ON THE MODULAR AFFINE HECKE CATEGORY

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1. Introduction

The goal of the series of 3 papers presented in this volume [BRR, BR2, BR3] is to construct an analogue for coefficients of positive characteristic of the main construction of [B2], and provide an application in the modular representation theory of reductive algebraic groups.

1.1. **First realization of the affine Hecke algebra, via p-adic groups.** Let G be an adjoint semisimple[1] algebraic group over an algebraically closed field $\mathbb{F}$ of characteristic $p > 0$, with a choice of Borel subgroup $B \subset G$ and maximal torus $T \subset B$. Let W_{f} be the Weyl group of (G, T); then the associated (extended) affine Weyl group is the semidirect product $W = W_{\mathrm{f}} \ltimes X_*(T)$, where $X_*(T)$ is the lattice of cocharacters of T. The group algebra $\mathbb{Z}[W]$ admits a natural deformation $\mathcal{H}$ called the *(extended) affine Hecke algebra*, which appears in different contexts in geometric representation theory. (Here by "deformation" we mean that $\mathcal{H}$ is an algebra over the ring $\mathbb{Z}[v, v^{-1}]$ where v is an indeterminate, which is free as a $\mathbb{Z}[v, v^{-1}]$-module, and we have an identification $\mathcal{H}/(v-1)\mathcal{H} = \mathbb{Z}[W]$.)

The first context (which provides one of the main motivations for studying this algebra) is that of representations of p-adic groups. Namely, consider a split semisimple group scheme G_0 over some finite subfield $\mathbb{F}_0 \subset \mathbb{F}$ whose base change to $\mathbb{F}$ is G, and assume that B and T are similarly obtained by base change from a Borel subgroup $B_0 \subset G_0$ and a split maximal torus $T_0 \subset B_0$. The theory of smooth representations of the group $G_0(\mathbb{F}_0((z)))$[2] on $\mathbb{K}$-vector spaces has attracted a lot of attention. (Here $\mathbb{K}$ is a fixed algebraically closed field of characteristic 0.) Bernstein's theory [Ber] allows one to break the category of such representations as a product indexed by $G_0(\mathbb{F}_0((z)))$-conjugacy classes of cuspidal data for Levi subgroups, and the factor corresponding to the pair consisting of the class of $T_0(\mathbb{F}_0((z)))$ and its trivial representation (the so-called "principal block") identifies with the subgroup of representations generated by their fixed points under the Iwahori subgroup $I_0 \subset G_0(\mathbb{F}_0((z)))$ consisting of elements whose image in $G_0(\mathbb{F}_0)$ belongs to $B_0(\mathbb{F}_0)$. Using results of Borel [Bo] and Iwahori–Matsumoto [IM], this subcategory therefore identifies with the category of modules for the algebra $\mathcal{H} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{K}$, where the algebra morphism $\mathbb{Z}[v, v^{-1}] \to \mathbb{K}$ sends v to the inverse of a fixed square root of q. In particular, in this way the classification the simple objects in this principal

[1]This assumption is made here for simplicity, but in the papers we consider more general connected reductive algebraic groups.

[2]Here we consider the case of the local field $\mathbb{F}_0((z))$ since it is closer to the geometry that will be considered later, but similar statements hold over other local fields, like finite extensions of $\mathbb{Q}_p$.

block (which is the subject of a part of Langlands' conjectures called the *Deligne–Langlands conjecture*) reduces to the classification of the simple $\mathcal{H} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{K}$-modules.

This identification relies of the description of $\mathcal{H} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{K}$ as the algebra of locally constant I_0-biinvariant functions with compact support on $G_0(\mathbb{F}_0((z)))$ (for an appropriate "convolution" product). Grothendieck's "faisceaux-fonctions" philosophy suggests to consider "categorifications" of this algebra, in the form of derived categories of Iwahori-equivariant constructible sheaves on the affine flag variety Fl_G of G; here Fl_G is an ind-scheme over $\mathbb{F}$ whose set of $\mathbb{F}$-points is the quotient $G(\mathbb{F}((z)))/\mathrm{I}(\mathbb{F})$, where $\mathrm{I}(\mathbb{F}) \subset G(\mathbb{F}[\![z]\!])$ is the preimage of B under the natural map $G(\mathbb{F}[\![z]\!]) \to G(\mathbb{F})$. (Grothendieck's original motivations involved $\overline{\mathbb{Q}}_\ell$-sheaves, but more general coefficients might be considered too.)

