On a Conjecture of Montgomery-Vaughan on Extreme Values of Automorphic *L*-Functions at 1

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ABSTRACT. In this paper, we prove a weaker form of a conjecture of Montgomery–Vaughan on extreme values of automorphic *L*-functions at 1.

1. Introduction

The automorphic *L*-functions constitute a powerful tool for studying arithmetic, algebraic or geometric objects. For squarefree integer N and even integer k, denote by $\mathrm{H}_k^*(N)$ the set of all newforms of level N and of weight k. It is known that

$$(1.1) \qquad |\,\mathrm{H}_k^*(N)| = \frac{k-1}{12}\varphi(N) + O\big((kN)^{2/3}\big),$$

where $\varphi(N)$ is the Euler function and the implied constant is absolute. Let $m \geq 1$ be an integer and let $L(s, \mathrm{sym}^m f)$ be the symmetric mth power *L*-function of $f \in \mathrm{H}_k^*(N)$ normalized so that the critical strip is given by $0 < \mathrm{Re}\, s < 1$. The values of these functions at the edge of the critical strip contain information of great interest. For example, Serre [17] showed that the Sato–Tate conjecture is equivalent to $L(1 + \mathrm{i}\tau, \mathrm{sym}^m f) \neq 0$ for all $m \in \mathbb{N}$ and $\tau \in \mathbb{R}$. The distribution of the values $L(1, \mathrm{sym}^m f)$ has received attention of many authors, including Goldfeld, Hoffstein & Lieman [6, Appendix], Hoffstein & Lockhart [6], Luo [11], Royer [13,14], Royer & Wu [15,16], Cogdell & Michel [1], Habsieger & Royer [4] and Lau & Wu [9,10]. In particular, Lau & Wu [9,10] proved the following results:

(i) For every fixed integer $m \geq 1$, there are four positive constants $A_m^\pm$ and $B_m^\pm$ such that for any newform $f \in \mathrm{H}_k^*(1)$, under the Great Riemann Hypothesis (GRH) for $L(s, \mathrm{sym}^m f)$, we have, for $k \to \infty$,

$$(1.2) \quad \{1 + o(1)\}(2B_m^- \log_2 k)^{-A_m^-} \leq L(1, \mathrm{sym}^m f) \leq \{1 + o(1)\}(2B_m^+ \log_2 k)^{A_m^+}.$$

Here (and in the sequel) $\log_j$ denotes the j-fold iterated logarithm. For most values of m, the constants $A_m^{\pm}$ and $B_m^{\pm}$ can be explicitly evaluated, for example,

$$\begin{cases} A_m^+ = m+1, & B_m^+ = e^{\gamma} & (m \in \mathbb{N}), \\ A_m^- = m+1, & B_m^- = e^{\gamma}\zeta(2)^{-1} & (\text{odd } m), \\ A_2^- = 1, & B_2^- = e^{\gamma}\zeta(2)^{-2}, \\ A_4^- = \frac{5}{4}, & B_4^- = e^{\gamma}B_4'^-, \end{cases}$$

where $\zeta(s)$ is the Riemann zeta-function, γ denotes the Euler constant and $B_4'^-$ is a positive constant given by a rather complicated Euler product [9, Theorem 3].

(ii) In the opposite direction, it was shown unconditionally that for $m \in \{1,2,3,4\}$ there are newforms $f_m^{\pm} \in H_k^*(1)$ such that for $k \to \infty$ [9, Theorem 2],

$$(1.3) \qquad \begin{cases} L(1, \operatorname{sym}^m f_m^+) \geq \{1 + o(1)\}(B_m^+ \log_2 k)^{A_m^+}, \\ L(1, \operatorname{sym}^m f_m^-) \leq \{1 + o(1)\}(B_m^- \log_2 k)^{-A_m^-}. \end{cases}$$

(iii) In the aim of removing GRH and closing up the gap coming from the factor 2 in (1.2) (comparing it with (1.3)), an almost all result was established. Let $\varepsilon > 0$ be an arbitrarily small positive number, $m \in \{1,2,3,4\}$ and $2 \mid k$. Then there is a subset E_k^* of $H_k^*(1)$ such that $|E_k^*| \ll H_k^*(1)e^{-(\log k)^{1/2-\varepsilon}}$ and for each $f \in H_k^*(1) \setminus E_k^*$, we have, for $k \to \infty$,

$$(1.4) \quad \{1 + O(\varepsilon_k)\}(B_m^- \log_2 k)^{-A_m^-} \leq L(1, \operatorname{sym}^m f) \leq \{1 + O(\varepsilon_k)\}(B_m^+ \log_2 k)^{A_m^+},$$

where $\varepsilon_k := (\log k)^{-\varepsilon}$ and the implied constants depend on ε only [10, Corollary 2]. By comparing (1.3) with (1.4), the extreme values of $L(1, \operatorname{sym}^m f)$ seem to be given by (1.3). Clearly it is interesting to investigate further the size of exceptional set E_k^*. In the case of quadratic characters L-functions, Montgomery & Vaughan [12] proposed, based on a probabilistic model, three conjectures on the size of exceptional set. The first one has been proved recently by Granville & Soundararajan [3]. As Cogdell & Michel indicated in [1], it would be interesting to try to get, as close as possible, the analogues of the conjectures of Montgomery–Vaughan for automorphic L-functions. The analogue of Montgomery–Vaughan's first conjecture for the automorphic symmetric power L-functions can be stated as follows.

Conjecture. *Let $m \geq 1$ be a fixed integer and*

$$F_k(t, \operatorname{sym}^m) := \frac{1}{|H_k^*(1)|} \sum_{\substack{f \in H_k^*(1) \\ L(1, \operatorname{sym}^m f) \geq (B_m^+ t)^{A_m^+}}} 1,$$

$$G_k(t, \operatorname{sym}^m) := \frac{1}{|H_k^*(1)|} \sum_{\substack{f \in H_k^*(1) \\ L(1, \operatorname{sym}^m f) \leq (B_m^- t)^{-A_m^-}}} 1.$$

Then there are positive constants $c_i = c_i(m)$ $(i = 1, 2)$ such that for $k \to \infty$,

$$(1.5) \qquad \begin{cases} e^{-c_1 (\log k)/\log_2 k} \ll F_k(\log_2 k, \operatorname{sym}^m) \ll e^{-c_2 (\log k)/\log_2 k}, \\ e^{-c_1 (\log k)/\log_2 k} \ll G_k(\log_2 k, \operatorname{sym}^m) \ll e^{-c_2 (\log k)/\log_2 k}. \end{cases}$$

The aim of this paper is to prove a weaker form of this conjecture for $m = 1$. In this case, we write, for simplification of notation,

$$L(s, f) = L(s, \operatorname{sym}^1 f), \quad F_k(t) = F_k(t, \operatorname{sym}^1), \quad G_k(t) = G_k(t, \operatorname{sym}^1).$$

In view of the trace formula of Petersson [7, Theorem 3.6], it is more convenient to consider the weighted arithmetic distribution function. As usual, denote by

$$\omega_f := \frac{\Gamma(k-1)}{(4\pi)^{k-1}\|f\|}$$

the harmonic weight in modular forms theory and define the weighted arithmetic distribution functions

$$\widetilde{F}_k(t) := \left(\sum_{f \in \mathrm{H}_k^*(1)} \omega_f \right)^{-1} \sum_{\substack{f \in \mathrm{H}_k^*(1) \\ L(1,f) \geq (\mathrm{e}^\gamma t)^2}} \omega_f,$$

$$\widetilde{G}_k(t) := \left(\sum_{f \in \mathrm{H}_k^*(1)} \omega_f \right)^{-1} \sum_{\substack{f \in \mathrm{H}_k^*(1) \\ L(1,f) \leq (6\pi^{-2}\mathrm{e}^\gamma t)^{-2}}} \omega_f.$$

By using (1.1), the classical estimate

(1.6) $$\sum_{f \in \mathrm{H}_k^*(1)} \omega_f = 1 + O(k^{-5/6})$$

and the bound of Goldfeld, Hoffstein & Lieman [6, Appendix]:

(1.7) $$1/(k \log k) \ll \omega_f \ll (\log k)/k,$$

we easily see that

(1.8) $$\begin{cases} \widetilde{F}_k(t)/\log k \ll F_k(t) \ll \widetilde{F}_k(t) \log k, \\ \widetilde{G}_k(t)/\log k \ll G_k(t) \ll \widetilde{G}_k(t) \log k. \end{cases}$$

This shows that in order to prove (1.5) it is sufficient to establish corresponding estimates of the same quality for $\widetilde{F}_k(t)$ and $\widetilde{G}_k(t)$.

Our main result is the following one.

Theorem 1.1. *For any $A \geq 1$ there are two positive constants $c = c(A)$ and $C = C(A)$ such that the estimate*

(1.9) $$\widetilde{F}_k(t) = \{1 + \Delta_k(t)\} \exp\left\{ -\frac{\mathrm{e}^{t-\gamma_0}}{t} \left(1 + O\left(\frac{1}{t}\right) \right) \right\}$$

holds uniformly for $k \geq 16, 2 \mid k$ and $t \leq T(k)$, where γ_0 is given by (1.26) below, $|\theta| \leq 1$ and

(1.10) $$\begin{cases} \Delta_k(t) := \theta \mathrm{e}^{t-T(k)-C}\left(t/T(k)\right)^{1/2} + O_A\left(\mathrm{e}^{-c\mathrm{e}^{t/5}} + (\log k)^{-A}\right), \\ T(k) := \log_2 k - \frac{5}{2}\log_3 k - \log_4 k - 3C. \end{cases}$$

In particular there are two positive constants c_1 and c_2 such that

(1.11) $$\mathrm{e}^{-c_1(\log k)/\{(\log_2 k)^{7/2}\log_3 k\}} \ll F_k\big(T(k)\big) \ll \mathrm{e}^{-c_2(\log k)/\{(\log_2 k)^{7/2}\log_3 k\}}.$$

The similar estimates for $\widetilde{G}_k(t)$ and $G_k\big(T(k)\big)$ hold also.

Remark 1.1. The estimates (1.11) of Theorem 1.1 can be considered as a weaker form of Montgomery–Vaughan's conjecture (1.5) for $m = 1$, since $T(k) \sim \log_2 k$ as $k \to \infty$. Moreover, if we could take $T(k) = \log_2 k$ in (1.11) then (1.9)

would lead to the Montgomery–Vaughan's conjecture (1.5). Hence we fail from a shift

$$\tfrac{5}{2}\log_3 k + \log_4 k + 3C.$$

It seems however to be rather difficult to resolve completely this conjecture. One of the main difficulties is that there are no analogues of the quadratic reciprocity law and Graham – Ringrose's estimates for short characters sums of friable moduli [2], which have been exploited by Granville & Soundararajan [3].

In order to prove Theorem 1.1, we need to introduce a probabilistic model as in [1]. Consider a probability space (Ω, μ), with measure μ. Let $\mathrm{SU}(2)^\natural$ be the set of conjugacy classes of $\mathrm{SU}(2)$. The group $\mathrm{SU}(2)$ is endowed with its Haar measure μ_{H} and

$$\mathrm{SU}(2)^\natural = \left\{ \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix} : \theta \in [0, \pi] \right\} \Big/ \sim$$

is endowed with the Sato – Tate measure $\mathrm{d}\mu_{\mathrm{st}}(\theta) := (2/\pi) \sin^2 \theta \, \mathrm{d}\theta$, i.e., the direct image of μ_{H} by the canonical projection $\mathrm{SU}(2) \to \mathrm{SU}(2)^\natural$. On the space (Ω, μ), define the sequence indexed by the prime numbers, $g^\natural(\omega) = \{g_p^\natural(\omega)\}_p$ of random matrices taking values in $\mathrm{SU}(2)^\natural$, given by

$$g_p^\natural(\omega) := \begin{pmatrix} \mathrm{e}^{\mathrm{i}\vartheta_p(\omega)} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\vartheta_p(\omega)} \end{pmatrix}^\natural.$$

We assume that each function $g_p^\natural(\omega)$ is distributed according to the Sato – Tate measure. This means that, for each integrable function $\phi \colon \mathrm{SU}(2)^\natural \to \mathbb{R}$, the expected value of $\phi \circ g_p^\natural$ is

$$\mathbb{E}(\phi \circ g_p^\natural) := \int_\Omega \phi \circ g_p^\natural(\omega) \, \mathrm{d}\mu(\omega) = \int_0^\pi \phi\left(\begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix} \right) \cdot (2/\pi) \sin^2 \theta \, \mathrm{d}\theta.$$

Moreover, we assume that the sequence $g^\natural(\omega)$ is made of independent random variables. This means that, for any sequence of integrable functions $\{G_p \colon \mathrm{SU}(2)^\natural \to \mathbb{R}\}_p$, we have

$$(1.12) \qquad \mathbb{E}\left(\prod_p G_p \circ g_p^\natural \right) := \int_\Omega \prod_p G_p \circ g_p^\natural(\omega) \, \mathrm{d}\mu(\omega) = \prod_p \int_\Omega G_p \circ g_p^\natural(\omega) \, \mathrm{d}\mu(\omega)$$

$$= \prod_p \int_0^\pi G_p\left(\begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix} \right) \cdot (2/\pi) \sin^2 \theta \, \mathrm{d}\theta.$$

Let I be the identity matrix. Then for $\operatorname{Re} s > \frac{1}{2}$, the random Euler product

$$(1.13) \qquad L\big(s, g^\natural(\omega)\big) := \prod_p \det\big(I - p^{-s} g_p^\natural(\omega)\big)^{-1} =: \prod_p L_p\big(s, g^\natural(\omega)\big)$$

turns out to be absolutely convergent a.s.

