Differential algebras of quasi-Jacobi forms of index 0

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Date: 2024, November 21.

1. Introduction : derivations of modular forms

1.1. **Modular forms.** *References:* [Ser78]

We recall that a modular form of weight $k \in \mathbb{Z}_{\geq 0}$ on $\mathrm{SL}(2,\mathbb{Z})$ is the vector space $\mathcal{M}_k$ of holomorphic functions f on $\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ that satisfies

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\ \forall \tau \in \mathcal{H} \qquad \underbrace{(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)}_{=: f|_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(\tau)} = f(\tau)$$

and

$$f(\tau) = \sum_{n=0}^{+\infty} \widehat{f(n)}\, \mathrm{e}(n\tau) \qquad \mathrm{e}(\xi) = \exp(2\pi \mathrm{i} \xi).$$

The algebra $\mathcal{M}$ of all modular forms is a polynomial algebra

$$\mathcal{M} = \bigoplus_{\substack{k \in 2\mathbb{Z}_{\geq 0} \\ k \neq 2}} \mathcal{M}_k = \mathcal{M} = \mathbb{C}[\mathrm{e}_4, \mathrm{e}_6]$$

where

$$\forall k \in 2\mathbb{Z}_{\geq 0}\ \ k \geq 4 \qquad \mathrm{e}_k(\tau) = \sum_{\omega \in \mathbb{Z} \oplus \tau\mathbb{Z}} \frac{1}{\omega^k}. \tag{1.1}$$

The algebra $\mathcal{M}$ is not stable by differentiation with respect to τ.

1.2. **Serre's derivative.** *References:* [Zag08]

Let

$$\partial_\tau = \frac{\pi}{2\mathrm{i}} \frac{\partial}{\partial \tau}.$$

We define the linear map

$$\mathrm{Se}_k : f \mapsto 4\, \partial_\tau(f) - k f\, \mathrm{e}_2$$

and prove that it satisfies $\mathrm{Se}_k(\mathcal{M}_k) = \mathcal{M}_{k+2}$. This is the restriction to $\mathcal{M}_k$ of a derivation Se of the algebra $\mathcal{M}$.

The introduction of Serre's derivative is a response to the lack of stability under differentiation in the algebra of modular forms.

1.3. **Quasimodular forms.** *References:* [Roy12]

Differentiating the definition of modular forms leads to

$$(c\tau+d)^{-k-2n}\frac{\partial^n f}{\partial\tau^n}\left(\frac{a\tau+b}{c\tau+d}\right)=\sum_{r=0}^{n} f_r(\tau)\left(\frac{c}{c\tau+d}\right)^r$$

for some (explicitly computable) holomorphic functions f_r not depending on $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$. This computation justifies the following definition implying the cocycle:

$$\begin{array}{rcccc} \mathrm{X} & : & \mathrm{SL}(2,\mathbb{Z}) & \to & \mathbb{C}^{\mathcal{H}} \\ & & \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) & \mapsto & \left(\tau\mapsto \frac{c}{c\tau+d}\right). \end{array}$$

Definition 1.1. *A holomorphic function $f\in\mathbb{C}^{\mathcal{H}}$ is a* quasimodular form *of weight k and depth s if there exist holomorphic functions $f_0,\dots,f_s$ with $f_s\neq 0$ such that*

$$\forall\gamma\in\mathrm{SL}(2,\mathbb{Z})\quad f|_k\gamma=\sum_{r=0}^{s} f_r\,\mathrm{X}(\gamma)^r$$

and

$$\forall r\quad f_r(\tau)=\sum_{n=0}^{+\infty}\widehat{f_r}(n)\,\mathrm{e}(n\tau).$$

Since

$$\frac{\partial\,\mathrm{X}^2}{\partial\tau}=-\mathrm{X}^2,$$

the definition of quasimodular forms implies that $\mathcal{M}^{\leq\infty}$ is stable by differentiation.