1.2. **Second realization of the affine Hecke algebra, via coherent sheaves.** The problem of classifying simple $\mathcal{H} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{K}$-modules (and, in fact, the classification of all simple modules over similar algebras where the image of v is not a root of unity) was solved by Kazhdan–Lusztig [KL] using a completely different geometric realization of $\mathcal{H}$, in terms of coherent sheaves on varieties attached to the Langlands dual group $G^\vee_\mathbb{C}$ over $\mathbb{C}$. Namely, by construction this group comes with a maximal torus $T^\vee_\mathbb{C} \subset G^\vee_\mathbb{C}$ whose lattice of characters identifies with $X_*(T)$, and we denote by $B^\vee_\mathbb{C} \subset G^\vee_\mathbb{C}$ the Borel subgroup containing $T^\vee_\mathbb{C}$ corresponding to B. The cotangent space $\widetilde{\mathcal{N}}$ to the flag variety $G^\vee_\mathbb{C}/B^\vee_\mathbb{C}$ admits a canonical morphism to the dual $(\mathfrak{g}^\vee_\mathbb{C})^*$ of the Lie algebra $\mathfrak{g}^\vee_\mathbb{C}$ of $G^\vee_\mathbb{C}$, and the Steinberg variety is the fiber product

$$\widetilde{\mathcal{N}} \times_{(\mathfrak{g}^\vee_\mathbb{C})^*} \widetilde{\mathcal{N}}.$$

It admits a canonical (diagonal) action of $G^\vee_\mathbb{C}$, and an action of the multiplicative group $\mathbb{G}_\mathrm{m}$ by (diagonal) dilation along the cotangent directions. Moreover, the equivariant K-theory

$$\mathsf{K}^{G^\vee_\mathbb{C} \times \mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}} \times_{(\mathfrak{g}^\vee_\mathbb{C})^*} \widetilde{\mathcal{N}})$$

admits a canonical product, given by a "convolution" procedure.[3] With this notation, a crucial step in Kazhdan–Lusztig's proof is the construction of a ring isomorphism

$$\mathcal{H} \xrightarrow{\sim} \mathsf{K}^{G^\vee_\mathbb{C} \times \mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}} \times_{(\mathfrak{g}^\vee_\mathbb{C})^*} \widetilde{\mathcal{N}}). \tag{1.1}$$

This isomorphism suggests to consider (derived) categories of equivariant coherent sheaves on $\widetilde{\mathcal{N}} \times_{(\mathfrak{g}^\vee_\mathbb{C})^*} \widetilde{\mathcal{N}}$ (or appropriate variants) as another categorical incarnation of the algebra $\mathcal{H}$. (Here again, the isomorphism (1.1) holds more generally when $\mathbb{C}$ is replaced by an algebraically closed field of very good characteristic, so that coefficients more general than $\mathbb{C}$ might be considered.)

[3] To be precise, this product was not considered by Kazhdan–Lusztig in [KL], and the result they actually prove is slightly weaker (although sufficient for the expected application to classification of simple $\mathcal{H} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{K}$-modules) than the one stated below. The convolution product was considered in parallel work by Ginzburg (see the reference [G] in [KL]), which was never published because it had a gap. A complete exposition of these constructions, which uses some of the results of Kazhdan–Lusztig to fill this gap, can be found in [CG]. For a slightly different treatment of these constructions, see [Lu2].

1.3. **The "two realizations" equivalence.** In [B2], the first author constructed several equivalences of monoidal categories relating these categorifications of $\mathcal{H}$, in the setting of étale $\overline{\mathbb{Q}}_\ell$-sheaves on Fl_G (here ℓ is a prime number distinct from p) and equivariant coherent sheaves on (variants of) the Steinberg variety of the dual group $G^\vee_{\overline{\mathbb{Q}}_\ell}$ over $\overline{\mathbb{Q}}_\ell$. We will not recall these equivalences explicitly here, but only mention that they have found important applications in representation theory (some of which are presented in [B1]) and in various geometric approaches to the Langlands program (in particular in work of Gaitsgory, Hemo–Zhu, and Ben-Zvi–Chen–Helm–Nadler).