Now we define our probabilistic distribution functions

$$\begin{cases} \Phi(t) := \operatorname{Prob}\big(\{L\big(1, g^\natural(\cdot)\big) \geq (\mathrm{e}^\gamma t)^2\}\big), \\ \Psi(t) := \operatorname{Prob}\big(\{L\big(1, g^\natural(\cdot)\big) \leq (6\pi^{-2}\mathrm{e}^\gamma t)^{-2}\}\big). \end{cases}$$

We shall prove Theorem 1.1 in two steps. The first one is to compare $\widetilde{F}_k(t)$ with $\Phi(t)$ (resp. $\widetilde{G}_k(t)$ with $\Psi(t)$).

Theorem 1.2. *For any $A \geq 1$ there are two positive constants $c = c(A)$ and $C = C(A)$ such that the asymptotic formulas*

$$(1.14) \qquad \widetilde{F}_k(t) = \Phi(t)\{1 + \Delta_k(t)\} \quad and \quad \widetilde{G}_k(t) = \Psi(t)\{1 + \Delta_k(t)\}$$

hold uniformly for $k \geq 16, 2 \mid k$ and $t \leq T(k)$, where $\Delta_k(t)$ and $T(k)$ are defined by (1.10).

The second step of the proof of Theorem 1.1 is the evaluation of $\Phi(t)$ (resp. $\Psi(t)$). For this, we consider a truncated random Euler product

$$(1.15) \qquad L(s, g^{\natural}(\omega); y) := \prod_{p \leq y} L_p\big(s, g^{\natural}(\omega)\big)$$

and the corresponding distribution functions

$$\begin{cases} \Phi(t, y) := \mathrm{Prob}(\{L(1, g^{\natural}(\omega); y) \geq (\mathrm{e}^{\gamma} t)^2\}), \\ \Psi(t, y) := \mathrm{Prob}(\{L(1, g^{\natural}(\omega); y) \leq (6\pi^{-2}\mathrm{e}^{\gamma} t)^{-2}\}). \end{cases}$$

We have

$$(1.16) \qquad \Phi(t) = \Phi(t, \infty) \quad and \quad \Psi(t) = \Psi(t, \infty).$$

We shall use the saddle-point method (introduced by Hildebrand & Tenenbaum [5]) to evaluate $\Phi(t, y)$ and $\Psi(t, y)$. For this, we need to introduce some notation. For $s \in \mathbb{C}$ and $y \geq 2$, define

$$(1.17) \qquad E(s, y) := \mathbb{E}(L(1, g^{\natural}(\omega); y)^s) \quad and \quad E(s) := E(s, \infty),$$

where $\mathbb{E}(\cdot)$ denotes the expected value. We define also

$$(1.18) \qquad \phi(s, y) := \log E(s, y), \quad \phi_n(s, y) := \frac{\partial^n \phi}{\partial s^n}(s, y) \ (n \geq 0).$$

According to Lemmas 2.3 and 8.1 below, there is an absolute constant $c \geq 2$ such that for $t \geq 4 \log c$ and $y \geq c\mathrm{e}^t$, the equation

$$(1.19) \qquad \phi_1(\kappa, y) = 2(\log t + \gamma)$$

has a unique positive solution $\kappa = \kappa(t, y)$ and for each integer $J \geq 1$, there are computable constants $\gamma_0, \gamma_1, \ldots, \gamma_J$ such that the asymptotic formula

$$(1.20) \qquad \kappa(t, y) = \mathrm{e}^{t-\gamma_0}\left\{1 + \sum_{j=1}^{J} \frac{\gamma_j}{t^j} + O_J\left(\frac{1}{t^{J+1}} + \frac{\mathrm{e}^t t}{y \log y}\right)\right\}$$

holds uniformly for $t \geq 1$ and $y \geq 2\mathrm{e}^t$, the constant γ_0 being given by (1.26) below.

Finally write $\sigma_n := \phi_n(\kappa, y)$.

Theorem 1.3. *We have*

$$\Phi(t, y) = \frac{E(\kappa, y)}{\kappa\sqrt{2\pi\sigma_2}(\mathrm{e}^{\gamma} t)^{2\kappa}}\left\{1 + O\left(\frac{t}{\mathrm{e}^t}\right)\right\}$$

uniformly for $t \geq 1$ and $y \geq 2\mathrm{e}^t$.

Theorem 1.4. *For each integer $J \geq 1$, we have*

$$(1.21) \qquad \Phi(t, y) = \exp\left\{-\kappa\left[\sum_{j=1}^{J} \frac{a_j}{(\log \kappa)^j} + O_J\big(R_J(\kappa, y)\big)\right]\right\}$$

uniformly for $t \geq 1$ and $y \geq 2\mathrm{e}^t$, where the error term $R_J(\kappa, y)$ is given by

$$(1.22) \qquad R_J(\kappa, y) := \frac{1}{(\log \kappa)^{J+1}} + \frac{\kappa}{y \log y}$$

and

$$(1.23) \qquad a_j := \int_0^\infty \left(\frac{h(u)}{u}\right)' (\log u)^{j-1} \, \mathrm{d}u$$

with

$$(1.24) \qquad h(u) := \begin{cases} \log\left(\dfrac{2}{\pi} \displaystyle\int_0^\pi \mathrm{e}^{2u\cos\theta} \sin^2\theta \, \mathrm{d}\theta\right) & \textit{if } 0 \leq u < 1, \\[2ex] \log\left(\dfrac{2}{\pi} \displaystyle\int_0^\pi \mathrm{e}^{2u\cos\theta} \sin^2\theta \, \mathrm{d}\theta\right) - 2u & \textit{if } u \geq 1. \end{cases}$$

As a corollary of Theorem 1.4, we can obtain an asymptotic development for $\log \Phi(t, y)$ in t^{-1}. In particular we see that the probabilistic distribution function $\Phi(t)$ decays double exponentially as $t \to \infty$.

Corollary 1.5. *For each integer $J \geq 1$, there are computable constants a_1^*, $\ldots, a_J^*$ such that the asymptotic formula*

$$(1.25) \qquad \Phi(t, y) = \exp\left\{-\mathrm{e}^{t-\gamma_0}\left[\sum_{j=1}^J \frac{a_j^*}{t^j} + O_J\big(R_J(\mathrm{e}^t, y)\big)\right]\right\}$$

holds uniformly for $t \geq 1$ and $y \geq 2\mathrm{e}^t$. Further we have

$$(1.26) \quad \gamma_0 := \frac{1}{2}\int_0^\infty \frac{h'(u)}{u} \, \mathrm{d}u, \quad a_1^* := 1, \quad a_2^* := \gamma_0 - \frac{\gamma_0^2}{2} - \int_0^\infty \frac{h(u)}{u^2}(\log u) \, \mathrm{d}u.$$

In particular for each integer $J \geq 1$, we have

$$(1.27) \qquad \Phi(t) = \exp\left\{-\mathrm{e}^{t-\gamma_0}\left[\sum_{j=1}^J \frac{a_j^*}{t^j} + O_J\left(\frac{1}{t^{J+1}}\right)\right]\right\}$$

uniformly for $t \geq 1$.

Remark 1.2. (i) The same results hold also for $\Psi(t, y)$.

(ii) Taking $t = \log_2 k$ and $J = 1$ in (1.27) of Corollary 1.5, we see that the probabilistic distribution function $\Phi(t)$ (resp. $\Psi(t)$) verifies Montgomery–Vaughan's conjecture (1.5). But (1.14) is too weak to derive this conjecture for $F_k(t)$ (resp. $G_k(t)$). This means that we must take $T(k) = \log_2 k$ in Theorem 1.2, which seems to be rather difficult.

(iii) Our method can be generalized (with a little extra effort) to prove that Theorems 1.1 and 1.2 hold for $L(1, \mathrm{sym}^m f)$ for $m \geq 1$ (unconditionally when $m = 1, 2, 3, 4$ and under Cogdell–Michel's hypothesis $\mathrm{Sym}^m(f)$ and $\mathrm{LSZ}^m(1)$ [1] when $m \geqslant 5$) and that Theorems 1.3, 1.4 and Corollary 1.5 are true for $L(1, \mathrm{sym}^m g^\natural(\omega); y)$ when $m \geq 1$.

2. Expression of $E(s, y)$ and Existence of Saddle-point

The aim of this section is to prove the existence of the saddle-point $\kappa(t, y)$, defined by equation (1.19). The first step is to give an explicit expression of $E(s, y)$, which is [1, (1.24)]. For the convenience of readers, we state it here as a lemma.

Lemma 2.1. *For prime p, real θ and complex number s, we define*

$$(2.1) \quad D_p(\theta) := \prod_{0 \leq j \leq 1} \left(1 - e^{i(1-2j)\theta} p^{-1}\right)^{-1} \quad and \quad E_p(s) := \frac{2}{\pi} \int_0^\pi D_p(\theta)^s \sin^2 \theta \, d\theta.$$

Then for all $s \in \mathbb{C}$ and $y \geq 2$, we have

$$(2.2) \qquad\qquad E(s, y) = \prod_{p \leq y} E_p(s).$$

PROOF. Taking

$$G_p(M^\natural) = \begin{cases} \det(I - p^{-s'} M^\natural)^{-s} & \text{if } p \leq y \\ 1 & \text{otherwise} \end{cases}$$

in (1.12), we get

$$\mathbb{E}(L(s', g^\natural(\omega); y)^s) = \prod_{p \leq y} \mathbb{E}\left(L_p\left(s', g_p^\natural(\omega)\right)^s\right) = \prod_{p \leq y} \int_\Omega \det\left(1 - p^{-s'} g_p^\natural(\omega)\right)^{-s} d\mu(\omega)$$

$$= \prod_{p \leq y} \frac{2}{\pi} \int_0^\pi (1 - 2p^{-s'} \cos \theta + p^{-2s'})^{-s} \sin^2 \theta \, d\theta.$$

Taking $s' = 1$ and noticing (1.17) and (2.1), we get the desired result. $\qquad\square$

Lemma 2.2. *For all p and $\sigma > 0$, we have*

$$E_p''(\sigma) E_p(\sigma) - E_p'(\sigma)^2 > 0.$$

In particular for all $\sigma > 0$ and $y \geq 2$, we have $\phi_2(\sigma, y) > 0$.

PROOF. By using the definition (2.1) of $E_p(\sigma)$, it is easy to see that

$$E_p''(\sigma) E_p(\sigma) - E_p'(\sigma)^2$$

$$= \frac{4}{\pi^2} \int_0^\pi D_p(\theta)^\sigma \log^2 D_p(\theta) \sin^2 \theta \, d\theta \int_0^\pi D_p(\theta)^\sigma \sin^2 \theta \, d\theta$$

$$- \left(\frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log D_p(\theta) \sin^2 \theta \, d\theta\right)^2$$

$$= \frac{4}{\pi^2} \int_0^\pi \int_0^\pi D_p(\theta_1)^\sigma D_p(\theta_2)^\sigma \left(\log^2 D_p(\theta_1) - \log D_p(\theta_1) \log D_p(\theta_2)\right)$$

$$\times \sin^2 \theta_1 \sin^2 \theta_2 \, d\theta_1 \, d\theta_2.$$

In view of the symmetry in θ_1 and θ_2, the same formula holds if we exchange the roles of θ_1 and θ_2. Thus it follows that

$$E_p''(\sigma) E_p(\sigma) - E_p'(\sigma)^2$$

$$= \frac{2}{\pi^2} \int_0^\pi \int_0^\pi D_p(\theta_1)^\sigma D_p(\theta_2)^\sigma \log^2\left(\frac{D_p(\theta_1)}{D_p(\theta_2)}\right) \sin^2 \theta_1 \sin^2 \theta_2 \, d\theta_1 \, d\theta_2.$$

This proves the first assertion and the second follows immediately. $\qquad\square$

Lemma 2.3. *There is an absolute constant $c \geq 2$ such that for $t \geq 4 \log c$ and $y \geq ce^t$, the equation $\phi_1(\sigma, y) = 2(\log t + \gamma)$ has a unique positive solution in σ. Denoting by $\kappa(t, y)$ this solution, we have $\kappa(t, y) \asymp e^t$ uniformly for $t \geq 4 \log c$ and $y \geq ce^t$.*

PROOF. According to Lemma 4.3 below with the choice of $J = 1$, we have

$$\phi_1(\sigma, y) = 2(\log_2 \sigma + \gamma) + O(1/\log \sigma)$$

for $y \geq \sigma \geq 2$. Thus

$$\phi(ce^t, y) = 2 \log(t + \log c) + 2\gamma + O\left(\frac{1}{t + \log c}\right) > 2 \log t + 2\gamma$$

and

$$\phi(c^{-1}e^t, y) = 2 \log(t - \log c) + 2\gamma + O\left(\frac{1}{t - \log c}\right) < 2 \log t + 2\gamma,$$

provided that c is a large constant and $t \geq 4 \log c$. On the other hand, in view of Lemma 2.2, we know that for any $y \geq 2$, $\phi_1(\sigma, y)$ is an increasing function of σ in $(0, \infty)$. Hence the equation $\phi_1(\sigma, y) = 2(\log t + \gamma)$ has a unique positive solution $\kappa(t, y)$ and $c^{-1}e^t \leq \kappa(t, y) \leq ce^t$ for $t \geq 4 \log c$ and $y \geq ce^t$. This completes the proof. $\square$

3. Preliminary Lemmas

This section is devoted to establish some preliminary lemmas, which will be useful later.