The derivatives of modular forms describe nearly all quasimodular forms. The vector space of quasimodular forms of weight k is

$$\mathcal{M}_k^{\leq\infty}=\bigoplus_{r=0}^{k/2-2}\frac{\partial^r}{\partial\tau^r}\mathcal{M}_{k-2r}\oplus\mathbb{C}\frac{\partial^{k/2-1}}{\partial\tau^{k/2-1}}\,\mathrm{e}_2$$

where e_2 is definied similarly to (1.1) but with extra care due to the lack of absolute convergence:

$$\mathrm{e}_2(\tau)=\lim_{N\to+\infty}\sum_{n=-N}^{N}\lim_{M\to+\infty}\sum_{\substack{m=-M\\(m,n)\neq(0,0)}}^{M}\frac{1}{(m\tau+n)^2}.$$

The algebra of quasimodular forms is also a polynomial algebra

$$\mathcal{M}^{\leq\infty} = \mathcal{M}[\mathrm{e}_2] = \mathbb{C}[\mathrm{e}_2, \mathrm{e}_4, \mathrm{e}_6].$$

> The introduction of the notion of quasi-modular forms is a response to the lack of stability under differentiation in the algebra of modular forms.

1.4. **Rankin-Cohen brackets.** *References:* [CS17]

Another notion provides us with a response, that has been initiated by Rankin and fully developed by Henri Cohen. The typical question is to find a bilinear form in the derivatives of two modular forms in such a way to obtain a new modular form. A prototypical example is the following: if $f \in \mathcal{M}_k$ and $g \in \mathcal{M}_\ell$, then

$$[f, g]_1 = kf\,\partial_\tau(g) - \ell g\,\partial_\tau(f) \in \mathcal{M}_{k+\ell+2}.$$

Cohen extended this showing that

$$[f, g]_n = \sum_{r=0}^{n} (-1)^r \binom{k+n-1}{n-r}\binom{\ell+n-1}{r} \partial_\tau^r(f)\,\partial_\tau^{n-r}(g) \in \mathcal{M}_{k+\ell+2n}$$

for any n. Note that $[\ ,\ ]_n$ can be extended to $\mathcal{M}$ by bilinear extension.

A fact conjectured by Eholzer and proved by the combination of efforts of Cohen, Manin & Zagier on the one hand and Yao on the other hand is that the family $([\ ,\ ]_n)_{n\in\mathbb{Z}_{\geq 0}}$ is a *formal deformation*.

Definition 1.2. *Le A be a commutative $\mathbb{C}$-algebra and $(\mu_j)_{j\in\mathbb{Z}_{\geq 0}}$ a family of bilinear maps from $A \times A$ to A such that μ_0 is the product on A. Let $A[[\hbar]]$ be the commutative algebra of formal power series in $\hbar$ with coefficients in A. Then, $(\mu_j)_{j\in\mathbb{Z}_{\geq 0}}$ is a* formal deformation *of A if the non-commutative product on $A[[\hbar]]$ defined by extension of*

$$f * g = \sum_{j\in\mathbb{Z}_{\geq 0}} \mu_j(f, g)\hbar^j \qquad (f, g \in A)$$

is associative.

This notion encodes a wide range of equalities since, the associativity of $*$ is equivalent to

$$\sum_{r=0}^{n} \mu_{n-r}(\mu_r(f, g), h) = \sum_{r=0}^{n} \mu_{n-r}(f, \mu_r(g, h)) \quad (f, g, h \in A).$$

The introduction of the notion of formal deformation is a response to the lack of stability under differentiation in the algebra of modular forms.

2. Derivations of Jacobi forms

2.1. **Jacobi forms.** *References:* [EZ85, DMR24]

The notion of modular form originates in the action of $\mathrm{SL}(2,\mathbb{Z})$ to $\mathcal{H}$ and the notion of weight is attached to the cocycle

$$\begin{array}{cccc} j & : & \mathrm{SL}(2,\mathbb{Z}) & \to & \mathbb{C}^{\mathcal{H}} \\ & & \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) & \mapsto & \left(\tau \mapsto \frac{c}{c\tau+d}\right). \end{array}$$