1.4. **The modular case.** The main results of the present series of papers is a counterpart of (some of) the equivalences of [B2] in the setting of étale sheaves on Fl_G with coefficients in an algebraic closure $\Bbbk$ of $\mathbb{F}_\ell$ and equivariant coherent sheaves on the Steinberg variety of the Langlands dual group $G^\vee_\Bbbk$ over $\Bbbk$. Although they might find applications in some modular aspects of the Geometric Langlands Program, our desire to construct such variants primarily comes from questions in the representation theory of reductive algebraic groups over $\Bbbk$; conceretely, in [BR3] we use them to confirm a conjecture of Finkelberg–Mirković giving a geometric incarnation of the (extended) principal block of representations of the reductive group $\mathbf{G}$ over $\Bbbk$ whose Frobenius twist is $G^\vee_\Bbbk$ in terms of perverse sheaves on the affine Grassmannian Gr_G of G.

1.5. **Outline of the proof.** Although the basic ingredients of our approach on the constructible side are rather similar to those used in [B2] (in particular, we make extensive use of the geometric Satake equivalence [MV], Gaitsgory's central functor [Ga], and Wakimoto sheaves), the strategy we follow differs significantly from that of [B2]. It is based on a general technique first advertised by Soergel: on each side we identify a monoidal additive subcategory from which the whole category can be reconstructed as the bounded homotopy category, and we compare the subcategories on the two sides via an intermediate category of "Soergel bimodules"[4] for the group W.

On the constructible side this subcategory consists of tilting perverse sheaves, which are related to Soergel bimodules via a "functor $\mathbb{V}$". This is similar to other results of this form, as e.g. in [BY], although the present setting combines the technical difficulties of handling infinite-dimensional flag varieties with the necessity to incorporate some specific aspects of the loop group setting (i.e. the comparison with the geometric Satake equivalence, which is not considered in [BY]), and occupies the first two parts [BRR, BR2] of the series.

To handle the coherent side we use a third geometric incarnation of the affine Hecke algebra, constructed using Harish-Chandra bimodules for the group $\mathbf{G}$ as above. (This third player has no counterpart in [B2].) The relation between this category and coherent sheaves on Steinberg's variety is provided by the "localization theory" developed by the first author with Mirković and Rumynin [BMR], which plays a crucial role in [BR3]. In a sense, this is what allows to "control" the coherent side of the picture; namely, the additive subcategory alluded to above is

[4]Technically the category we use is not a category of Soergel bimodules in any usual sense, but rather a certain category of representations of an affine group scheme constructed out of the regular centralizer for $G^\vee_\Bbbk$. But philosophically this category plays exactly the traditional role of Soergel bimodules.

the subcategory corresponding to the "wall crossing bimodules" studied in [BR1] in the category of Harish-Chandra bimodules, and the proof of the fact that the whole category can be reconstructed from this subcategory crucially builds on the comparison with Harish-Chandra bimodules.

2. Contents

Each paper in this series relies on the results obtained in the previous part, but they have been written at different periods, and we have tried our best to make them understandable without prior familiarity with the other parts (if the reader accepts to use the results of these parts as black boxes).

2.1. **The regular quotient and representations of the regular unipotent centralizer.** The first step in our approach, handled in [BRR], consists of a "coherent" description of the "regular quotient" category $\mathsf{P}^0_{\mathrm{I},\mathrm{I}}$. Namely, recall the flag variety Fl_G considered above, and fix an algebraic closure $\Bbbk$ of $\mathbb{F}_\ell$, where ℓ is a prime number different from p. The ind-scheme Fl_G is the quotient of the loop group $\mathrm{L}G$ (whose $\mathbb{F}$-points are $G(\mathbb{F}((z)))$ by the Iwahori subgroup I. The I-orbits on Fl_G are naturally parametrized by W, so that the simple objects in the abelian category $\mathsf{P}_{\mathrm{I},\mathrm{I}}$ of I-equivariant perverse sheaves on Fl_G are in a canonical bijection with W. This group has a canonical structure of "quasi-Coxeter" group, hence in particular a length function. With this notation, $\mathsf{P}^0_{\mathrm{I},\mathrm{I}}$ is the Serre quotient of $\mathsf{P}_{\mathrm{I},\mathrm{I}}$ by the Serre subcategory generated by the simple objects whose label has positive length. The natural convolution operation on the I-equivariant derived category of sheaves on Fl_G induces a monoidal product on this category.