Lemma 3.1. *Let $j \geq 0$ be a fixed real number. Then we have*

$$(3.1) \qquad \int_0^\pi e^{2u \cos \theta} (1 - \cos \theta)^j \sin^2 \theta \, d\theta \asymp_j e^{2u} u^{-(j+3/2)} \quad (u \geq 1).$$

The implied constant depends on j only.

PROOF. First we write

$$\int_0^\pi e^{2u \cos \theta} (1 - \cos \theta)^j \sin^2 \theta \, d\theta$$

$$= \int_0^{\pi/2} (e^{2u \cos \theta} (1 - \cos \theta)^j + e^{-2u \cos \theta} (1 + \cos \theta)^j) \sin^2 \theta \, d\theta$$

$$= \int_0^1 (e^{2ut}(1 - t)^j + e^{-2ut}(1 + t)^j)(1 - t^2)^{1/2} \, dt$$

$$\asymp \int_0^1 e^{2ut}(1 - t)^{j+1/2} \, dt + \int_0^1 e^{-2ut}(1 - t)^{1/2} \, dt.$$

By the change of variables $u(1 - t) = v$, it follows that

$$\int_0^1 e^{2ut}(1 - t)^{j+1/2} \, dt = e^{2u} u^{-(j+3/2)} \int_0^u e^{-2v} v^{j+1/2} \, dv \asymp e^{2u} u^{-(j+3/2)},$$

$$\int_0^1 e^{-2ut}(1 - t)^{1/2} \, dt \leq \int_0^1 e^{-2ut} \, dt \ll u^{-1}.$$

We obtain the desired result by insertion of these estimates into the preceding relation. $\square$

Lemma 3.2. *Let $j \geq 0$ be an integer and*

$$(3.2) \qquad E_{p,j}(\sigma) := \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma (1 - \cos\theta)^j \sin^2\theta \, d\theta.$$

(In particular $E_{p,0}(\sigma) = E_p(\sigma)$.) Then we have

$$E_{p,j}(\sigma) = \frac{2^{j+3}}{\pi} \int_0^1 \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1/2} (1-u)^{1/2} \, du$$

and the estimate

$$(3.3) \qquad E_{p,j}(\sigma)/E_p(\sigma) \ll (p/\sigma)^j$$

holds uniformly for all primes p and $\sigma > 0$. Further if $p \geq \sigma \geq 0$, we have

$$(3.4) \qquad E_p(\sigma) \asymp 1.$$

The implied constant in (3.3) depends on j only and the one in (3.4) is absolute.

PROOF. By the change of variables $u = \sin^2(\theta/2)$, a simple computation shows that the first assertion is true. Obviously (3.3) holds for $j = 0$.

Now assume that it is true for j. An integration by parts leads to

$$E_p(\sigma) \gg_j \left(\frac{\sigma}{p}\right)^j \int_0^1 \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1/2}(1-u)^{1/2} \, du$$

$$\gg_j \left(\frac{\sigma}{p}\right)^j \int_0^1 \left\{ \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-1} \frac{4\sigma}{p} + \frac{1}{2(1-u)} \right\}$$

$$\times \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1+1/2}(1-u)^{1/2} \, du.$$

On the other hand, we have

$$0 < u < 1 \implies \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-1} \frac{4\sigma}{p} + \frac{1}{2(1-u)} \geq \left(1 + \frac{1}{p}\right)^{-2} \frac{4\sigma}{p} \geq \frac{16\sigma}{9p}.$$

Inserting it into the preceding estimate, we see that

$$E_p(\sigma) \gg_j \left(\frac{\sigma}{p}\right)^{j+1} \int_0^1 \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1+1/2}(1-u)^{1/2} \, du$$

$$\asymp_j \left(\frac{\sigma}{p}\right)^{j+1} E_{p,j+1}(\sigma).$$

Thus (3.3) holds also for $j + 1$.

Since $(1 + 1/p)^{-2} \leq D_p(\theta) \leq (1 - 1/p)^{-2}$ for all primes p and any $\theta \in \mathbb{R}$, we have $D_p(\theta)^\sigma \asymp 1$ uniformly for $p \geq \sigma \geq 0$ and $\theta \in \mathbb{R}$. This implies (3.4). $\qquad \square$

Introduce the function

$$(3.5) \qquad g(u) := \log\left(\frac{2}{\pi} \int_0^\pi e^{2u\cos\theta} \sin^2\theta \, d\theta \right) \quad (u \geq 0)$$

and let $h(u)$ be defined as in (1.24). Clearly we have

$$(3.6) \qquad h(u) = \begin{cases} g(u) & \text{if } 0 \le u < 1, \\ g(u) - 2u & \text{if } u \ge 1, \end{cases}$$

$$(3.7) \qquad h'(u) = \begin{cases} g'(u) & \text{if } 0 \le u < 1, \\ g'(u) - 2 & \text{if } u \ge 1, \end{cases}$$

$$(3.8) \qquad h''(u) = g''(u), \quad (u \ge 0, u \ne 1).$$

Lemma 3.3. *We have*

$$(3.9) \qquad h(u) \asymp \begin{cases} u^2 & \text{if } 0 \le u < 1, \\ \log(2u) & \text{if } u \ge 1, \end{cases}$$

$$(3.10) \qquad h'(u) \asymp \begin{cases} u & \text{if } 0 \le u < 1, \\ u^{-1} & \text{if } u \ge 1, \end{cases}$$

$$(3.11) \qquad h''(u) \asymp \begin{cases} 1 & \text{if } 0 \le u < 1, \\ u^{-2} & \text{if } u \ge 1, \end{cases}$$

$$(3.12) \qquad h'''(u) \asymp \begin{cases} u & \text{if } 0 \le u < 1, \\ u^{-3} & \text{if } u \ge 1. \end{cases}$$

PROOF. When $0 \le u < 1$, we have

$$e^{2u \cos \theta} = \sum_{n=0}^{\infty} \frac{(2u \cos \theta)^n}{n!}.$$

From this we deduce that

$$(3.13) \quad h(u) = \log \left(\frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(2u)^n}{n!} \int_0^\pi (\cos \theta)^n \sin^2 \theta \, d\theta \right) = \log \sum_{\ell=0}^{\infty} \frac{1}{\ell!(\ell+1)!} u^{2\ell}$$

where we have used the following facts:

$$\int_0^\pi (\cos \theta)^{2\ell+1} \sin^2 \theta \, d\theta = 0$$

and

$$\frac{2}{\pi} \int_0^\pi (\cos \theta)^{2\ell} \sin^2 \theta \, d\theta = \frac{(2\ell)!}{2^\ell \ell!(\ell+1)!}.$$

Now we easily deduce, from (3.13), the desired results (3.9) – (3.12) in the case of $0 \le u < 1$.

The estimates of (3.9) – (3.12) for $u > 1$ are simple consequences of (3.1), by noticing the following relations

$$h'(u) = -2 \frac{\int_0^\pi e^{2u \cos \theta} (1 - \cos \theta) \sin^2 \theta \, d\theta}{\int_0^\pi e^{2u \cos \theta} \sin^2 \theta \, d\theta},$$

$$h''(u) = 4 \frac{\int_0^\pi e^{2u \cos \theta} (1 - \cos \theta)^2 \sin^2 \theta \, d\theta}{\int_0^\pi e^{2u \cos \theta} \sin^2 \theta \, d\theta} - 4 \left(\frac{\int_0^\pi e^{2u \cos \theta} (1 - \cos \theta) \sin^2 \theta \, d\theta}{\int_0^\pi e^{2u \cos \theta} \sin^2 \theta \, d\theta} \right)^2.$$

This completes the proof. □

4. Estimates of $\phi_n(\sigma, y)$

The aim of this section is to prove some estimates of $\phi_n(\sigma, y)$ for $n = 0, 1, 2, 3, 4$.

Lemma 4.1. *For any fixed integer $J \geq 1$, we have*

$$(4.1) \qquad \phi_0(\sigma, y) = \sigma \left\{ 2\log_2 \sigma + 2\gamma + \sum_{j=1}^{J} \frac{b_{j,0}}{(\log \sigma)^j} + O_J\big(R_J(\sigma, y)\big) \right\}$$

uniformly for $y \geq \sigma \geq 3$, where $R_J(\sigma, y)$ is defined as in (1.22) and

$$(4.2) \qquad b_{j,0} := \int_0^\infty \frac{h(u)}{u^2}(\log u)^{j-1}\, \mathrm{d}u.$$

PROOF. By the definition (2.1) of $D_p(\theta)$ and the one of $E_p(\sigma)$, it is easy to see that for $p \geq \sigma^{1/2}$, we have

$$(4.3) \qquad D_p(\theta)^\sigma = e^{2(\sigma/p)\cos\theta} \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\},$$

$$(4.4) \qquad E_p(\sigma) = \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{2}{\pi} \int_0^\pi e^{2(\sigma/p)\cos\theta} \sin^2\theta\, \mathrm{d}\theta.$$

From these, we deduce that

$$(4.5) \qquad \sum_{\sigma^{1/2} < p \leq y} \log E_p(\sigma) = \sum_{\sigma^{1/2} < p \leq y} g(\sigma/p) + O(\sigma^{1/2}/\log\sigma)$$

where $g(u)$ is defined as in (3.5).

In order to treat the sum over $p \leq \sigma$, we write

$$E_p(\sigma) = (1 - 1/p)^{-2\sigma} E_p^*(\sigma),$$

where

$$E_p^*(\sigma) := \frac{2}{\pi} \int_0^\pi \left\{ 1 + \frac{2(1-\cos\theta)}{p}\left(1 - \frac{1}{p}\right)^{-2} \right\}^{-\sigma} \sin^2\theta\, \mathrm{d}\theta.$$

By using the change of variables $u = \sin^2(\theta/2)$, we have

$$E_p^*(\sigma) = \frac{8}{\pi} \int_0^\pi \left\{ 1 + \frac{4}{p}\left(1 - \frac{1}{p}\right)^{-2} \sin^2(\theta/2) \right\}^{-\sigma} \sin^2(\theta/2)\cos^2(\theta/2)\, \mathrm{d}\theta$$

$$\geq \frac{8}{\pi} \int_0^{p/2\sigma} \left\{ 1 + \frac{4}{p}\left(1 - \frac{1}{p}\right)^{-2} u \right\}^{-\sigma} \sqrt{u(1-u)}\, \mathrm{d}u$$

$$\geq \frac{8}{\pi}\left(1 + \frac{8}{\sigma}\right)^{-\sigma} \int_0^{p/2\sigma} \sqrt{u(1-u)}\, \mathrm{d}u \geq C\left(\frac{p}{\sigma}\right)^{3/2},$$

where $C > 0$ is a constant. On the other hand, we have trivially $E_p^*(\sigma) \leq 1$ for all p and $\sigma > 0$. Thus $|\log E_p^*(\sigma)| \ll \log(\sigma/p)$ for $p \leq \sigma^{1/2}$ and

$$(4.6) \qquad \sum_{p \leq \sigma^{1/2}} \big|\log E_p^*(\sigma)\big| \ll \sum_{p \leq \sigma^{1/2}} \log(\sigma/p) \ll \sigma^{1/2}.$$

Combining (4.5) and (4.6), we can write

$$\sum_{p \leq y} \log E_p(\sigma) = 2\sigma \sum_{p \leq \sigma^{1/2}} \log(1 - 1/p)^{-1} + \sum_{\sigma^{1/2} < p \leq y} g(\sigma/p) + O(\sigma^{1/2}).$$

In view of (3.6) and the following estimate

$$\sum_{\sigma^{1/2} < p \le \sigma} (2\sigma \log(1 - 1/p)^{-1} - 2\sigma/p) \ll \sum_{\sigma^{1/2} < p \le \sigma} \sigma/p^2 \ll \sigma^{1/2}/\log \sigma,$$

the preceding estimate can be written as

$$(4.7) \qquad \sum_{p \le y} \log E_p(\sigma) = 2\sigma \sum_{p \le \sigma} \log(1 - 1/p)^{-1} + \sum_{\sigma^{1/2} < p \le y} h(\sigma/p) + O(\sigma^{1/2}).$$

By using the prime number theorem in the form

$$(4.8) \qquad \pi(t) := \sum_{p \le t} 1 = \int_2^t \frac{\mathrm{d}v}{\log v} + O\big(t e^{-8\sqrt{\log t}}\big),$$

it follows that

$$(4.9) \qquad \sum_{\sigma^{1/2} < p \le y} h\left(\frac{\sigma}{p}\right) = \int_{\sigma^{1/2}}^y \frac{h(\sigma/t)}{\log t}\, \mathrm{d}t + O(R_0),$$

where

$$R_0 := h\left(\frac{\sigma}{y}\right) y e^{-8\sqrt{\log y}} + h(\sigma^{1/2})\sigma^{1/2} e^{-4\sqrt{\log \sigma}} + \int_{\sigma^{1/2}}^y (\sigma/t)|h'(\sigma/t)| e^{-8\sqrt{\log t}}\, \mathrm{d}t$$

$$\ll \frac{\sigma^2}{y} e^{-8\sqrt{\log y}} + \sigma^{1/2} e^{-2\sqrt{\log \sigma}} + \int_{\sigma^{1/2}}^\sigma e^{-2\sqrt{\log t}}\, \mathrm{d}t + \sigma^2 \int_\sigma^y \frac{e^{-8\sqrt{\log t}}}{t^2}\, \mathrm{d}t$$

$$\ll \sigma e^{-\sqrt{\log \sigma}}$$

by use of Lemma 3.3.