This is a cocycle of $\mathrm{SL}(2,\mathbb{Z})$ for its action of weight 1 on $\mathcal{H}$, meaning

$$j(\gamma\gamma)(\tau) = j(\gamma)(\gamma'\tau)j(\gamma')(\tau).$$

The multiplicative groupe $\mathrm{SL}(2,\mathbb{Z})$ acts on the additive group $\mathbb{Z}^2$ (whose elements are identified with 1×2 matrices) by right multiplication

$$\left((\lambda,\mu),\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \mapsto (\lambda\mu)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (\lambda a+\mu c, \lambda b+\mu d)$$

and on $\mathcal{H}\times\mathbb{C}$ by

$$\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),(\tau,z)\right) \mapsto \left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)$$

whereas $\mathbb{Z}^2$ acts on $\mathcal{H}\times\mathbb{C}$

$$(\lambda,\mu)(\tau,z) \mapsto (\tau, z+\lambda\tau+\mu).$$

The semi-direct product $\mathrm{SL}(2,\mathbb{Z})\ltimes\mathbb{Z}^2$ is the set $\mathrm{SL}(2,\mathbb{Z})\times\mathbb{Z}^2$ with the group operation

$$(\gamma,x)\cdot(\gamma',x') = (\gamma\gamma', x\gamma'+x').$$

It acts on $\mathcal{H}\times\mathbb{C}$ the following way:

$$\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),(\lambda,\mu)\right) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)((\lambda,\mu)(\tau,z)) = \left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right).$$

Let G be a group acting on the right on the group H via $\odot$. This action defines a morphism from G into $\mathrm{Aut}(H)$: $g \mapsto (h \mapsto h \odot g)$, and thus a group $G \ltimes H$, called the semidirect product of G and H, whose product is given by

$$(g, h) \ltimes (g', h') = \left(gg', (h \odot g')h'\right).$$

Let F be a set on which G acts on the left via $|_G$, and H acts on the left via $|_H$. Assume that the actions are compatible in the following sense:

$$\forall (g, h) \in G \times H \ \ \forall f \in F \qquad g|_G\left((h \odot g)|_H f\right) = h|_H\left(g|_G f\right).$$

Then, a left action of $G \ltimes H$ on F is defined by setting

$$\forall (g, h) \in G \times H \ \ \forall f \in F \qquad (g, h)|f = g|_G\left(h|_H f\right).$$

We have two cocycles of $\mathrm{SL}(2, \mathbb{Z})$ into $\mathbb{C}^{\mathcal{H}\times\mathbb{C}}$ described by

$$j\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(\tau, z) = c\tau + d \qquad \ell\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(\tau, z) = \mathrm{e}\left(-\frac{cz^2}{c\tau + d}\right)$$

and one of $\mathbb{Z}^2$ into $\mathbb{C}^{\mathcal{H}\times\mathbb{C}}$ described by

$$p(\lambda, \mu)(\tau, z) = e(\lambda^2\tau + 2\lambda z).$$

$$p\left((\lambda, \mu) + (\lambda', \mu')\right)(\tau, z) = p\left((\lambda, \mu)\right)\left((\lambda', \mu')(\tau, z)\right) \cdot p\left((\lambda', \mu')\right)(\tau, z)$$

By a general method, one deduces a cocycle of $\mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$ into $\mathbb{C}^{\mathcal{H}\times\mathbb{C}}$ described by

$$\nu\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (\lambda, \mu)\right)(\tau, z) = (c\tau + d)^{-k} \underbrace{\mathrm{e}^m}_{\exp(2\pi i m\cdot)}\left(-\frac{c(z + \lambda\tau + \mu)^2}{c\tau + d} + \lambda^2\tau + 2\lambda z\right).$$

Let G and H be two groups written multiplicatively.
Assume that G acts on the right on H.
Let A be an abelian group on which G acts on the right via $|_G$ and H acts on the right via $|_H$, with the actions of G and H on A respecting the group structures.
Assume that the actions are compatible in the following sense:

$$\forall (g, h) \in G \times H \ \ \forall a \in A \qquad (a|_G g)\,|_H(hg) = (a|_H h)\,|_G g.$$

Let ν_G be a cocycle of G in A, and let ν_H be a cocycle of H in A. Define

$$\begin{array}{rccc} \nu & : & G \ltimes H & \to & A \\ & & (g, h) & \mapsto & (\nu_G(g)|_H h) \cdot \nu_H(h). \end{array}$$