On the other hand, consider a regular unipotent element $\mathsf{u} \in G^\vee_\Bbbk$, and its centralizer $\mathrm{Z}_{G^\vee_\Bbbk}(\mathsf{u})$. (Under mild assumptions on ℓ, this is a smooth group scheme over $\Bbbk$.) Let $\mathsf{Rep}(\mathrm{Z}_{G^\vee_\Bbbk}(\mathsf{u}))$ be the abelian category of finite-dimensional algebraic representations of $\mathrm{Z}_{G^\vee_\Bbbk}(\mathsf{u})$; it is a monoidal category for the tensor product of representations. Then the main result of [BRR] states that, under mild assumptions on ℓ, there exists an equivalence of monoidal abelian categories

$$\mathsf{P}^0_{\mathrm{I},\mathrm{I}} \cong \mathsf{Rep}(\mathrm{Z}_{G^\vee_\Bbbk}(\mathsf{u})). \tag{2.1}$$

By construction, this equivalence is compatible with the geometric Satake equivalence [MV] in the following sense. Let $\mathrm{L}^+G \subset \mathrm{L}G$ be the arc group of G, and let $\mathsf{P}_{\mathrm{L}^+G,\mathrm{L}^+G}$ be the category of L^+G-equivariant perverse sheaves on the affine Grassmannian $\mathrm{Gr}_G = \mathrm{L}G/\mathrm{L}^+G$. Convolution endows this category with a monoidal structure, and the geometric Satake equivalence is a canonical equivalence of monoidal abelian categories

$$\mathsf{P}_{\mathrm{L}^+G,\mathrm{L}^+G} \xrightarrow{\sim} \mathsf{Rep}(G^\vee_\Bbbk). \tag{2.2}$$

Gaitsgory has constructed a "central" functor $\mathsf{Z} : \mathsf{P}_{\mathrm{L}^+G,\mathrm{L}^+G} \to \mathsf{P}_{\mathrm{I},\mathrm{I}}$, and we denote by Z^0 its composition with the natural quotient functor to $\mathsf{P}^0_{\mathrm{I},\mathrm{I}}$. Then the following diagram commutes, where the right-hand vertical arrow is the obvious forgetful functor:

$$\begin{array}{ccc}
\mathsf{P}_{\mathrm{L}^+G,\mathrm{L}^+G} & \xrightarrow{(2.2)} & \mathsf{Rep}(G^\vee_\Bbbk) \\
\downarrow{\scriptstyle \mathsf{Z}^0} & & \downarrow{\scriptstyle \mathsf{For}^{G^\vee_\Bbbk}_{\mathrm{Z}_{G^\vee_\Bbbk}(\mathsf{u})}} \\
\mathsf{P}^0_{\mathrm{I},\mathrm{I}} & \xrightarrow{(2.1)} & \mathsf{Rep}(\mathrm{Z}_{G^\vee_\Bbbk}(\mathsf{u})).
\end{array}$$