In order to evaluate the integral of (4.9), we use the change of variables $u = \sigma/t$ to write

$$\int_{\sigma^{1/2}}^y \frac{h(\sigma/t)}{\log t}\, \mathrm{d}t = \sigma \int_{\sigma/y}^{\sigma^{1/2}} \frac{h(u)}{u^2 \log(\sigma/u)}\, \mathrm{d}u = \sigma \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2 \log(\sigma/u)}\, \mathrm{d}u + O(R_0')$$

where

$$R_0' := \sigma \int_0^{\sigma/y} \frac{|h(u)|}{u^2 \log(\sigma/u)}\, \mathrm{d}u + \sigma \int_0^{\sigma^{-1/2}} \frac{|h(u)|}{u^2 \log(\sigma/u)}\, \mathrm{d}u \ll \frac{\sigma^2}{y \log y} + \frac{\sigma^{1/2}}{\log \sigma}.$$

On the other hand, we have

$$\int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2 \log(\sigma/u)}\, \mathrm{d}u = \frac{1}{\log \sigma} \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2(1 - (\log u)/\log \sigma)}\, \mathrm{d}u$$

$$= \sum_{j=1}^J \frac{1}{(\log \sigma)^j} \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2} (\log u)^{j-1}\, \mathrm{d}u + O\left(\frac{1}{(\log \sigma)^{J+1}}\right).$$

Extending the interval of integration $[\sigma^{-1/2}, \sigma^{1/2}]$ to $(0, \infty)$ and bounding the contributions of $(0, \sigma^{-1/2}]$ and $[\sigma^{1/2}, \infty)$ by using (3.9) of Lemma (3.3), we have

$$\int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2} (\log u)^{j-1}\, \mathrm{d}u = b_{j,0} + O\left(\frac{(\log \sigma)^j}{\sigma^{1/2}}\right).$$

Combining these estimates, we find that

$$(4.10) \qquad \sum_{\sigma^{1/2} < p \le y} h\left(\frac{\sigma}{p}\right) = \sigma\left\{\sum_{j=1}^J \frac{b_{j,0}}{(\log \sigma)^j} + O_J\big(R_J(\sigma, y)\big)\right\}.$$

Now the desired result follows from (4.7), (4.10) and the prime number theorem in the form

$$(4.11) \qquad \sum_{p \leq \sigma} \log(1 - 1/p)^{-1} = \log_2 \sigma + \gamma + O\big(\mathrm{e}^{-2\sqrt{\log \sigma}}\big).$$

This completes the proof. □

Remark 4.1. In view of (1.3), we can write (4.1) as

$$\phi_0(\sigma, y) = \sigma \left\{ \log(B_1^+ \log \sigma)^{A_1^+} + \sum_{j=1}^{J} \frac{b_{j,0}}{(\log \sigma)^j} + O_J\big(R_J(\sigma, y)\big) \right\}$$

uniformly for $y \geq \sigma \geq 3$. In the case $\sigma < 0$, a similar asymptotic formula (with A_1^-, B_1^- and corresponding $b_{j,0}^-$ in place of A_1^+, B_1^+ and $b_{j,0}$) can be established uniformly for $y \geq -\sigma \geq 3$. As indicated in the introduction, Lemma 4.1 can be easily generalized to the general case $m \geq 1$. Thus we give an improvement and generalization of [4, Theorem B; 14, Corollaries A and C], and an improvement of [1, Theorem 1.12]. It is worthy to indicate that our method seems to be simpler and more natural.

Lemma 4.2. *We have*

$$(4.12) \qquad \frac{E_p'(\sigma)}{E_p(\sigma)} = \begin{cases} \log D_p(0) + O\big(\frac{1}{\sigma}\big) & \text{for all } p \text{ and } \sigma > 0, \\ \frac{1}{2}g'\big(\frac{\sigma}{p}\big) \log D_p(0) + O\big(\frac{1}{p^2} + \frac{\sigma}{p^3}\big) & \text{if } p \geq \sigma^{1/2}, \end{cases}$$

where $g(u)$ is defined as in (3.5).

PROOF. First we write

$$(4.13) \qquad E_p'(\sigma) = \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log D_p(\theta) \sin^2 \theta \, \mathrm{d}\theta = E_p(\sigma) \log D_p(0) + R',$$

where

$$(4.14) \qquad R' := \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log\left(\frac{D_p(\theta)}{D_p(0)}\right) \sin^2 \theta \, \mathrm{d}\theta.$$

Since

$$\left| \log\left(\frac{D_p(\theta)}{D_p(0)}\right) \right| = \left| -\log\left(1 + \frac{2p(1 - \cos \theta)}{(p-1)^2}\right) \right| \leq \frac{2p(1 - \cos \theta)}{(p-1)^2} \leq \frac{8(1 - \cos \theta)}{p},$$

it follows from (3.3) of Lemma 3.2 with $j = 1$ that

$$\frac{R'}{E_p(\sigma)} \ll \frac{E_{p,1}(\sigma)}{pE_p(\sigma)} \ll \frac{1}{\sigma}$$

for all p and $\sigma > 0$. This implies, via (4.13), the first estimate of (4.12).

We have

$$\log D_p(\theta) = (\cos \theta)(2/p) + O(1/p^2) = (\cos \theta) \log D_p(0) + O(1/p^2).$$

Inserting it and (4.3) into the first relation of (4.13) and in view of (4.4), we can write, for $p \geq \sigma^{1/2}$,

$$E_p'(\sigma) = \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{2}{\pi} \int_0^\pi \mathrm{e}^{2(\sigma/p)\cos \theta} \left\{ (\cos \theta) \log D_p(0) + O\left(\frac{1}{p^2}\right) \right\} \sin^2 \theta \, \mathrm{d}\theta$$

$$= \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{2}{\pi} \int_0^\pi \mathrm{e}^{2(\sigma/p)\cos \theta} (\cos \theta) \sin^2 \theta \, \mathrm{d}\theta \log D_p(0) + O\left(\frac{E_p(\sigma)}{p^2}\right).$$

From this and (4.4), we deduce

$$\frac{E_p'(\sigma)}{E_p(\sigma)} = \left\{1 + O\left(\frac{\sigma}{p^2}\right)\right\}\frac{1}{2}g'\left(\frac{\sigma}{p}\right)\log D_p(0) + O\left(\frac{1}{p^2}\right),$$

which implies the second estimate of (4.12). This completes the proof. $\qquad\square$

Lemma 4.3. *Let $J \geq 1$ be a fixed integer. Then we have*

$$\phi_1(\sigma, y) = 2\log_2 \sigma + 2\gamma + \sum_{j=1}^{J}\frac{b_{j,1}}{(\log \sigma)^j} + O_J\big(R_J(\sigma, y)\big)$$

uniformly for $y \geq \sigma \geq 3$, where the constant $b_{j,1}$ is given by

$$(4.15) \qquad\qquad b_{j,1} := \int_0^\infty \frac{h'(u)}{u}(\log u)^{j-1}\,\mathrm{d}u$$

and $R_J(\sigma, y)$ is defined as in (1.22).

Proof. We have

$$\phi_1(\sigma, y) = \sum_{p \leq y} E_p'(\sigma)/E_p(\sigma).$$

Using the first relation of (4.12) for $p \leq \sigma^{2/3}$ and the second for $\sigma^{2/3} < p \leq y$, we obtain

$$\phi_1(\sigma, y) = \sum_{p \leq \sigma^{2/3}}\log D_p(0) + \frac{1}{2}\sum_{\sigma^{2/3} < p \leq y}g'\left(\frac{\sigma}{p}\right)\log D_p(0) + O\left(\frac{1}{\sigma^{1/3}}\right).$$

In view of (3.7), the preceding formula can be written as

$$(4.16) \quad \phi_1(\sigma, y) = \sum_{p \leq \sigma}\log D_p(0) + \sum_{\sigma^{2/3} < p \leq y}h'\left(\frac{\sigma}{p}\right)\log\left(1 - \frac{1}{p}\right)^{-1} + O\left(\frac{1}{\sigma^{1/3}}\right).$$

Similarly to (4.10), we can prove that

$$(4.17) \qquad \sum_{\sigma^{2/3} < p \leq y}h'\left(\frac{\sigma}{p}\right)\log\left(1 - \frac{1}{p}\right)^{-1} = \sum_{j=1}^{J}\frac{b_{j,1}}{(\log \sigma)^j} + O_J\big(R_J(\sigma, y)\big),$$

using (3.10), (3.11) and (4.11) instead of (3.9), (3.10) and (4.8). Now the desired result follows from (4.16), (4.10) and (4.17). $\qquad\square$

Lemma 4.4. *We have*

$$(4.18) \quad \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \begin{cases} O\left(\frac{1}{\sigma^2}\right) & \text{if } p \leq \sigma^{1/2}, \\ \frac{1}{p^2}g''\left(\frac{\sigma}{p}\right) + O\left(\min\left\{\frac{1}{\sigma^2 p}, \frac{1}{\sigma p^2}\right\}\right) & \text{if } p > \sigma^{1/2}, \end{cases}$$

where $g(u)$ is defined as in (3.5).

Proof. First we write

$$(4.19) \qquad E_p''(\sigma) = \frac{2}{\pi}\int_0^\pi D_p(\theta)^\sigma \log^2 D_p(\theta)\sin^2\theta\,\mathrm{d}\theta = E_p(\sigma)\log^2 D_p(0) + R'',$$

where

$$R'' := \frac{2}{\pi}\int_0^\pi D_p(\theta)^\sigma\big(\log^2 D_p(\theta) - \log^2 D_p(0)\big)\sin^2\theta\,\mathrm{d}\theta.$$

Using (4.13) and (4.19), we can deduce

$$(4.20) \qquad \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \frac{R'' - 2R' \log D_p(0)}{E_p(\sigma)} - \left(\frac{R'}{E_p(\sigma)}\right)^2,$$

where R' is defined as in (4.14).

From the definitions of R' and R'', a simple calculation shows that

$$R'' - 2R' \log D_p(0) = \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log^2\left(\frac{D_p(\theta)}{D_p(0)}\right) \sin^2 \theta \, d\theta.$$

Since

$$\log^2\left(\frac{D_p(\theta)}{D_p(0)}\right) = \log^2\left(1 + \frac{2p(1-\cos\theta)}{(p-1)^2}\right) = \frac{4(1-\cos\theta)^2}{p^2} + O\left(\frac{(1-\cos\theta)^2}{p^3}\right),$$

we have

$$R'' - 2R' \log D_p(0) = \frac{4}{p^2} E_{p,2}(\sigma) + O\left(\frac{E_{p,2}(\sigma)}{p^3}\right),$$

where $E_{p,j}(\sigma)$ is defined as in (3.2). By using (3.3) with the choice of $j = 2$ and the trivial estimate $E_{p,2}(\sigma) \leq 4E_p(\sigma)$, we deduce

$$(4.21) \qquad \frac{R'' - 2R' \log D_p(0)}{E_p(\sigma)} = \frac{4}{p^2} \frac{E_{p,2}(\sigma)}{E_p(\sigma)} + O\left(\min\left\{\frac{1}{\sigma^2 p}, \frac{1}{p^3}\right\}\right).$$

Similarly we have

$$\log\left(\frac{D_p(\theta)}{D_p(0)}\right) = -\log\left(1 + \frac{2p(1-\cos\theta)}{(p-1)^2}\right) = -\frac{2(1-\cos\theta)}{p} + O\left(\frac{(1-\cos\theta)}{p^2}\right),$$

and therefore

$$R' = -\frac{2}{p} E_{p,1}(\sigma) + O\left(\frac{E_{p,1}(\sigma)}{p^2}\right).$$

Now (3.3) with $j = 1$ and the trivial estimate $E_{p,1}(\sigma) \leq 2E_p(\sigma)$ imply

$$(4.22) \qquad \left(\frac{R'}{E_p(\sigma)}\right)^2 = \frac{4}{p^2}\left(\frac{E_{p,1}(\sigma)}{E_p(\sigma)}\right)^2 + O\left(\frac{E_{p,1}(\sigma)^2}{p^3 E_p(\sigma)^2}\right)$$

$$= \frac{4}{p^2}\left(\frac{E_{p,1}(\sigma)}{E_p(\sigma)}\right)^2 + O\left(\min\left\{\frac{1}{\sigma^2 p}, \frac{1}{p^3}\right\}\right).$$

Inserting (4.21) and (4.22) into (4.20) and in view of (4.14), we deduce

$$(4.23) \qquad \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \frac{4}{p^2} h_p(\sigma) + O\left(\min\left\{\frac{1}{\sigma^2 p}, \frac{1}{p^3}\right\}\right)$$

for all p and $\sigma > 0$, where

$$h_p(\sigma) := \frac{E_{p,2}(\sigma)}{E_p(\sigma)} - \left(\frac{E_{p,1}(\sigma)}{E_p(\sigma)}\right)^2.$$

When $p \leq \sigma^{1/2}$, the inequality (3.3) of Lemma 3.2 implies that $h_p(\sigma) \ll (p/\sigma)^2$. From this and (4.23) we deduce the first estimate of (4.18).