The map is a cocycle of $G \ltimes H$ in A if and only if it satisfies the cocycle condition on $(e_G, H) \ltimes (G, e_H)$, that is, if and only if

$$\forall (g, h) \in G \times H \qquad \frac{\nu_G(g)|_H(hg)}{\nu_G(g)} = \frac{\nu_H(h)|_G g}{\nu_H(hg)}.$$

Finally, we have an action of $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ on $\mathbb{C}^{\mathcal{H}\times\mathbb{C}}$,of weight k and depth m described by

$$f|_{k,m}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (\lambda,\mu)\right)(\tau,z) =$$
$$(c\tau+d)^{-k}\,\mathrm{e}^{m}\left(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z\right) f\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right).$$

Note that if f in invariant under this action, then it is 1-periodic both in the τ and z aspects. In particular, if it has a Laurent expansion around 0 given by

$$f(\tau,z) = \sum_{n=-N}^{+\infty} A_n(\tau) z^n$$

then, the Laurent coefficients are 1-periodic in the τ aspect.

The notion of singularity entails the analytic conditions we shall add to the invariant functions under the action of $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$. A function $f \in \mathbb{C}^{\mathcal{H}\times\mathbb{C}}$ is *singular* if

- For any τ, the function $z \mapsto f(\tau,z)$ is 1-periodic, meromorphic with poles in $\mathbb{Z} \oplus \tau\mathbb{Z}$, all having same order not depending on τ,
- The function $\tau \mapsto f(\tau,z)$ is 1-periodic
- THe laurent coefficients A_n are holomorphic on $\mathcal{H}$ and have a Fourier expansion of the form

$$A_n(\tau) = \sum_{r=0}^{+\infty} \widehat{A_n}(r)\,\mathrm{e}(r\tau).$$

A *singular Jacobi form* of weight k and index m is then a function $f \in \mathbb{C}^{\mathcal{H}\times\mathbb{C}}$ that is invariant under the action of $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ of weight k and index m and singular.

We focus on the case $m = 0$ and shall omit to say “of index 0" at any time we should. We denote by $\mathcal{J}$ the algebra of all singular Jacobi forms of index 0. Examples are

(1) Any modular form,

(2) The Weierstrass function

$$\wp(\tau,z)=\frac{1}{z^2}+\sum_{\substack{\omega\in\mathbb{Z}\oplus\tau\mathbb{Z}\\ \omega\neq 0}}\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}$$

that satisfies

$$\wp(\tau,z)=\frac{1}{z^2}+\sum_{n=1}^{+\infty}(2n+1)\,\mathrm{e}_{2n+2}(\tau)z^{2n}$$

is a singular Jacobi form of weight 2 and index 0,

(3) its derivatives with respect to the second variable

$$\underbrace{\partial_z}_{\partial/\partial z}\wp$$

is a singular Jacobi form of weight 3 and index 0.

Proposition 2.1 (van Ittersum ; Dumas, Martin & Royer)**.** *The three singular Jacobi forms* $\wp$, $\partial_z\wp$ *and* e_4 *are algebraically independent and generate the algebra of singular Jacobi forms:*

$$\mathcal{J}=\mathbb{C}[\wp,\partial_z\wp,\mathrm{e}_4].$$

$$\mathrm{e}_6=-\frac{1}{140}(\partial_z\wp)^2+\frac{1}{35}\wp^3-\frac{3}{7}\wp\,\mathrm{e}_4\,.$$

2.2. **Oberdieck's derivative.** *References:* [Obe14, CDMR21a]

If $\mathcal{J}$ is trivially stable by ∂_z, it can be seen that it is not stable by ∂_τ, for example by remarking that $\partial_\tau\,\mathrm{e}_4$ is not a modular form. Oberdieck's derivative plays for $\mathcal{J}$ the role that Serre's derivative plays for modular forms.