2.2. **Tilting perverse sheaves and the regular centralizer.** The next step is handled in [BR2]. For this, we consider the pro-unipotent radical $\mathrm{I_u} \subset \mathrm{I}$ (i.e. the preimage of the unipotent radical of B via the natural map $\mathrm{I} \to B$), and the category $\mathsf{D}_{\mathrm{I_u},\mathrm{I_u}}$ obtained as the triangulated subcategory in the $\mathrm{I_u}$-equivariant derived category of sheaves on $\mathrm{L}G/\mathrm{I_u}$ generated by complexes obtained by pullback from I-equivariant complexes on Fl_G. This category admits a convolution product which makes it a *nonunital* monoidal category, and Yun [BY] has explained how to construct a "completion" $\mathsf{D}^{\wedge}_{\mathrm{I_u},\mathrm{I_u}}$ which (in particular) becomes a unital monoidal category. The perverse t-structure on $\mathsf{D}_{\mathrm{I_u},\mathrm{I_u}}$ extends to a t-structure on $\mathsf{D}^{\wedge}_{\mathrm{I_u},\mathrm{I_u}}$ whose heart has a structure which is close to that of a highest weight category. In particular, there is a notion of *tilting perverse sheaves*, and we will denote by $\mathsf{T}^{\wedge}_{\mathrm{I_u},\mathrm{I_u}}$ the full subcategory of $\mathsf{D}^{\wedge}_{\mathrm{I_u},\mathrm{I_u}}$ whose objects are those tilting perverse sheaves. Classical arguments show that this category is closed under the monoidal product, and that there exists a natural equivalence of monoidal triangulated categories

$$K^{\mathrm{b}}\mathsf{T}^{\wedge}_{\mathrm{I_u},\mathrm{I_u}} \xrightarrow{\sim} \mathsf{D}^{\wedge}_{\mathrm{I_u},\mathrm{I_u}}. \tag{2.3}$$

On the dual side, consider a Steinberg section $\Sigma \subset G^{\vee}_{\Bbbk}$ to the adjoint quotient $G^{\vee}_{\Bbbk} \to G^{\vee}_{\Bbbk}/G^{\vee}_{\Bbbk}$ sending u to the base point in the quotient, and the restriction $\mathbb{J}_{\Sigma}$ of the universal centralizer group scheme (i.e. the affine group scheme over $G^{\vee}_{\Bbbk}$ whose fiber at an element g is the centralizer of g). Under mild assumptions on ℓ, this Steinberg section is well defined, and the group scheme is smooth. We have a canonical identification

$$G^{\vee}_{\Bbbk}/G^{\vee}_{\Bbbk} \xrightarrow{\sim} T^{\vee}_{\Bbbk}/W_{\mathrm{f}},$$

hence a canonical morphism $T^{\vee}_{\Bbbk} \times_{T^{\vee}_{\Bbbk}/W_{\mathrm{f}}} T^{\vee}_{\Bbbk} \to \Sigma$, and we denote by $\mathbb{I}^{\wedge}_{\Sigma}$ the pullback of $\mathbb{J}_{\Sigma}$ to the spectrum of the completion of $\mathscr{O}(T^{\vee}_{\Bbbk} \times_{T^{\vee}_{\Bbbk}/W_{\mathrm{f}}} T^{\vee}_{\Bbbk})$ with respect to the ideal corresponding to the point (e,e). Then the category $\mathsf{Rep}(\mathbb{I}^{\wedge}_{\Sigma})$ of representations of $\mathbb{I}^{\wedge}_{\Sigma}$ on coherent sheaves over this spectrum admits a canonical monoidal product. In [BR2] we construct a monoidal additive subcategory $\mathsf{SRep}(\mathbb{I}^{\wedge}_{\Sigma}) \subset \mathsf{Rep}(\mathbb{I}^{\wedge}_{\Sigma})$ of "Soergel representations," and the main result of [BR2] is an equivalence of monoidal categories

$$\mathsf{T}^{\wedge}_{\mathrm{I_u},\mathrm{I_u}} \cong \mathsf{SRep}(\mathbb{I}^{\wedge}_{\Sigma}) \tag{2.4}$$

(again, under some mild technical assumptions on ℓ). This equivalence enjoys some compatibility properties with (2.1) and with the geometric Satake equivalence (2.2) (via a "completed" variant of Gaitsgory's functor) that we will not state explicitly here.