If $p \geq \sigma^{1/2}$, we can use (4.3), (3.11) and (3.8) to write

$$4h_p(\sigma) = g''\left(\frac{\sigma}{p}\right)\left\{1 + O\left(\frac{\sigma}{p^2}\right)\right\} = g''\left(\frac{\sigma}{p}\right) + O\left(\min\left\{\frac{\sigma}{p^2}, \frac{1}{\sigma}\right\}\right).$$

Inserting it into (4.23) and in view of Lemma 3.1, we get, for $p \geq \sigma^{1/2}$,

$$\frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \frac{1}{p^2}g''\left(\frac{\sigma}{p}\right) + O\left(\min\left\{\frac{1}{\sigma^2 p}, \frac{1}{\sigma p^2}\right\}\right).$$

This completes the proof. □

Lemma 4.5. *Let $J \geq 1$ be a fixed integer. Then we have*

$$\phi_2(\sigma, y) = \frac{1}{\sigma}\left\{\sum_{j=1}^{J}\frac{b_{j,2}}{(\log \sigma)^j} + O_J\big(R_J(\sigma, y)\big)\right\}$$

uniformly for $y \geq \sigma \geq 2$, where

$$b_{j,2} := \int_0^\infty h''(u)(\log u)^{j-1}\, \mathrm{d}u.$$

In particular $b_{1,2} = 2$.

PROOF. From Lemma 4.4 and (3.8), we deduce easily that

$$\begin{aligned}
\phi_2(\sigma, y) &= \sum_{p \leq y}\frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} \\
&= \sum_{\sigma^{1/2} < p \leq y}\frac{g''(\sigma/p)}{p^2} + O\left(\frac{1}{\sigma^{3/2}\log \sigma}\right) \\
&= \sum_{\sigma^{1/2} < p \leq y}\frac{h''(\sigma/p)}{p^2} + O\left(\frac{1}{\sigma^{3/2}\log \sigma}\right).
\end{aligned}$$

Similarly to (4.10), we can prove that

$$\sum_{\sigma^{1/2} < p \leq y}\frac{h''(\sigma/p)}{p^2} = \frac{1}{\sigma}\left\{\sum_{j=1}^{J}\frac{b_{j,2}}{(\log \sigma)^j} + O_J\big(R_J(\sigma, y)\big)\right\},$$

by using (3.11), (3.12) and (4.8). Now the desired result follows from the preceding two estimates.

Finally

$$b_{1,2} = \int_0^1 h''(u)\, \mathrm{d}u + \int_1^\infty h''(u)\, \mathrm{d}u = h'(1-) - h'(1+) = h'(1-) - (h'(1-) - 2) = 2.$$

This completes the proof. □

Similarly (even more easily, since we only need an upper bound instead of an asymptotic formula), we can prove the following result.

Lemma 4.6. *We have*

$$(4.24) \qquad\qquad \phi_n(\sigma, y) \ll 1/(\sigma^{n-1}\log \sigma) \quad (n = 3, 4)$$

uniformly for $y \geq \sigma \geq 3$.

5. Estimate of $|E(\kappa + i\tau, y)|$

Lemma 5.1. *For any $\delta \in (0, \frac{1}{4})$, there are two absolute positive constants c_1, c_2 and a positive constant $c_3 = c_3(\delta)$ such that for all $y \geq \sigma \geq 3$ we have*

$$(5.1) \qquad \left| \frac{E(\sigma + i\tau, y)}{E(\sigma, y)} \right| \leq \begin{cases} 1 & \text{if } |\tau| \leq c_1 \sigma^{1/2} \log \sigma \ \text{ or } |\tau| \geq y^{1/\delta}, \\ e^{-c_2 \tau^2 / [\sigma (\log \sigma)^2]} & \text{if } c_1 \sigma^{1/2} \log \sigma \leq |\tau| \leq \sigma, \\ e^{-c_3 |\tau|^\delta} & \text{if } \sigma \leq |\tau| \leq y^{1/\delta}. \end{cases}$$

PROOF. First we write

$$E_p(s) = \frac{2}{\pi} \int_0^\pi \left(D_p(\theta)^{-1} \right)^{-s} \sin^2 \theta \, d\theta = \frac{2}{\pi} \int_0^\pi \frac{\sin^2 \theta}{(1-s)(D_p(\theta)^{-1})'} \, d\left(D_p(\theta)^{-1} \right)^{1-s}.$$

Since $(D_p(\theta)^{-1})' = 2p^{-1} \sin \theta$, after a simplification and an integration by parts it follows that

$$E_p(s) = \frac{p}{\pi(s-1)} \int_0^\pi D_p(\theta)^{s-1} \cos \theta \, d\theta$$

$$= \frac{p}{\pi(s-1)} \int_0^{\pi/2} \left\{ D_p(\theta)^{s-1} - D_p(\pi - \theta)^{s-1} \right\} \cos \theta \, d\theta.$$

This implies that

$$(5.2) \qquad \left| \frac{E_p(s)}{E_p(\sigma)} \right| = \left| \frac{\sigma - 1}{s - 1} \right| \left| \frac{E_p^*(s)}{E_p^*(\sigma)} \right|$$

with

$$E_p^*(s) := \int_0^{\pi/2} \left\{ D_p(\theta)^{s-1} - D_p(\pi - \theta)^{s-1} \right\} \cos \theta \, d\theta.$$

(1) Case of $\sigma^{1/\delta} < |\tau| \leq y^{1/\delta}$

Write

$$E_p^*(s) = \int_0^{\pi/2} D_p(\theta)^{s-1} \{ 1 - \Delta_p(\theta)^{s-1} \} \cos \theta \, d\theta$$

with

$$\Delta_p(\theta) := \frac{1 - 2p^{-1} \cos \theta + p^{-2}}{1 + 2p^{-1} \cos \theta + p^{-2}}.$$

It is clear that for all p, the function $\theta \mapsto \Delta_p(\theta)$ is increasing on $[0, \pi/2]$. It follows that

$$E_p^*(\sigma) \geq \int_0^{\pi/4} D_p(\theta)^{\sigma-1} \{ 1 - \Delta_p(\theta)^{\sigma-1} \} \cos \theta \, d\theta$$

$$\geq \{ 1 - \Delta_p(\pi/4)^{\sigma-1} \} \int_0^{\pi/4} D_p(\theta)^{\sigma-1} \cos \theta \, d\theta$$

for all p and $\sigma \geq 1$. This implies that

$$(5.3) \qquad \left| \frac{1}{E_p^*(\sigma)} \int_0^{\pi/4} D_p(\theta)^{\sigma-1} \cos \theta \, d\theta \right| \leq \frac{1}{1 - \Delta_p(\pi/4)^{\sigma-1}}.$$

Similarly since the function $\theta \mapsto D_p(\theta)^{\sigma-1} \cos \theta$ is decreasing on $[0, \pi/2]$ for all p and $\sigma \geq 2$, we can deduce, via (5.3), that

$$(5.4) \qquad \left| \frac{1}{E_p^*(\sigma)} \int_{\pi/4}^{\pi/2} D_p(\theta)^{\sigma-1} \cos \theta \, d\theta \right| \leq \frac{1}{1 - \Delta_p(\pi/4)^{\sigma-1}}.$$

From (5.3) and (5.4), we deduce that

$$\left|\frac{E_p^*(s)}{E_p^*(\sigma)}\right| \leq \frac{2}{1 - \Delta_p(\pi/4)^{\sigma-1}}.$$

It is easy to verify that for all $p \geq \sigma \geq 2$, we have

$$\Delta_p\left(\frac{\pi}{4}\right)^{\sigma-1} \leq \left(1 - \frac{\sqrt{2}}{p} + \frac{1}{p^2}\right)^{\sigma-1} \leq 1 - \frac{\sigma-1}{4p}.$$

Combining these estimates with (5.2), we obtain

$$\left|\frac{E_p(s)}{E_p(\sigma)}\right| \leq \frac{8p}{|s-1|} \leq \frac{p^4}{|\tau|} \quad (p \geq \sigma).$$

By multiplying this inequality for $\sigma < p \leq |\tau|^\delta \ (\leq y)$ and the trivial inequality $|E_p(s)| \leq |E_p(\sigma)|$ for the others p, we deduce, via the prime number theorem, that

$$\left|\frac{E(s,y)}{E(\sigma,y)}\right| \leq \exp\left\{-\sum_{\sigma < p \leq |\tau|^\delta} \log|\tau| + 4\sum_{\sigma < p \leq |\tau|^\delta} \log p\right\} \leq e^{-\{1/\delta - 4 + o(1)\}|\tau|^\delta}.$$

(2) Case of $c_1\sigma^{1/2}\log\sigma \leq |\tau| \leq \sigma^{1/\delta}$

For $p \geq \sigma^{1/2} \geq 2$, we can write

$$|E_p^*(s)| \leq \int_0^{\pi/2} \{D_p(\theta)^{\sigma-1} + D_p(\pi-\theta)^{\sigma-1}\}\cos\theta\,d\theta$$

$$= \left\{1 + O\left(\frac{\sigma}{p^2}\right)\right\}\int_0^{\pi/2}\left(e^{2[(\sigma-1)/p]\cos\theta} + e^{-2[(\sigma-1)/p]\cos\theta}\right)\cos\theta\,d\theta$$

and

$$|E_p^*(\sigma)| = \int_0^{\pi/2} \{D_p(\theta)^{\sigma-1} - D_p(\pi-\theta)^{\sigma-1}\}\cos\theta\,d\theta$$

$$= \left\{1 + O\left(\frac{\sigma}{p^2}\right)\right\}\int_0^{\pi/2}\left(e^{2[(\sigma-1)/p]\cos\theta} - e^{-2[(\sigma-1)/p]\cos\theta}\right)\cos\theta\,d\theta.$$

From these, we deduce that

(5.5) $$\left|\frac{E_p^*(s)}{E_p^*(\sigma)}\right| \leq \left\{1 + O\left(\frac{\sigma}{p^2} + \frac{1}{e^{\sigma/p}}\right)\right\} \quad (2 \leq \sigma^{1/2} \leq p \leq \sigma)$$

where we have used the following facts

$$\int_0^{\pi/2} e^{2[(\sigma-1)/p]\cos\theta}\cos\theta\,d\theta \gg e^{\sigma/p} \quad \text{and} \quad \int_0^{\pi/2} e^{-2[(\sigma-1)/p]\cos\theta}\cos\theta\,d\theta \ll 1.$$

Inserting (5.5) into (5.2), for $2 \leq \sigma^{1/2} \leq p \leq \sigma$ we obtain

$$\left|\frac{E_p(s)}{E_p(\sigma)}\right| \leq \exp\left\{-\log\left|\frac{s-1}{\sigma-1}\right| + C\left(\frac{\sigma}{p^2} + \frac{1}{e^{\sigma/p}}\right)\right\}$$

$$\leq \begin{cases} e^{-\tau^2/(2\sigma^2) + C\sigma/p^2 + Ce^{-\sigma/p}} & \text{if } 3 \leq |\tau| \leq \sigma, \\ e^{-\log(1+\tau^2/\sigma^2)/2 + C\sigma/p^2 + Ce^{-\sigma/p}} & \text{if } \sigma \leq |\tau| \leq \sigma^{1/\delta}, \end{cases}$$

where $C > 0$ is an absolute constant.

Now by multiplying these inequalities for $\sigma/(4\log\sigma) \le p \le \sigma/(2\log\sigma)$ and the trivial inequality $|E_p(s)| \le E_p(\sigma)$ for the other p, we get

$$\left|\frac{E(s,y)}{E(\sigma,y)}\right| \le \exp\left\{-\sum_{\sigma/(4\log\sigma)\le p\le\sigma/(2\log\sigma)}\left(\frac{\tau^2}{2\sigma^2}-\frac{C\sigma}{p^2}-\frac{C}{\mathrm{e}^{\sigma/p}}\right)\right\}$$

$$\le \exp\left\{-\left(\frac{\tau^2}{16\sigma(\log\sigma)^2}-10C-\frac{10C}{\sigma\log\sigma}\right)\right\} \le \exp\left\{-\frac{c_2\tau^2}{\sigma(\log\sigma)^2}\right\}$$

if $c_1\sigma^{1/2}\log\sigma \le |\tau| \le \sigma$, and

$$(5.6) \quad \left|\frac{E(s,y)}{E(\sigma,y)}\right| \le \exp\left\{-\sum_{\sigma/(4\log\sigma)\le p\le\sigma/(2\log\sigma)}\left[\frac{1}{2}\log\left(1+\frac{\tau^2}{\sigma^2}\right)-\frac{C\sigma}{p^2}-\frac{C}{\mathrm{e}^{\sigma/p}}\right]\right\}$$

$$\le \exp\left\{-\left[\frac{\sigma}{8\log\sigma}\log\left(1+\frac{\tau^2}{\sigma^2}\right)-10C-\frac{10C}{\sigma\log\sigma}\right]\right\}$$

$$\le \exp\left\{-c_3|\tau|^\delta\right\}$$

if $\sigma \le |\tau| \le \sigma^{1/\delta}$. This completes the proof. $\qquad\square$

6. Proof of Theorem 1.3

We follow the argument of Granville & Soundararajan [3] to prove Theorem 1.3. We shall divide the proof in several steps which are embodied in the following lemmas.