Let E_1 be defined by

$$\mathrm{E}_1(\tau,z)=\lim_{N\to+\infty}\sum_{n=-N}^{N}\lim_{M\to+\infty}\sum_{\substack{m=-M\\(m,n)\neq(0,0)}}^{M}\frac{1}{z+m\tau+n}$$

$$=\frac{1}{z}-\sum_{r=0}^{+\infty}\mathrm{e}_{2r+2}(\tau)z^{2r+1}.$$

Oberdieck's derivation is defined by over $\mathcal{J}_k$ by

$$\mathrm{Ob}_k(f) = \underbrace{4\,\partial_\tau(f) - k\,\mathrm{e}_2\,f}_{\mathrm{Se}_k(f)} + \mathrm{E}_1\,\partial_z(f) \qquad (f \in \mathcal{J}_k)$$

and its linear extension Ob to $\mathcal{J}$ satisfies (Oberdiecks : Choie, Dumas, Martin & Royer) $\mathrm{Ob}(\mathcal{J}) \subset \mathcal{J}$, and more precisely $\mathrm{Ob}(\mathcal{J}_k) \subset \mathcal{J}_{k+2}$.

By dimension consideration, $\mathrm{Ob}(\wp)$ belongs to the space $\mathcal{J}_4$ generated by $\wp$ and e_4. One deduces that $\mathrm{Ob}(\wp) = -2(\wp^2 - 10\,\mathrm{e}_4)$ which leads to the well known

$$\begin{aligned}2(2n+1)\,\partial_\tau\,\mathrm{e}_{2n+2} = (n+1)(2n+1)\,\mathrm{e}_{2n+2}\,\mathrm{e}_2 - (n+2)(2n+5)\,\mathrm{e}_{2n+4}\\ + \sum_{\substack{a\geq 1, b\geq 1\\ a+b=n}} (2a+1)(a-2b-1)\,\mathrm{e}_{2a+2}\,\mathrm{e}_{2b+2}\,.\end{aligned}$$

2.3. **Quasi-Jacobi forms.** *References:* [vI23, DMR24]

The action of $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ ton $\mathcal{H} \times \mathbb{C}$ is described by

$$\begin{array}{cccc} \mathrm{H} & : & \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2 & \to & (\mathcal{H}\times\mathbb{C})^{\mathcal{H}\times\mathbb{C}} \\ & & \left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right), (\lambda,\mu)\right) & \mapsto & \begin{array}{ccc} \mathcal{H}\times\mathbb{C} & \to & \mathcal{H}\times\mathbb{C} \\ (\tau, z) & \mapsto & \left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right). \end{array}\end{array}$$

that satisfies

$$\frac{\partial\,\mathrm{H}}{\partial\tau} = \left(\frac{1}{j^2}, -\frac{\mathrm{Y}}{j}\right) \qquad \frac{\partial\,\mathrm{H}}{\partial z} = \left(0, \frac{1}{j}\right)$$

where Y is defined by:

$$\mathrm{Y}\left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right), (\lambda,\mu)\right)(\tau,z) = \frac{cz + c\mu - d\lambda}{c\tau + d}.$$

Moreover (X is the natural extension to $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ of the previously defined X function)

$$\frac{\partial j}{\partial\tau} = \mathrm{X}\,j \quad \frac{\partial j}{\partial z} = 0 \qquad \frac{\partial\,\mathrm{Y}}{\partial\tau} = -\mathrm{X}\,\mathrm{Y} \quad \frac{\partial\,\mathrm{Y}}{\partial z} = \mathrm{X} \qquad \frac{\partial\,\mathrm{X}}{\partial\tau} = -\mathrm{X}^2 \quad \frac{\partial\,\mathrm{X}}{\partial z} = 0.$$

This remark justifies, since our goal is the stability by ∂_τ and ∂_z to introduce the following notion of quasi-Jacobi form.

Definition 2.2. *A singular function* $f \in \mathbb{C}^{\mathcal{H}\times\mathbb{C}}$ *is a* quasi-Jacobi form *of weight* k *and depth* (s_1, s_2) *if there exist singular functions* $(f_{r_1,r_2})_{\substack{0\leq r_1\leq s_1\\ 0\leq r_2\leq s_2}}$ *with* $f_{s_1,s_2} \neq 0$ *such that*

$$\forall A \in \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2 \quad f|_{k,0}A = \sum_{r_1=0}^{s_1}\sum_{r_2=0}^{s_2} f_{r_1,r_2}\,\mathrm{X}(A)^{r_1}\,\mathrm{Y}(A)^{r_2}.$$

The corresponding notation are $\mathcal{J}_k^{\leq s_1,s_2}$ for the vector space of quasi-Jacobi forms of weight k and depth (u, v) with $u \leq s_1$ and $v \leq s_2$ and $\mathcal{J}^{\leq\infty}$ for the algebra of all the quasi-Jacobi forms.