2.3. **Coherent sheaves on the Steinberg variety.** For the last step of our program [BR3], we pass to the coherent side of the picture. Consider the multiplicative Grothendieck resolution $\widetilde{G^{\vee}_{\Bbbk}} = G^{\vee}_{\Bbbk} \times^{B^{\vee}_{\Bbbk}} B^{\vee}_{\Bbbk}$ (where $B^{\vee}_{\Bbbk}$ acts on itself by conjugation), with its natural morphism to $G^{\vee}_{\Bbbk}$ and the "multiplicative and Grothendieck" variant of the Steinberg variety given by

$$\mathrm{St_m} = \widetilde{G^{\vee}_{\Bbbk}} \times_{G^{\vee}_{\Bbbk}} \widetilde{G^{\vee}_{\Bbbk}}$$

This variety has a natural (diagonal) action of $G^{\vee}_{\Bbbk}$, and a canonical morphism to $T^{\vee}_{\Bbbk} \times_{T^{\vee}_{\Bbbk}/W_{\mathrm{f}}} T^{\vee}_{\Bbbk}$. Let $\mathrm{St}^{\wedge}_{\mathrm{m}}$ be the fiber product of $\mathrm{St_m}$ with the spectrum of the completion of $\mathscr{O}(T^{\vee}_{\Bbbk} \times_{T^{\vee}_{\Bbbk}/W_{\mathrm{f}}} T^{\vee}_{\Bbbk})$ with respect to the ideal corresponding to the point (e,e). This scheme again has a canonical action of $G^\vee_\Bbbk$, and the derived category

$$D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk}(\mathrm{St}^\wedge_{\mathrm{m}})$$

of $G^\vee_\Bbbk$-equivariant coherent sheaves on $\mathrm{St}^\wedge_{\mathrm{m}}$ admits a canonical monoidal product given by convolution. The main result of [BR3] is the construction of a monoidal additive subcategory

$$\mathsf{A} \subset D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk}(\mathrm{St}^\wedge_{\mathrm{m}})$$

such that we have a natural equivalence of monoidal triangulated categories

$$K^{\mathrm{b}}(\mathsf{A}) \xrightarrow{\sim} D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk}(\mathrm{St}^\wedge_{\mathrm{m}}), \tag{2.5}$$

and such that restriction to the preimage of $\Sigma \times_{G^\vee_\Bbbk/G^\vee_\Bbbk} \Sigma \subset G^\vee_\Bbbk \times_{G^\vee_\Bbbk/G^\vee_\Bbbk} G^\vee_\Bbbk$ induces an equivalence of monoidal categories

$$\mathsf{A} \xrightarrow{\sim} \mathsf{SRep}(\mathbb{I}^\wedge_\Sigma). \tag{2.6}$$

(Again, our proofs require some mild and explicit assumptions on ℓ.)

Combining the equivalences (2.3), (2.4), (2.5) and (2.6) we finally obtain an equivalence of monoidal categories

$$\mathsf{D}^\wedge_{\mathrm{I_u},\mathrm{I_u}} \xrightarrow{\sim} D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk}(\mathrm{St}^\wedge_{\mathrm{m}}) \tag{2.7}$$

which can be considered as a categorical version of the comparison between the two geometric realizations of $\mathcal{H}$ discussed in Section 1.

Remark 2.1.
1. Technically, the categories considered in (2.7) are rather categorical realizations of the group algebra of W, since we do not consider any counterpart of the parameter v.
2. In [BR3] we deduce from (2.7) several variants, and in particular an equivalence of nonunital monoidal categories

$$\mathsf{D}_{\mathrm{I_u},\mathrm{I_u}} \xrightarrow{\sim} D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk}_{\mathcal{U}}(\mathrm{St}_{\mathrm{m}})$$

where the right-hand side is the derived category of the category of equivariant coherent sheaves on St_{m} supported (set-theoretically) on the preimage of $(e,e) \in T^\vee_\Bbbk \times_{T^\vee_\Bbbk/W_{\mathrm{f}}} T^\vee_\Bbbk$.

The construction of the subcategory A (and, more importantly, the proof of its properties) is indirect; it requires a passage to the Lie algebra $\mathfrak{g}^\vee_\Bbbk$ of $G^\vee_\Bbbk$, and the consideration of the reductive group $\mathbf{G}$ whose Frobenius twist $\mathbf{G}^{(1)}$ is $G^\vee_\Bbbk$. Let also $\mathbf{B},\mathbf{T} \subset \mathbf{G}$ be the subgroups whose Frobenius twists are $B^\vee_\Bbbk, T^\vee_\Bbbk$ respectively, let $\mathbf{b},\mathbf{t}$ be their Lie algebras, let $\mathbf{u}$ be the Lie algebra of the unipotent radical of $\mathbf{B}$, and let $\mathbf{g}$ be the Lie algebra of $\mathbf{G}$. If the completed Steinberg variety $\mathbf{St}^\wedge_{\mathrm{m}}$ is defined in terms of $\mathbf{G}$ as above for $G^\vee_\Bbbk$, then we have $\mathbf{St}^{\wedge(1)}_{\mathrm{m}} = \mathrm{St}^\wedge_{\mathrm{m}}$. We also consider the additive Grothendieck resolution $\widetilde{\mathbf{g}} = \mathbf{G} \times^{\mathbf{B}} (\mathbf{g}/\mathbf{u})^*$. This smooth variety has a canonical morphism to $\mathbf{g}^*$, and we consider the fiber product