The first one is a classic integration formula (see [3, p. 1019]).

Lemma 6.1. Let $c > 0$, $\lambda > 0$ and $N \in \mathbb{N}$. Then we have

$$(6.1) \qquad \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} y^s\left(\frac{\mathrm{e}^{\lambda s}-1}{\lambda s}\right)^N\frac{\mathrm{d}s}{s} = \begin{cases} 0 & \text{if } 0 < y < \mathrm{e}^{-\lambda N}, \\ \in [0,1] & \text{if } \mathrm{e}^{-\lambda N} \le y < 1, \\ 1 & \text{if } y \ge 1. \end{cases}$$

The second one is an analogue for [3, (3.6) and (3.7)] (see also [19, Lemma 3.1]).

Lemma 6.2. Let $t \ge 1$, $y \ge 2\mathrm{e}^t$ and $0 < \lambda \le \mathrm{e}^{-t}$. Then we have

$$(6.2) \qquad \Phi(t,y) \le \frac{1}{2\pi\mathrm{i}}\int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty}\frac{E(s,y)}{(\mathrm{e}^\gamma t)^{2s}}\frac{\mathrm{e}^{\lambda s}-1}{\lambda s}\frac{\mathrm{d}s}{s} \le \Phi(t\mathrm{e}^{-\lambda},y),$$

$$(6.3) \qquad \Phi(t\mathrm{e}^{-\lambda},y)-\Phi(t,y) \le \frac{1}{2\pi\mathrm{i}}\int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty}\frac{E(s,y)}{(\mathrm{e}^\gamma t)^{2s}}\frac{\mathrm{e}^{\lambda s}-1}{\lambda s}\left(\mathrm{e}^{2\lambda s}-\mathrm{e}^{-2\lambda s}\right)\frac{\mathrm{d}s}{s}.$$

PROOF. Denote by $\mathbf{1}_X(\omega)$ the characteristic function of the set $X \subset \Omega$. Then by Lemma 6.1 with $N = 1$ and $c = \kappa$, we have

$$\mathbf{1}_{\{\omega\in\Omega:L(1,g^\natural(\omega);y)>(\mathrm{e}^\gamma t)^2\}}(\omega) \le 12\pi\mathrm{i}\int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty}\left(\frac{L(1,g^\natural(\omega);y)}{(\mathrm{e}^\gamma t)^2}\right)^s\frac{\mathrm{e}^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s.$$

Integrating over Ω and interchanging the order of integrations yield

$$\Phi(t,y) \le \int_\Omega\left\{\frac{1}{2\pi\mathrm{i}}\int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty}\left(\frac{L(1,g^\natural(\omega);y)}{(\mathrm{e}^\gamma t)^2}\right)^s\frac{\mathrm{e}^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s\right\}\mathrm{d}\mu(\omega)$$

$$= \frac{1}{2\pi\mathrm{i}}\int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty}\frac{E(s,y)}{(\mathrm{e}^\gamma t)^{2s}}\frac{\mathrm{e}^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s.$$

This proves the first inequality of (6.2). The second can be treated by noticing that

$$\mathbf{1}_{\{\omega\in\Omega:L(1,g^\natural(\omega);y)>(\mathrm{e}^{\gamma-\lambda}t)^2\}}(\omega)$$

$$= \mathbf{1}_{\{\omega\in\Omega:L(1,g^\natural(\omega);y)>(\mathrm{e}^\gamma t)^2\}}(\omega) + \mathbf{1}_{\{\omega\in\Omega:(\mathrm{e}^\gamma t)^2\geq L(1,g^\natural(\omega);y)>(\mathrm{e}^{\gamma-\lambda}t)^2\}}(\omega)$$

$$\geq \frac{1}{2\pi\mathrm{i}} \int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty} \left(\frac{L(1,g^\natural(\omega);y)}{(\mathrm{e}^\gamma t)^2}\right)^s \frac{\mathrm{e}^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s.$$

From (6.2), we can deduce

$$\Phi(t\mathrm{e}^{-\lambda},y)-\Phi(t,y)$$

$$\leq \frac{1}{2\pi\mathrm{i}} \int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty} \frac{E(s,y)}{(\mathrm{e}^{\gamma-\lambda}t)^{2s}} \frac{\mathrm{e}^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s - \frac{1}{2\pi\mathrm{i}} \int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty} \frac{E(s,y)}{(\mathrm{e}^{\gamma+\lambda}t)^{2s}} \frac{\mathrm{e}^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s$$

$$= \frac{1}{2\pi\mathrm{i}} \int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty} \frac{E(s,y)}{(\mathrm{e}^\gamma t)^{2s}} \frac{\mathrm{e}^{\lambda s}-1}{\lambda s^2}\big(\mathrm{e}^{2\lambda s}-\mathrm{e}^{-2\lambda s}\big)\,\mathrm{d}s.$$

This completes the proof. □

Lemma 6.3. *Let $t\geq 1$, $y\geq 2\mathrm{e}^t$ and $0<\kappa\lambda\leq 1$. Then we have*

$$\frac{1}{2\pi\mathrm{i}} \int_{\kappa-\mathrm{i}\kappa}^{\kappa+\mathrm{i}\kappa} \frac{E(s,y)}{(\mathrm{e}^\gamma t)^{2s}} \frac{\mathrm{e}^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s = \frac{E(\kappa,y)}{\kappa\sqrt{2\pi\sigma_2}(\mathrm{e}^\gamma t)^{2\kappa}}\left\{1+O\Big(\kappa\lambda+\frac{\log\kappa}{\kappa}\Big)\right\}.$$

PROOF. First in view of (4.24) we write, for $s=\kappa+\mathrm{i}\tau$ and $|\tau|\leq\kappa$,

$$E(s,y) = \exp\left\{\sigma_0 + \mathrm{i}\sigma_1\tau - \frac{\sigma_2}{2}\tau^2 - \mathrm{i}\frac{\sigma_3}{6}\tau^3 + O(\sigma_4\tau^4)\right\}$$

and

$$\frac{\mathrm{e}^{\lambda s}-1}{\lambda s^2} = \frac{1}{\kappa}\left\{1 - \frac{\mathrm{i}}{\kappa}\tau + O\Big(\kappa\lambda+\frac{\tau^2}{\kappa^2}\Big)\right\}.$$

Since $\sigma_1=\log t+\gamma$, we have

$$\frac{E(s,y)}{(\mathrm{e}^\gamma t)^{2s}} \frac{\mathrm{e}^{\lambda s}-1}{\lambda s^2} = \frac{E(\kappa,y)}{\kappa(\mathrm{e}^\gamma t)^{2\kappa}}\mathrm{e}^{-(\sigma_2/2)\tau^2}\left\{1 - \frac{\mathrm{i}}{\kappa}\tau - \mathrm{i}\frac{\sigma_3}{6}\tau^3 + O\big(R(\tau)\big)\right\}$$

with

$$R(\tau) := \kappa\lambda + \kappa^{-2}\tau^2 + \sigma_4\tau^4 + \sigma_3^2\tau^6.$$

Now we integrate the last expression over $|\tau|\leq\kappa$ to obtain

$$(6.4)\quad \frac{1}{2\pi\mathrm{i}} \int_{\kappa-\mathrm{i}\kappa}^{\kappa+\mathrm{i}\kappa} \frac{E(s,y)}{(\mathrm{e}^\gamma t)^{2s}} \frac{\mathrm{e}^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s = \frac{E(\kappa,y)}{2\pi\kappa(\mathrm{e}^\gamma t)^{2\kappa}} \int_{-\kappa}^{\kappa} \mathrm{e}^{-(\sigma_2/2)\tau^2}\big\{1+O\big(R(\tau)\big)\big\}\,\mathrm{d}\tau,$$

where we have used the fact that the integrals involving $(\mathrm{i}/\kappa)\tau$ and $(\mathrm{i}\sigma_3/6)\tau^3$ vanish.

On the other hand, using Lemmas 4.5 and 4.6 we have

$$\int_{-\kappa}^{\kappa} \mathrm{e}^{-(\sigma_2/2)\tau^2}\,\mathrm{d}\tau = \sqrt{\frac{2\pi}{\sigma_2}}\{1+O(\exp\{-\tfrac{1}{2}\kappa^2\sigma_2\})\},$$

$$\int_{\kappa-\mathrm{i}\kappa}^{\kappa+\mathrm{i}\kappa} \mathrm{e}^{-(\sigma_2/2)\tau^2} R(\tau)\,\mathrm{d}\tau \ll \frac{1}{\sqrt{\sigma_2}}\Big(\kappa\lambda + \frac{1}{\kappa^2\sigma_2} + \frac{\sigma_3^2}{\sigma_2^3} + \frac{\sigma_4}{\sigma_2^2}\Big) \ll \frac{1}{\sqrt{\sigma_2}}\Big(\kappa\lambda + \frac{\log\kappa}{\kappa}\Big).$$

Inserting these into (6.4), we obtain the desired result. □

Lemma 6.4. *Let δ and c_3 be two constants determined by Lemma 5.1. Then we have*

$$(6.5) \qquad \int_{\kappa\pm i\kappa}^{\kappa\pm i\infty} \frac{E(s,y)}{(e^\gamma t)^{2s}} \frac{e^{\lambda s}-1}{\lambda s^2}\, ds \ll \frac{E(\kappa,y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^{2\kappa}} R_1,$$

$$(6.6) \qquad \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s,y)}{(e^\gamma t)^{2s}} \frac{e^{\lambda s}-1}{\lambda s^2}\left(e^{2\lambda s}-e^{-2\lambda s}\right) ds \ll \frac{E(\kappa,y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^{2\kappa}} R_2,$$

uniformly for $t \geq 1$, $y \geq 2e^t$, $\kappa \geq 2$ and $0 < \lambda\kappa \leq 1$, where

$$R_1 := \lambda^{-1}e^{-c_3\kappa^\delta} + \lambda^{-1}(\kappa/\log\kappa)^{1/2}y^{-1/\delta},$$

$$R_2 := \lambda\kappa(\log\kappa)^{1/2} + e^{-(c_3/2)\kappa^\delta} + \lambda^{-1}(\kappa/\log\kappa)^{1/2}y^{-1/\delta}.$$

PROOF. We split the integral in (6.5) into two parts according to $\kappa \leq |\tau| \leq y^{1/\delta}$ or $|\tau| \geq y^{1/\delta}$. Using Lemma 5.1 with $\sigma = \kappa$ and the inequality $(e^{\lambda s}-1)/s^2 \ll 1/\tau^2$, the integral in (6.5) is

$$\ll \frac{E(\kappa,y)}{(e^\gamma t)^{2\kappa}\lambda}\left(\frac{e^{-c_3\kappa^\delta}}{\kappa} + \frac{1}{y^{1/\delta}}\right),$$

which implies (6.5), in view of Lemma 4.5 with $J = 1$.

Similarly we split the integral in (6.6) into four parts according to

$$|\tau| \leq c_1\kappa^{1/2}\log\kappa, \quad c_1\kappa^{1/2}\log\kappa < |\tau| \leq \kappa, \quad \kappa < |\tau| \leq y^{1/\delta}, \quad |\tau| \geq y^{1/\delta}.$$

By Lemma 5.1 with $\sigma = \kappa$ and the inequalities

$$(e^{\lambda s}-1)/\lambda s \ll \min\{1, 1/(\lambda|\tau|)\}, \quad (e^{2\lambda s}-e^{-2\lambda s})/s \ll \min\{\lambda, 1/|\tau|\},$$

the integral in (6.6) is, as before,

$$\ll_\varepsilon \frac{E(\kappa,y)}{(e^\gamma t)^{2\kappa}}\left(\lambda\kappa^{1/2}\log\kappa + e^{-c_3\kappa^\delta} + \lambda^{-1}y^{-1/\delta}\right),$$

which implies (6.6), as before. $\qquad\square$

Now we are ready to complete the proof of Theorem 1.3. Lemma 6.3 and (6.5) of Lemma 6.4 give

$$(6.7) \qquad \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s,y)}{(e^\gamma t)^{2s}} \frac{e^{\lambda s}-1}{\lambda s^2}\, ds = \frac{E(\kappa,y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa}}\left\{1 + O(R')\right\}$$

where

$$R' := \frac{\log\kappa}{\kappa} + \kappa\lambda + \frac{e^{-c_3\kappa^\delta} + (\kappa/\log\kappa)^{1/2}y^{-1/\delta}}{\lambda}.$$

Taking $\lambda = \kappa^{-2}$ and noticing $y \geq 2e^t \asymp \kappa$ and $1/\delta > 4$, we deduce

$$(6.8) \qquad R' \ll t/e^t.$$

Combining (6.7) and (6.8) with (6.2), we obtain

$$(6.9) \qquad \Phi(t,y) \leq \frac{E(\kappa,y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa}}\left\{1 + O\left(\frac{t}{e^t}\right)\right\} \leq \Phi(te^{-\lambda},y)$$

uniformly for $t \geq 1$, $y \geq 2e^t$ and $0 < \lambda \leq e^{-t}$.