This algebra is stable by the derivations with respect to both variables:

$$\partial_\tau\left(\mathcal{J}_k^{\leq s_1,s_2}\right) \subset \mathcal{J}_{k+2}^{\leq s_1+1,s_2+1} \text{ and } \partial_z\left(\mathcal{J}_k^{\leq s_1,s_2}\right) \subset \mathcal{J}_{k+1}^{\leq s_1+1,s_2}.$$

A prototypical example, beside all quasimodular forms and all Jacobi forms is E_1 since

$$\mathrm{E}_1\,|_{1,1}A = \mathrm{E}_1 + 2\pi\mathrm{i}\,\mathrm{Y}(A)$$

and hence $\mathrm{E}_{1}1$ has weight 1 and depth $(0,1)$. Together with e_2 whose depth is $(1,0)$, one can recursively decrease the depth of any quasi-jacobi form and prove

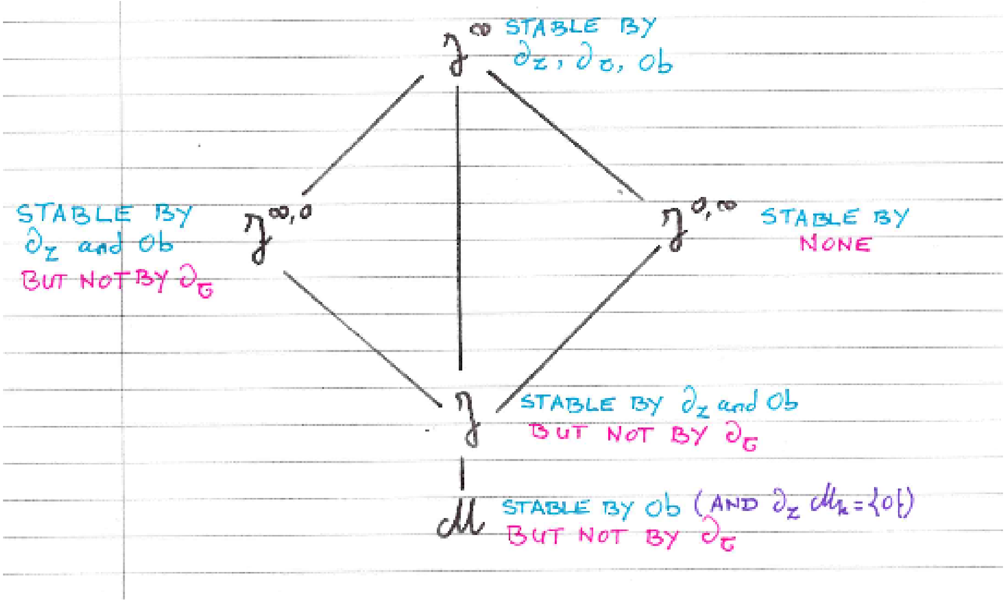
$$\mathcal{J}^{\leq\infty} = \mathcal{J}\,[\mathrm{E}_1, \mathrm{e}_2] = \mathbb{C}[\wp, \partial_z\wp, \mathrm{e}_4, \mathrm{E}_1, \mathrm{e}_2].$$

From the notion of a *bi*-depth emerge two remarkable subalgebras of quasi-Jacobi forms:

$$\mathcal{J}^{\leq\infty,0} = \mathbb{C}[\wp, \partial_z\wp, \mathrm{e}_4, \mathrm{e}_2] \qquad \text{(quasimodular type)}$$

and

$$\mathcal{J}^{\leq 0,\infty} = \mathbb{C}[\wp, \partial_z\wp, \mathrm{e}_4, \mathrm{E}_1] \qquad \text{(elliptic type).}$$



2.4. **Bilinear combinations of derivatives.** *Reference:* [DMR24]