$$\mathbf{St} = \widetilde{\mathbf{g}} \times_{\mathbf{g}^*} \widetilde{\mathbf{g}}.$$

There exists a canonical morphism $\mathbf{St} \to \mathbf{t}^* \times_{\mathbf{t}^*/W_{\mathrm{f}}} \mathbf{t}^*$, and we denote by $\mathbf{St}^\wedge$ the fiber product of $\mathbf{St}$ with the spectrum of the completion of $\mathscr{O}(\mathbf{t}^* \times_{\mathbf{t}^*/W_{\mathrm{f}}} \mathbf{t}^*)$ with respect to the ideal corresponding to the point $(0,0)$. Then, under suitable technical assumptions we show that there exists a $\mathbf{G}$-equivariant isomorphism of schemes

$$\mathbf{St}^\wedge_{\mathrm{m}} \cong \mathbf{St}^\wedge,$$

hence finally an equivalence of monoidal triangulated categories

$$D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}}(\mathrm{St}^{\wedge}_{\mathrm{m}}) \cong D^{\mathrm{b}}\mathsf{Coh}^{\mathbf{G}^{(1)}}(\mathbf{St}^{\wedge(1)}).$$

In [BR3] we consider a certain category $\mathsf{HC}^{\widehat{0},\widehat{0}}$ of Harish-Chandra bimodules for $\mathbf{G}$ "completed at the trivial Harish-Chandra central character," whose derived category has a natural monoidal product induced by tensor product of bimodules, and using a variant of the localization theory developed in [BMR] we construct an equivalence of monoidal triangulated categories

$$D^{\mathrm{b}}\mathsf{HC}^{\widehat{0},\widehat{0}} \cong D^{\mathrm{b}}\mathsf{Coh}^{\mathbf{G}^{(1)}}(\mathbf{St}^{\wedge(1)}).$$

The subcategory A is obtained as the image under these equivalences of the Karoubian additive monoidal subcategory of $D^{\mathrm{b}}\mathsf{HC}^{\widehat{0},\widehat{0}}$ generated by the "wall-crossing bimodules," i.e. the bimodules which induce the standard wall-crossing functors on the derived category of representations of $\mathbf{G}$. These bimodules also play a major role in the previous paper [BR1] by the first two authors, and the proof of the desired properties of A follows from considerations close to those used in that paper.

2.4. **Application to the Finkelberg–Mirković conjecture.** The Weyl group of $(\mathbf{G},\mathbf{T})$ identifies with W_{f}, and acts naturally on $X^*(\mathbf{T})$. Pullback under the Frobenius morphism also induces a natural group morphism $X^*(T^\vee_{\Bbbk}) \to X^*(\mathbf{T})$, hence an action of $X_*(T) = X^*(T^\vee_{\Bbbk})$ on $X^*(\mathbf{T})$. Combining these actions we deduce an action of W on $X^*(\mathbf{T})$. The "dot-action" is the twist of this action defined by

$$w \bullet \lambda = w \cdot (\lambda + \rho) - \rho,$$

where $\rho \in \frac{1}{2}X^*(\mathbf{T})$ is the half-sum of the positive roots (chosen as the $\mathbf{T}$-weights in $\mathbf{g}/\mathbf{b}$). It is a standard fact that the simple objects in the category $\mathsf{Rep}(\mathbf{G})$ of finite-dimensional algebraic $\mathbf{G}$-modules are parametrized by the dominant weights $X^*(\mathbf{T})^+$. If ℓ is larger than the Coxeter number h of $\mathbf{G}$, the "extended principal block" $\mathsf{Rep}_{\langle 0\rangle}(\mathbf{G})$ in $\mathsf{Rep}(\mathbf{G})$ is the Serre subcategory generated by the simple objects whose label belongs to $W \bullet 0$. (This is a direct summand in $\mathsf{Rep}(\mathbf{G})$, which captures all the relevant combinatorial information on this category in a precise sense.) Pullback under the Frobenius morphism defines a fully faithful functor