On the other hand, (6.3) of Lemma 6.2 and (6.6) of Lemma 6.4 imply

$$\Phi(te^{-\lambda}, y) - \Phi(t, y) \ll \frac{E(\kappa, y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^{2\kappa}} \left(\lambda\kappa(\log \kappa)^{1/2} + \frac{(\kappa/\log \kappa)^{1/2}}{e^{c_3\kappa^\delta}} + \frac{(\kappa/\log \kappa)^{1/2}}{\lambda y^{1/\delta}} \right)$$

$$\ll \frac{E(\kappa, y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^{2\kappa}} \left(\lambda\kappa(\log \kappa)^{1/2} + \frac{(\kappa/\log \kappa)^{1/2}}{e^{c_3\kappa^\delta}} \right)$$

when $y^{-1/(2\delta)}\kappa^{-1/2}(\log \kappa)^{-1} \leq \lambda \leq \kappa^{-1}$. Since $\Phi(te^{-\lambda}, y) - \Phi(t, y)$ is a nondecreasing function of λ, we deduce

$$(6.10) \quad \Phi(te^{-\lambda}, y) - \Phi(t, y)$$

$$\ll \frac{E(\kappa, y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^{2\kappa}} \left(\lambda\kappa(\log \kappa)^{1/2} + \frac{(\kappa/\log \kappa)^{1/2}}{e^{c_3\kappa^\delta}} + \frac{\kappa(\log \kappa)^{1/2}}{y^{1/(2\delta)}} \right)$$

uniformly for $t \geq 1$, $y \geq 2e^t$ and $0 < \lambda \leq e^{-t}$. Obviously the estimates (6.9) and (6.10) imply the desired result. This completes the proof of Theorem 1.3. $\qquad\square$

7. Proof of Theorem 1.4

Using Lemmas 4.1 and 4.5, we can write

$$\frac{E(\kappa, y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa}} = \exp\{\phi(\kappa, y) - 2\kappa(\gamma + \log t) + O(\log \kappa)\}$$

$$= \exp\left\{ \kappa\left(2\log_2 \kappa - 2\log t + \sum_{j=1}^J \frac{b_{j,0}}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) \right) \right\}.$$

On the other hand, Lemma 4.3 and (1.19) imply that

$$2\log_2 \kappa + 2\gamma + \sum_{j=1}^J \frac{b_{j,1}}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) = 2(\log t + \gamma).$$

Combining these estimates, we can obtain

$$\frac{E(\kappa, y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa}} = \exp\left\{ -\kappa\left[\sum_{j=1}^J \frac{b_{j,1} - b_{j,0}}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) \right] \right\}.$$

In view of (1.23), (4.2) and (4.15), we have $b_{j,1} - b_{j,0} = a_j$. This completes the proof. $\qquad\square$

8. Proof of Corollary 1.5

We first prove an asymptotic development of $\kappa(t, y)$ in t.

Lemma 8.1. *For each integer $J \geq 1$, there are computable constants γ_0, $\gamma_1 \ldots, \gamma_J$ such that the asymptotic formula*

$$(8.1) \qquad \kappa(t, y) = e^{t-\gamma_0}\left\{ 1 + \sum_{j=1}^J \frac{\gamma_j}{t^j} + O_J(R_J^*(t, y)) \right\}$$

holds uniformly for $t \geq 1$ and $y \geq 2e^t$, where

$$R_N^*(t, y) := \frac{1}{t^{N+1}} + e^t ty \log y.$$

Further γ_0 is given by (1.26) and $\gamma_1 = -\frac{1}{8}b_{1,1}^2 - \frac{1}{4}b_{2,1}$.

PROOF. By Lemma 4.3 and (1.19), we have

$$(8.2) \qquad 2\log t = 2\log_2 \kappa + \sum_{j=1}^{J+1} \frac{b_{j,1}}{(\log \kappa)^j} + O_J\big(R_{J+1}(\kappa, y)\big),$$

where $R_J(\kappa, y)$ is defined as in (1.22). From (8.2), we easily deduce that

$$t = (\log \kappa) \prod_{j=1}^{J+1} \exp\Big\{ \frac{b_{j,1}}{2(\log \kappa)^j} \Big\} \exp\big\{ O_J\big(R_{J+1}(\kappa, y)\big) \big\}$$

$$= (\log \kappa) \prod_{j=1}^{J+1} \Big\{ \sum_{m_j=0}^{J+1} \frac{1}{m_j!} \Big(\frac{b_{j,1}}{2(\log \kappa)^j} \Big)^{m_j} + O_J\big(R_{J+1}(\kappa, y)\big) \Big\}.$$

Developing the product, we get

$$t = (\log \kappa) \Big\{ \sum_{j=0}^{J+1} \frac{b'_j}{(\log \kappa)^j} + O_J\big(R_{J+1}(\kappa, y)\big) \Big\},$$

where

$$b'_j := \sum_{\substack{m_1 \geq 0, \ldots, m_{J+1} \geq 0 \\ m_1 + 2m_2 + \cdots + (J+1)m_{J+1} = j}} \frac{b_{1,1}^{m_1} \cdots b_{J+1,1}^{m_{J+1}}}{(2m_1)!! \cdots (2m_{J+1})!!}$$

$$= \sum_{\substack{m_1 \geq 0, \ldots, m_j \geq 0 \\ m_1 + 2m_2 + \cdots + jm_j = j}} \frac{b_{1,1}^{m_1} \cdots b_{j,1}^{m_j}}{(2m_1)!! \cdots (2m_j)!!}.$$

Since $b'_0 = 1$ and $b'_1 = b_{1,1}/2 = \gamma_0$, the preceding asymptotic formula can be written as

$$(8.3) \qquad t = \log \kappa + \gamma_0 + \sum_{j=1}^{J} \frac{b'_{j+1}}{(\log \kappa)^j} + O_J\big(R_J^*(t, y)\big),$$

where we have used the fact that $\kappa(t, y) \asymp \mathrm{e}^t$ (see Lemma 2.3) and $(\log k)R_{J+1}(\kappa, y) \asymp R_J^*(t, y)$.

With the help of (8.3), a simple recurrence argument shows that there are constants γ'_n such that

$$(8.4) \qquad t = \log \kappa + \sum_{j=0}^{J} \frac{\gamma'_j}{t^j} + O_J\big(R_J^*(t, y)\big).$$

In fact taking $J = 0$ in (8.3), we see that (8.4) holds for $J = 0$. Suppose that it holds for $0, \ldots, J-1$, i.e.,

$$t = \log \kappa + \sum_{i=0}^{J-j-1} \frac{\gamma'_i}{t^i} + O\big(R_{J-j-1}^*(t, y)\big) \quad (j = 0, \ldots, J-1),$$

which is equivalent to

$$(8.5) \qquad \log \kappa = t \Big\{ 1 - \sum_{i=1}^{J-j} \frac{\gamma'_{i-1}}{t^i} + O\Big(\frac{R_{J-j-1}^*(t, y)}{t} \Big) \Big\} \quad (j = 0, \ldots, J-1).$$

This holds also for $j = J$ if we use the convention:

$$\sum_{i=0}^{-1} = 0 \quad \text{and} \quad R_{-1}^*(t,y) := 1,$$

since $\log \kappa = t + O(1)$. Inserting it into (8.3), we easily see that (8.4) holds also for J. In particular we have

$$\gamma_1' = b_2' = \tfrac{1}{8}b_{1,1}^2 + \tfrac{1}{4}b_{2,1}.$$

Now (8.1) is an immediate consequence of (8.4) with

$$\gamma_j := \sum_{\substack{m_1 \geq 0, \ldots, m_J \geq 0 \\ m_1 + 2m_2 + \cdots + Jm_J = j}} (-1)^{m_1 + \cdots + m_J} \frac{\gamma_1'^{m_1} \cdots \gamma_J'^{m_J}}{m_1! \cdots m_J!}.$$

This completes the proof.   □

Now we are ready to prove Corollary 1.5.

Using (8.5), we have

$$(8.6) \qquad \sum_{j=1}^{J} \frac{a_j}{(\log \kappa)^j} = \sum_{j=1}^{J} \frac{a_j}{t^j} \left\{ 1 - \sum_{i=1}^{J-j} \frac{\gamma_{i-1}'}{t^i} + O_N\left(\frac{R_{J-j-1}^*(t,y)}{t} \right) \right\}^{-j}$$

$$= \sum_{j=1}^{J} \frac{\rho_j}{t^j} + O_J\left(\frac{R_{J-2}^*(t,y)}{t^2} \right),$$

where the ρ_n are constants. In particular we have $\rho_1 = a_1 = 1$ and $\rho_2 = \gamma_0 + a_2$.

Now Theorem 1.4, (8.1) and (8.6) imply the result of corollary with

$$a_1^* = \rho_1 = 1, \quad a_j^* = \rho_j + \sum_{i=1}^{j-1} \gamma_i \rho_{j-i} \quad (j \geq 2).$$

This completes the proof of Corollary 1.5.   □

9. Proof of Theorem 1.2

For each $\eta \in (0, \tfrac{1}{2})$, define

$$H_k^+(1; \eta) := \{ f \in H_k^*(1) : L(s, f) \neq 0, s \in \mathcal{S} \},$$

where $\mathcal{S} := \{ s := \sigma + i\tau : \sigma \geq 1 - \eta, |\tau| \leq 100k^\eta \} \cup \{ s := \sigma + i\tau : \sigma \geq 1, \tau \in \mathbb{R} \}$, and

$$H_k^-(1; \eta) := H_k^*(1) \setminus H_k^+(1; \eta).$$

Then we have (see [9, (1.11)])

$$(9.1) \qquad \left| H_k^-(1; \eta) \right| \ll_\eta k^{31\eta}.$$

Our starting point in the proof of Theorem 1.2 is the evaluation of the moments of $L(1, f)$. For this, we recall a particular case of [9, Proposition 6.1].

Lemma 9.1. *Let* $\eta \in (0, 1/31)$ *be fixed. There are two positive constants* $c_i = c_i(\eta)$ $(i = 4, 5)$ *such that*

$$(9.2) \qquad \sum_{f \in H_k^+(1; \eta)} \omega_f L(1, f)^s = E(s) + O_\eta\left(e^{-c_4 \log k / \log_2 k} \right)$$

uniformly for

$$(9.3) \qquad k \geq 16, \quad 2 \mid k \quad \text{and} \quad |s| \leq 2T_k$$

with

$$T_k := c_5 \log k / (\log_2 k \log_3 k).$$

Here $E(s)$ is defined by (1.17).

Let $\kappa(t, y)$ be the saddle-point determined by (1.19) and $\kappa_t := \kappa(t, \infty)$. For $k \geq 16$, $2 \mid k$, $\lambda > 0$, $N \in \mathbb{N}$ and $t > 0$, introduce the two integrals

$$I_1(k, t; \lambda, N) := \frac{1}{2\pi i} \int_{\kappa_t - i\infty}^{\kappa_t + i\infty} \sum_{f \in \mathrm{H}_k^+(1;\eta)} \omega_f \left(\frac{L(1, f)}{(e^\gamma t)^2} \right)^s \left(\frac{e^{\lambda s} - 1}{\lambda s} \right)^{2N} \frac{ds}{s}$$

and

$$I_2(k, t; \lambda, N) := \frac{1}{2\pi i} \int_{\kappa_t - i\infty}^{\kappa_t + i\infty} \frac{E(s)}{(e^\gamma t)^{2s}} \left(\frac{e^{\lambda s} - 1}{\lambda s} \right)^{2N} \frac{ds}{s}.$$

Lemma 9.2. *Let $\eta \in (0, 1/200]$ be fixed. Then we have*

$$(9.4) \qquad \widetilde{F}_k(t) + O_\eta\big(k^{-5/6}\big) \leq I_1(k, t; \lambda, N) \leq \widetilde{F}_k(te^{-\lambda N}) + O_\eta\big(k^{-5/6}\big),$$

$$(9.5) \qquad\qquad \Phi(t) \leq I_2(k, t; \lambda, N) \leq \Phi(te^{-\lambda N})$$

uniformly for $k \geq 16$, $2 \mid k$, $\lambda > 0$, $N \in \mathbb{N}$ and $t > 0$. The implied constants depend on η only.

PROOF. By exchanging the order of summation and by using Lemma 6.1 with $c = \kappa_t$, we obtain

$$I_1(k, t; \lambda, N) = \sum_{f \in \mathrm{H}_k^+(1;\eta)} \frac{\omega_f}{2\pi i} \int_{\kappa_t - i\infty}^{\kappa_t + i\infty} \left(\frac{L(1, f)}{(e^\gamma t)^2} \right)^s \left(\frac{e^{\lambda s} - 1}{\lambda s} \right)^{2N} \frac{ds}{s} \geq \sum_{\substack{f \in \mathrm{H}_k^+(1;\eta) \\ L(1,f) \geq (e^\gamma t)^2}} \omega_f.$$

In view of the second estimate of (1.7) and of (9.1), we reintroduce the missing forms

$$I_1(k, t; \lambda, N) \geq \sum_{\substack{f \in \mathrm{H}_k^*(1) \\ L(1,f) \geq (e^\gamma t)^2}} \omega_f + O\left(\sum_{f \in \mathrm{H}_k^* \setminus \mathrm{H}_k^+(1;\eta)} \omega_f \right) \geq \sum_{\substack{f \in \mathrm{H}_k^*(1) \\ L(1,f) \geq (e^\gamma t)^2}} \omega_f + O\big(k^{-1+31\eta} \log k\big).$$

Clearly this implies the first inequality of (9.4), thanks to (1.6) and (1.7).