2.4.1. *Rankin-Cohen brackets of elliptic type.* Since $\mathcal{J}^{\leq\infty}$ is stable by ∂_τ, then

$$[f,g]_n = \sum_{r=0}^{n} (-1)^r \binom{k+n-1}{n-r}\binom{\ell+n-1}{r} \partial_\tau^r(f)\,\partial_\tau^{n-r}(g)$$

(with $f \in \mathcal{J}_k^{\leq\infty}$ and $g \in \mathcal{J}_\ell^{\leq\infty}$) extends to a sequence of bilinear maps from $\mathcal{J}^{\leq\infty} \times \mathcal{J}^{\leq\infty}$ to $\mathcal{J}^{\leq\infty}$, and indeed this remains true if we replace the binomial coefficients by any other coefficients... However, the particular choice we made for the coefficients implies that $([\ ,\ ]_n)_{n\in\mathbb{Z}_{\geq 0}}$ is a *formal deformation* of $\mathcal{J}^{\leq\infty}$. This results from a general result we established with Choie, Dumas & Martin in 2021 [CDMR21b] and whose proof relies on a 2004 result due to Connes & Moscovici [CM04].

Let $A = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} A_k$ be a graded commutative $\mathbb{C}$-algebra, and D a derivation of A such that $D(A_k) \subset A_{k+2}$ for any $k \geq 0$. Let us consider the sequence $([\ ,\ ]_n^D)_{n\geq 0}$ of bilinear maps $A \times A \to A$ defined by bilinear extension of

$$[f,g]_n^D = \sum_{r=0}^{n} (-1)^r \binom{k+n-1}{n-r}\binom{\ell+n-1}{r} D^r(f) D^{n-r}(g),$$

for any $f \in A_k,\ g \in A_\ell$. Then, $([\ ,\ ]_n^D)_{n\geq 0}$ is a formal deformation of A.

A bit more surprising is the fact that $\mathcal{J}^{\leq\infty}0,\infty$ is also stable by $([\ ,\ ]_n)_{n\in\mathbb{Z}_{\geq 0}}$. To prove this result, we develope,d,a gain with Choie, Dumaus & Martin a general method called *extension-restriction*.

Let A a commutative $\mathbb{C}$-algebra, and Δ and D two $\mathbb{C}$-derivations of A satisfying

$$\Delta D - D\Delta = D.$$

The Connes-Moscovici deformation on A associated to (D,Δ) is the sequence $(\mathrm{CM}_n^{D,\Delta})_{n\geq 0}$ of bilinear maps $A\times A\to A$ defined for any $f,g\in A$ by

$$\mathrm{CM}_n^{D,\Delta}(f,g)=\sum_{r=0}^{n}\frac{(-1)^r}{r!(n-r)!}D^r(2\Delta+r)^{\langle n-r\rangle}(f)D^{n-r}(2\Delta+n-r)^{\langle r\rangle}(g),$$

with convention $1=\mathrm{Id}_A$ and for any function $F\colon A\to A$ the Pochhammer notation:

$$F^{\langle 0\rangle}=1 \quad\text{and}\quad F^{\langle m\rangle}=F(F+1)\cdots(F+m-1)\ \text{ for any } m\geq 1.$$

Théorème 2.3. *Consider a commutative $\mathbb{C}$-algebra R and a subalgebra A of R. Let Δ and θ be two $\mathbb{C}$-derivations of R such that $\Delta\theta-\theta\Delta=\theta$. We assume that*

(1) *$\Delta(A)\subseteq A$ and $\theta(A)\subseteq A$;*

(2) *there exists $h\in A$ such as $\Delta(h)=2h$;*

(3) *there exists $x\in R, x\notin A$ such that $\Delta(x)=x$ and $\theta(x)=-x^2+h$.*

Then, the derivation $D:=\theta+2x\Delta$ of R satisfies $\Delta D-D\Delta=D$ and the Connes-Moscovici deformation $(\mathrm{CM}_n^{D,\Delta})_{n\geq 0}$ of R defines by restriction to A a formal deformation of A.

$$A=\mathcal{J}^{\leq 0,\infty}\subset\mathcal{J}^{\leq\infty}=R,\Delta(f)=\frac{k}{2}f,\theta=\frac{1}{4}(\mathrm{Ob}-\mathrm{E}_1\,\partial_z),x=\frac{1}{4}\,\mathrm{e}_2,h=-\frac{5}{16}\,\mathrm{e}_4\,.$$

However, $\mathcal{J}$ and $\mathcal{J}<\infty,0>$ are not stable by $([\ ,\ ]_n)_{n\in\mathbb{Z}_{\geq 0}}$.

