are equivalent to (45). Next we prove $\widetilde{h}_m(t) \ll t^{-1}$ for $t \geqslant 1$, i.e.

$$\frac{\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)}(1-\cos\theta)\sin^2\theta\,\mathrm{d}\theta}{\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta} \ll \frac{1}{t}. \tag{56}$$

From the power series expansion, we have

$$X_m(2\cos\theta) = (m+1) - \frac{m(m+1)(m+2)}{6}\theta^2 + O_m\left(\theta^4\right),$$

and hence there exists $\delta = \delta_m \in (0, \pi/(3(m+1)))$ such that for all $0 \leqslant \theta \leqslant \delta$,

$$(m+1) - \frac{(m+2)^3}{6}\theta^2 < X_m(2\cos\theta) < (m+1) - \frac{1}{6}\theta^2. \tag{57}$$

Since $\theta \mapsto X_m(2\cos\theta)$ is continuous on the compact $[\delta, 2]$ where its values are strictly less than $m+1$, there exists $\alpha_m \in (0, m+1)$ such that

$$\left|X_m(2\cos\theta)\right| \leqslant \alpha_m \quad (\delta \leqslant \theta \leqslant \pi/2). \tag{58}$$

We give a lower bound to the denominator of the fraction in (56). As the integrand is nonnegative, we infer from (57) that

$$\int_0^{\pi} e^{tX_m(2\cos\theta)} \sin^2\theta\, d\theta \geqslant \int_0^{\delta} e^{tX_m(2\cos\theta)} \sin^2\theta\, d\theta$$
$$\gg e^{(m+1)t} \int_0^{\delta} e^{-c_m t\theta^2} \theta^2\, d\theta \gg_m \frac{e^{(m+1)t}}{t^{3/2}} \tag{59}$$

where the implied constant in $\gg_m$ depends on m only. For the numerator in the left-hand side of (56), we write

$$\int_0^{\pi} e^{tX_m(2\cos\theta)}(1-\cos\theta)\sin^2\theta\, \mathrm{d}\theta = \int_0^{\pi/2} e^{tX_m(2\cos\theta)}(1-\cos\theta)\sin^2\theta\, \mathrm{d}\theta$$
$$+ \int_0^{\pi/2} e^{-tX_m(2\cos\theta)}(1+\cos\theta)\sin^2\theta\, \mathrm{d}\theta.$$

Since $X_m(2\cos\theta) \geqslant 0$ for $\theta \in [0, \pi/(2(m+1))]$, we deduce with (58) that

$$\int_0^{\pi/2} e^{-tX_m(2\cos\theta)}(1+\cos\theta)\sin^2\theta\, \mathrm{d}\theta \ll \int_0^{\pi/(2(m+1))} \mathrm{d}\theta + \int_{\pi/(2(m+1))}^{\pi/2} e^{t\alpha_m}\, \mathrm{d}\theta \ll e^{\alpha_m t},$$

which is negligible in comparison with (59). Splitting at $\theta = \delta$ and applying (57) and (58), we have

$$\int_0^{\pi/2} e^{tX_m(2\cos\theta)}(1-\cos\theta)\sin^2\theta\, \mathrm{d}\theta \ll e^{(m+1)t} \int_0^{\delta} e^{-\frac{1}{6}t\theta^2}\theta^4 \mathrm{d}\theta + \int_{\delta}^{\pi/2} e^{\alpha_m t}\, \mathrm{d}\theta$$
$$\ll t^{-5/2} e^{(m+1)t} + e^{\alpha_m t}.$$

The desired estimate in (56) follows with (59) and the fact $\alpha_m < m+1$.

A direct differentiation shows that

$$\widetilde{h}'_m(t) = \frac{\int_0^{\pi} e^{tX_m(2\cos\theta)} X_m(2\cos\theta)\sin^2\theta\, \mathrm{d}\theta \int_0^{\pi} e^{tX_m(2\cos\theta)}(1-\cos\theta)\sin^2\theta\, \mathrm{d}\theta}{(\int_0^{\pi} e^{tX_m(2\cos\theta)}\sin^2\theta\, \mathrm{d}\theta)^2}$$
$$- \frac{\int_0^{\pi} e^{tX_m(2\cos\theta)} X_m(2\cos\theta)(1-\cos\theta)\sin^2\theta\, \mathrm{d}\theta}{\int_0^{\pi} e^{tX_m(2\cos\theta)}\sin^2\theta\, \mathrm{d}\theta} \quad (t \geqslant 1).$$