Similarly, using Lemma 6.1 with $c = \kappa_t$, we find

$$I_1(k, t; \lambda, N) \leq \sum_{\substack{f \in \mathrm{H}_k^+(1;\eta) \\ L(1,f) \geq (e^\gamma t)^2}} \omega_f + \sum_{\substack{f \in \mathrm{H}_k^+(1;\eta) \\ (e^\gamma te^{-\lambda N})^2 \leq L(1,f) < (e^\gamma t)^2}} \omega_f \quad = \sum_{\substack{f \in \mathrm{H}_k^+(1;\eta) \\ L(1,f) \geq (e^\gamma te^{-\lambda N})^2}} \omega_f.$$

As before, we can easily show that the last sum is $\leq \widetilde{F}_k(te^{-\lambda N}) + O,\big(k^{-5/6}\big)$.

The estimates (9.5) can be proved in the same way as (6.2). □

Lemma 9.3. *Let $\eta \in (0, 1/200]$ be fixed and c_4 be the positive constant given by Lemma 9.1. Then we have*

$$(9.6) \quad |I_1(k, t; \lambda, N) - I_2(k, t; \lambda, N)| \ll e^{-c_4 (\log k)/\log_2 k} \frac{(1 + e^{\lambda \kappa_t})^{2N} \log T_k}{(e^\gamma t)^{2\kappa_t}}$$

$$+ \frac{E(\kappa_t) + e^{-c_4 (\log k)/\log_2 k}}{N (e^\gamma t)^{2\kappa_t}} \left(\frac{1 + e^{\lambda \kappa_t}}{\lambda T_k} \right)^{2N}$$

uniformly for $\lambda > 0$, $N \in \mathbb{N}$, $k \geq 16$, $2 \mid k$ and $t \leq T(k)$, where $T(k)$ is given by (1.10). The implied constant depends on η only.

PROOF. By the definitions of I_1 and I_2, we can write

$$I_1(k, t; \lambda, N) - I_2(k, t; \lambda, N)$$
$$= \frac{1}{2\pi \mathrm{i}} \int_{\kappa_t - \mathrm{i}\infty}^{\kappa_t + \mathrm{i}\infty} \left(\sum_{f \in \mathrm{H}_k^+(1;\eta)} \omega_f L(1, f)^s - E(s) \right) \left(\frac{\mathrm{e}^{\lambda s} - 1}{\lambda s} \right)^{2N} \frac{\mathrm{d}s}{s(\mathrm{e}^{\gamma}t)^{2s}}.$$

In order to estimate the last integral, we split it into two parts according to $|\tau| \leq T_k$ or $|\tau| > T_k$.

In view of (1.20), it is easy to see that $\kappa_t \leq T_k$ for $t \leq T(k)$. Thus we may apply (9.2) of Lemma (9.1) for $s = \kappa_t + \mathrm{i}\tau$ with $|\tau| \leq T_k$. Note that $|(\mathrm{e}^{\lambda s} - 1)/(\lambda s)| \leq 1 + \mathrm{e}^{\lambda \kappa_t}$ for $s = \kappa_t + \mathrm{i}\tau$, which is easily seen by looking at the cases $|\lambda s| \leq 1$ and $|\lambda s| > 1$. The contribution of $|\tau| \leq T_k$ to $|I_1(k, t; \lambda, N) - I_2(k, t; \lambda, N)|$ is

$$(9.7) \qquad\qquad \ll \mathrm{e}^{-c_4(\log k)/\log_2 k} \frac{(1 + \mathrm{e}^{\lambda \kappa_t})^{2N} \log T_k}{(\mathrm{e}^{\gamma}t)^{2\kappa_t}}.$$

Since $\kappa_t \leq T_k$ for $t \leq T(k)$, we can apply (9.2) of Lemma (9.1) to write, for $s = \kappa_t + \mathrm{i}\tau$ with $\tau \in \mathbb{R}$,

$$\left| \sum_{f \in \mathrm{H}_k^+(1;\eta)} \omega_f L(1, f)^s - E(s) \right| \leq \sum_{f \in \mathrm{H}_k^+(1;\eta)} \omega_f L(1, f)^{\kappa_t} + E(\kappa_t)$$
$$\leq 2E(\kappa_t) + O\left(\mathrm{e}^{-c_4(\log k)/\log_2 k} \right).$$

Thus the contribution of $|\tau| > T_k$ to $|I_1(k, t; \lambda, N) - I_2(k, t; \lambda, N)|$ is

$$(9.8) \qquad \ll \frac{E(\kappa_t) + \mathrm{e}^{-c_4(\log k)/\log_2 k}}{(\mathrm{e}^{\gamma}t)^{2\kappa_t}} \int_{|\tau| \geq T_k} \left(\frac{1 + \mathrm{e}^{\lambda \kappa_t}}{\lambda |\tau|} \right)^{2N} \frac{\mathrm{d}\tau}{|\tau|}$$
$$\ll \frac{E(\kappa_t) + \mathrm{e}^{-c_4(\log k)/\log_2 k}}{N(\mathrm{e}^{\gamma}t)^{2\kappa_t}} \left(\frac{1 + \mathrm{e}^{\lambda \kappa_t}}{\lambda T_k} \right)^{2N}.$$

Combining (9.7) and (9.8) yields to the required estimate. □

END OF THE PROOF OF THEOREM 1.2. For simplicity of notation, we write

$$I_j := I_j(k, t; \lambda, N) \quad \text{and} \quad I_j^+ := I_j(k, t\mathrm{e}^{\lambda N}; \lambda, N) \quad (j = 1, 2).$$

By using Lemma 9.2, we have

$$(9.9) \quad \widetilde{F}_k(t) \leq I_1 + O\left(k^{-5/6} \right) = I_2 + O\left(|I_1 - I_2| + k^{-5/6} \right)$$
$$\leq \Phi(t\mathrm{e}^{-\lambda N}) + O\left(|I_1 - I_2| + k^{-5/6} \right)$$
$$\leq \Phi(t) + |\Phi(t\mathrm{e}^{-\lambda N}) - \Phi(t)| + O\left(|I_1 - I_2| + k^{-5/6} \right)$$

and

$$(9.10) \qquad \widetilde{F}_k(t) \geq I_1^+ + O\left(k^{-5/6} \right) = I_2^+ + O\left(|I_1^+ - I_2^+| + k^{-5/6} \right)$$
$$\geq \Phi(t\mathrm{e}^{\lambda N}) + O\left(|I_1^+ - I_2^+| + k^{-5/6} \right)$$
$$\geq \Phi(t) - |\Phi(t) - \Phi(t\mathrm{e}^{\lambda N})| + O\left(|I_1^+ - I_2^+| + k^{-5/6} \right).$$

In view of (6.10) and Theorem 1.3, we have

$$\left|\Phi(t) - \Phi(te^{-\lambda N})\right| \ll \Phi(t)\left\{\lambda N \kappa_t (\log \kappa_t)^{1/2} + e^{-(c_3/2)\kappa_t^\delta}\right\}$$

for $\lambda N \leq e^{-t}$. Take

(9.11) $$\lambda = e^{5A}/T_k \quad \text{and} \quad N = [\log_2 k].$$

Since $T_k = e^{T(k) + 3(\log_3 k)/2 + 2C + \log c_5}$, it is easy to see that

$$\lambda N \leq e^{-T(k) - 2C} T(k)^{-1/2} \quad \text{and} \quad \kappa_t \asymp e^t.$$

Inserting these estimates into the preceding inequality, a simple calculation shows that

(9.12) $$\left|\Phi(t) - \Phi(te^{-\lambda N})\right| \leq \Phi(t)\left\{e^{t - T(k) - C}\left(t/T(k)\right)^{1/2} + O\left(e^{-c_6 e^{\delta t}}\right)\right\},$$

provided the constant C is suitably large, where $c_6 = c_6(\eta, \delta)$ is a positive constant.

Similarly by using (6.10) with $te^{\lambda N}$ in place of t, we have

$$\left|\Phi(t) - \Phi(te^{\lambda N})\right| \ll \Phi(te^{\lambda N})\{\lambda N \kappa_{te^{\lambda N}}(\log \kappa_{te^{\lambda N}})^{1/2} + e^{-(c_3/2)\kappa_{te^{\lambda N}}^\delta}\}.$$

Since for $t \leq T(k)$ we have

$$te^{\lambda N} = t + O\left((\log_2 k)^3 (\log_3 k)/\log k\right) \quad \text{and} \quad \kappa_{te^{\lambda N}} \asymp e^{te^{\lambda N}} \asymp e^t,$$

the preceding estimate can be written as

$$
\begin{aligned}
\left|\Phi(t) - \Phi(te^{\lambda N})\right| &\leq \tfrac{1}{4}\Phi(te^{\lambda N})\left\{e^{t - T(k) - C}\left(t/T(k)\right)^{1/2} + O(e^{-c_6 e^{\delta t}})\right\}\\
&\leq \tfrac{1}{4}\Phi(t)\left\{e^{t - T(k) - C}\left(t/T(k)\right)^{1/2} + O(e^{-c_6 e^{\delta t}})\right\}\\
&\quad + \tfrac{1}{4}\left|\Phi(t) - \Phi(te^{\lambda N})\right|\left\{e^{t - T(k) - C}\left(t/T(k)\right)^{1/2} + O(e^{-c_6 e^{\delta t}})\right\},
\end{aligned}
$$

from which we deduce that

(9.13) $$\left|\Phi(t) - \Phi(te^{\lambda N})\right| \leq \Phi(t)\left\{e^{t - T(k) - C}\left(t/T(k)\right)^{1/2} + O(e^{-c_6 e^{\delta t}})\right\}.$$

By using Lemma 9.3 with $te^{\lambda N}$ in place of t, we have

$$
\begin{aligned}
|I_1^+ - I_2^+| &\ll e^{-c_4(\log k)/\log_2 k}\frac{(1 + e^{\lambda \kappa_{te^{\lambda N}}})^{2N}\log T_k}{(e^\gamma te^{\lambda N})^{2\kappa_{te^{\lambda N}}}}\\
&\quad + \frac{E(\kappa_{te^{\lambda N}}) + e^{-c_4(\log k)/\log_2 k}}{N(e^\gamma te^{\lambda N})^{2\kappa_{te^{\lambda N}}}}\left(\frac{1 + e^{\lambda \kappa_{te^{\lambda N}}}}{\lambda T_k}\right)^{2N}.
\end{aligned}
$$

On the other hand, by using Theorem 1.3 and (1.27), it is easy to see that there is a positive constant c such that

$$\Phi(te^{\lambda N}) \asymp \Phi(t) \sim \frac{E(\kappa_t)}{\kappa_t \sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa_t}} \gg e^{-c_8 e^t/t} \gg e^{-c_9(\log k)/[(\log_2 k)^{7/2}\log_3 k]}$$

for $t \leq T(k)$. Thanking to Lemma 4.5, the previous estimate can be written as

(9.14) $$|I_1^+ - I_2^+| \ll \Phi(t)\frac{1}{N}\left(\frac{\kappa_{te^{\lambda N}}}{\log \kappa_{te^{\lambda N}}}\right)^{1/2}\left(\frac{1 + e^{\lambda \kappa_{te^{\lambda N}}}}{\lambda T_k}\right)^{2N} \ll \frac{\Phi(t)}{(\log k)^A}.$$

Similarly we can prove (even more easily)

(9.15) $$|I_1 - I_2| \ll \Phi(t)/(\log k)^A.$$

Inserting (9.12) and (9.6) into (9.9) and (9.13) and (9.15) into (9.10), we obtain

$$\widetilde{F}_k(t) \leq \Phi(t)\left\{1 + e^{t - T(k) - C}\left(t/T(k)\right)^{1/2} + O\left(e^{-c_6 e^{\delta t}} + (\log k)^{-A}\right)\right\}$$

and

$$\widetilde{F}_k(t) \geq \Phi(t)\big\{1 - \mathrm{e}^{t-T(k)-C}\big(t/T(k)\big)^{1/2} + O(\mathrm{e}^{-c_6\mathrm{e}^{\delta t}} + (\log k)^{-A})\big\}.$$

This implies the first asymptotic formula of (1.14) by taking $\eta = 1/200$ and $\delta = \frac{1}{5}$.

The second can be established similarly. This completes the proof of Theorem 1.2. $\qquad\qquad\square$

10. Proof of Theorem 1.1

The formula (1.9) is an immediate consequence of Theorem 1.2 and (1.27).

Taking $t = T(k)$ in (1.9), we find that

$$(10.1) \qquad \mathrm{e}^{-c_1'(\log k)/\{(\log_2 k)^{7/2}\log_3 k\}} \ll \widetilde{F}_k\big(T(k)\big) \ll \mathrm{e}^{-c_2'(\log k)/\{(\log_2 k)^{7/2}\log_3 k\}},$$

where c_1' and c_2' are two positive constants. Clearly (10.1) and (1.8) imply (1.11).

The related results on $\widetilde{G}_k(t)$ and $G_k\big(T(k)\big)$ can be proved similarly. This completes the proof of Theorem 1.1.

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