2.4.2. *Rankin-Cohen brackets of quasimodular type.* Consider

$$d=\partial_\tau+\frac{1}{4}\,\mathrm{E}_1\,\partial_z=\frac{1}{4}\,\mathrm{Ob}+\frac{1}{2}\,\mathrm{e}_2\,\Delta$$

and consider the sequence $(\llbracket\ ,\ \rrbracket_n)_{n\geq 0}$ of applications from $\mathcal{J}^{\leq\infty}\times\mathcal{J}^{\leq\infty}$ to $\mathcal{J}^{\leq\infty}$ defined by bilinear extension of

$$\llbracket f,g\rrbracket_n=\sum_{r=0}^{n}(-1)^r\binom{k+n-1}{n-r}\binom{\ell+n-1}{r}d^r(f)d^{n-r}(g)$$

for all $f\in\mathcal{J}_k^{\leq\infty}$, $g\in\mathcal{J}_\ell^{\leq\infty}$.

Since Ob stabilises $\mathcal{J}^{\leq\infty,0}$, then $\mathcal{J}^{\leq\infty}$ and $\mathcal{J}^{\leq\infty,0}$ are stable by any linear combination of $d^r(f)d^{n-r}(g)$. Again, applying our general method we find that the particular choice of coefficients implies that the sequence we have built is a formal deformation of $\mathcal{J}^{\leq\infty}$ and $\mathcal{J}^{\leq\infty,0}$.

Our extension-restriction method implies the more remarkable following statement : $(\llbracket\ ,\ \rrbracket_n)_n$ is a formal deformation of $\mathcal{J}$.

2.4.3. *The transvectant approach. Reference:* [Olv99, DMR24]

Finally, to build a sequence of bilinear maps that stabilises again $\mathcal{J}^{\leq\infty,0}$ *but not trivially* we use the notion of transvectant due to Cayley.

The r-th transvectant of $f,g\in C^\infty(\mathbb{C}^2)$ is

$$\begin{array}{cccc}\{f,g\}_n & : & \mathbb{C}^2 & \to & \mathbb{C}\\ & & (x,y) & \mapsto & \Omega^n\left(((x_1,y_1),(x_2,y_2))\mapsto f(x_1,y_1)g(x_2,y_2)\right)(x,y)\end{array}$$

where

$$\Omega=\det\begin{pmatrix}\partial/\partial x_1 & \partial/\partial y_1\\ \partial/\partial x_2 & \partial/\partial y_2\end{pmatrix}.$$

One can compute an explicit form:

$$\{f,g\}_n=\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{\partial^n f}{\partial x^{n-r}\partial y^r}\frac{\partial^n g}{\partial x^r\partial y^{n-r}}$$

and that the sequence $\left(\frac{1}{n!}\{\ ,\ \}_n\right)_n$ is a formal deformation of $C^\infty(\mathbb{C}^2)$.

Two other properties are:

(1) a recurrence formula (just the binomial theorem...):

$$\{f,g\}_{n+1}=\{\partial_x f,\partial_y g\}_n-\{\partial_y f,\partial_x g\}_n$$

that allows to compute recuresively all the brackets one we have seen that the 0 bracket is the product

(2) the formal deformation property is equivalent to

$$\sum_{r=0}^{n}\binom{n}{r}\{\{f,g\}_r,h\}_{n-r}=\sum_{r=0}^{n}\binom{n}{r}\{f,\{g,h\}_r\}_{n-r}.$$

These two properties are our main tool to prove that $\left(\frac{1}{n!}\{\ ,\ \}_n\right)_n$ is indeed a formal deformation of $\mathcal{J}^{\leq\infty,0}$.

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