Using the nonnegativity, we see that

$$\int_0^{\pi} e^{tX_m(2\cos\theta)} X_m(2\cos\theta)\sin^2\theta\, \mathrm{d}\theta \ll \int_0^{\pi} e^{tX_m(2\cos\theta)}\sin^2\theta\, \mathrm{d}\theta,$$

and

$$\int_0^\pi e^{tX_m(2\cos\theta)} X_m(2\cos\theta)(1-\cos\theta)\sin^2\theta\,d\theta \ll \int_0^\pi e^{tX_m(2\cos\theta)}(1-\cos\theta)\sin^2\theta\,d\theta.$$

Therefore (56) implies $\widetilde{h}'_m(t) \ll t^{-1}$ for $t \geqslant 1$. □

Lemma 4.7. *Let $\ell \geqslant 0$ and $m \geqslant 1$ be two fixed integers. Then we have*

$$\frac{B'_m(0)}{B_m(0)} \ll \frac{\log p}{p^{1/2}} \tag{60}$$

uniformly for all p and $z \geqslant 1$; and

$$\frac{B'_m(0)}{B_m(0)} = -2\ell\frac{\log p}{p^{1/2}} h_m\left(\frac{z}{p}\right) - \ell(\ell-1)\frac{\log p}{p} + O\left(\frac{\log p}{p^{3/2}} + \frac{z\log p}{p^2}\right) \tag{61}$$

uniformly for $p \geqslant z^{2/3}$. The implied constants depend on ℓ and m only.

Proof. We have

$$B'_m(0) = -2\ell\frac{2}{\pi}\int_0^\pi D_1\big(\theta, p^{-1/2}\big)^{\ell+1}\left(\frac{\cos\theta}{p^{1/2}} - \frac{1}{p}\right)(\log p) D_m\big(\theta, p^{-1}\big)^z \sin^2\theta\,d\theta \tag{62}$$

hence

$$B'_m(0) \ll \frac{\log p}{p^{1/2}}\int_0^\pi D_m\big(\theta, p^{-1}\big)^z \sin^2\theta\,d\theta.$$

This implies (60), since

$$B_m(0) = \left\{1 + O\left(\frac{1}{p^{1/2}}\right)\right\}\frac{2}{\pi}\int_0^\pi D_m\big(\theta, p^{-1}\big)^z \sin^2\theta\,d\theta.$$

In view of (50), it follows that

$$D_1\big(\theta, p^{-1/2}\big)^{\ell+1}\left(\frac{\cos\theta}{p^{1/2}} - \frac{1}{p}\right) = \frac{\cos\theta}{p^{1/2}} + \frac{2(\ell+1)\cos^2\theta - 1}{p} + O\left(\frac{1}{p^{3/2}}\right). \tag{63}$$

By using it, (49) and (46) of Lemma 4.3, we can deduce, for $p \geqslant \sqrt{z}$,

$$\begin{aligned} B'_m(0) = {} & -2\ell\frac{\log p}{p^{1/2}}\frac{2}{\pi}\int_0^\pi e^{(z/p)X_m(2\cos\theta)}\cos\theta\sin^2\theta\,d\theta \\ & - \left\{\ell(\ell-1)\frac{\log p}{p} + O\left(\frac{\log p}{p^{3/2}} + \frac{z\log p}{p^2}\right)\right\}\frac{2}{\pi}\int_0^\pi e^{(z/p)X_m(2\cos\theta)}\sin^2\theta\,d\theta. \end{aligned}$$

Under the same condition, thanks to (49) and (45), we have

$$B_m(0)=\left\{1+O\left(\frac{1}{p}+\frac{z}{p^{3/2}}\right)\right\}\frac{2}{\pi}\int_0^\pi \mathrm{e}^{(z/p)X_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta.$$

Combining these, we obtain (61). □

Lemma 4.8. *Let $\ell \geqslant 0$ and $m \equiv 0 \pmod 2$ be two fixed integers. Then we have*

$$\frac{B'_m(0)}{B_m(0)} \ll \frac{\log p}{p} \tag{64}$$

uniformly for all p and $z \geqslant 1$, and

$$\frac{B'_m(0)}{B_m(0)} = -\ell(\ell-1)\frac{\log p}{p}+O\left(\frac{\log p}{p^{3/2}}+\frac{z\log p}{p^2}\right) \tag{65}$$

uniformly for $p \geqslant z^{2/3}$. The implied constants depend on ℓ and m only.

Proof. Eq. (64) follows from (62) and (63) since, by parity consideration we have

$$\int_0^\pi (\cos\theta)D_m\big(\theta,p^{-1}\big)^z\sin^2\theta\,\mathrm{d}\theta=0.$$

Eq. (65) is an immediate consequence of (61) since $h_m(t)=0$ when m is even. □

Now we are ready to prove Proposition 4.5. If m is even, we apply Lemma 4.8 to

$$\sum_{p\leqslant y}\frac{\mathrm{d}}{\mathrm{d}w}_{|w=0}\log F_m^{\ell,z}\left(w,0;\frac{1}{p}\right)=\sum_{p\leqslant y}\ell(\ell-1)\frac{\log p}{p-1}+\sum_{p\leqslant y}\frac{B'_m(0)}{B_m(0)}$$

and obtain

$$\begin{aligned}&\sum_{p\leqslant y}\frac{\mathrm{d}}{\mathrm{d}w}_{|w=0}\log F_m^{\ell,z}\left(w,0;\frac{1}{p}\right)\\&\quad=\sum_{p\leqslant z}\ell(\ell-1)\frac{\log p}{p-1}+\sum_{p\leqslant z}\frac{B'_m(0)}{B_m(0)}\\&\qquad+\sum_{z<p\leqslant y}\left\{\ell(\ell-1)\left(\frac{\log p}{p-1}-\frac{\log p}{p}\right)+O\left(\frac{\log p}{p^{3/2}}+\frac{z\log p}{p^2}\right)\right\}\ll\log z.\end{aligned}$$

When m is odd, by using (60) of Lemma 4.7 for $p\leqslant z^{2/3}$ and (61) for $z^{2/3}<p\leqslant y$, we obtain

$$\sum_{p\leqslant y}\frac{\mathrm{d}}{\mathrm{d}w}_{|w=0}\log F_m^{\ell,z}\left(w,0;\frac{1}{p}\right)=-2\ell\sum_{z^{2/3}<p\leqslant y}\frac{\log p}{p^{1/2}}h_m\left(\frac{z}{p}\right)+O\big(z^{1/3}\log z\big)$$

so that

$$\sum_{p\leqslant y} \frac{\mathrm{d}}{\mathrm{d}w}_{|w=0} \log F_m^{\ell,z}\left(w, 0; \frac{1}{p}\right)$$

$$= -2\ell\left\{\sum_{p\leqslant z} \frac{\log p}{p^{1/2}} + \sum_{z^{2/3}<p\leqslant y} \frac{\log p}{p^{1/2}} \widetilde{h}_m\left(\frac{z}{p}\right)\right\} + O\left(z^{1/3}\log z\right). \tag{66}$$

By using the prime number theorem, it follows by integration by parts that

$$\sum_{z^{2/3}<p\leqslant y} \frac{\log p}{p^{1/2}} \widetilde{h}_m\left(\frac{z}{p}\right) = \int_{z^{2/3}}^{y} \frac{\widetilde{h}_m(z/u)}{u^{1/2}}\,\mathrm{d}u + O\left(\sqrt{z}\mathrm{e}^{-\sqrt{\log z}}\right)$$

$$= \sqrt{z}\int_0^{+\infty} \frac{\widetilde{h}_m(t)}{t^{3/2}}\,\mathrm{d}t + O\left(\sqrt{z}\mathrm{e}^{-\sqrt{\log z}}\right)$$

with the help of Lemma 4.6, provided $y \geqslant z\mathrm{e}^{2\sqrt{\log z}}$. Combining these yields

$$\sum_{z^{2/3}<p\leqslant y} \frac{\log p}{p^{1/2}} \widetilde{h}_m\left(\frac{z}{p}\right) = \sqrt{z}\int_0^{+\infty} \frac{\widetilde{h}_m(t)}{t^{3/2}}\,\mathrm{d}t + O\left(\sqrt{z}\mathrm{e}^{-\sqrt{\log z}}\right). \tag{67}$$

Now the required result is a simple consequence of (66) and (67) and the prime number theorem.

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