$$\mathsf{Rep}(G^\vee_{\Bbbk}) \to \mathsf{Rep}(\mathbf{G}),$$

which takes values in $\mathsf{Rep}_{\langle 0\rangle}(\mathbf{G})$. In this way, the latter category can be considered an "enlargement" of $\mathsf{Rep}(G^\vee_{\Bbbk})$. The Finkelberg–Mirković conjecture [FM] gives an answer to the question of enlarging the equivalence (2.2) to a geometric description of $\mathsf{Rep}_{\langle 0\rangle}(\mathbf{G})$. More precisely, denoting by $\mathsf{P}_{\mathrm{L}^+G,\mathrm{I_u}}$ the category of $\mathrm{I_u}$-equivariant perverse sheaves on the "opposite affine Grassmannian"[5] $\mathrm{L}^+G\backslash\mathrm{L}G$, it postulates an equivalence of abelian categories

$$\mathsf{P}_{\mathrm{L}^+G,\mathrm{I_u}} \xrightarrow{\sim} \mathsf{Rep}_{\langle 0\rangle}(\mathbf{G})$$

such that the natural action of $\mathsf{P}_{\mathrm{L}^+G,\mathrm{L}^+G}$ by convolution on the left-hand side corresponds via the geometric Satake equivalence to the action of $\mathsf{Rep}(G^\vee_{\Bbbk})$ by tensor product with pullbacks under the Frobenius morphism. One of the nice aspects of

[5]Introducing this space allows to obtain an equivalence which behaves in the simplest possible way with respect to the natural parametrizations of simple objects on both sides. In any case, the assignment $g \mapsto g^{-1}$ induces an isomorphism $\mathrm{L}^+G\backslash\mathrm{L}G \xrightarrow{\sim} \mathrm{Gr}_G$, so that one can also describe this category in terms of perverse sheaves on the usual affine Grassmannian.

this equivalence is that it enlightens the meaning of Lusztig's character formula for simple modules in large characteristic. (For a detailed discussion of this topic, see e.g. [Wi, CW].)

As an application of the results discussed above, we prove this conjecture (under appropriate technical assumptions) in [BR3].

References

[Ber] J. Bernstein, *Le "centre" de Bernstein*, edited by P. Deligne, in Travaux en Cours, *Representations of reductive groups over a local field*, 1–32, Hermann, Paris, 1984.

[B1] R. Bezrukavnikov, *Noncommutative counterparts of the Springer resolution*, in *International Congress of Mathematicians. Vol. II*, 1119–1144, Eur. Math. Soc., Zürich, 2006.

[B2] R. Bezrukavnikov, *On two geometric realizations of an affine Hecke algebra*, Publ. Math. IHES **123** (2016), 1–67.

[BMR] R. Bezrukavnikov, I. Mirković, and D. Rumynin, *Localization of modules for a semisimple Lie algebra in prime characteristic*, with an appendix by R. Bezrukavnikov and S. Riche, Ann. of Math. **167** (2008), 945–991.

[BR1] R. Bezrukavnikov and S. Riche, *Hecke action on the principal block*, Compos. Math. **158** (2022), 953–1019.

[BR2] R. Bezrukavnikov and S. Riche, *Modular affine Hecke category and regular centralizer*, preprint arXiv:2206.03738.

[BR3] R. Bezrukavnikov and S. Riche, *On two modular geometric realizations of an affine Hecke algebra*, preprint arXiv:2402.08281.

[BRR] R. Bezrukavnikov, S. Riche, and L. Rider, *Modular affine Hecke category and regular unipotent centralizer*, preprint arXiv:2005.05583.

[BY] R. Bezrukavnikov and Z. Yun, *On Koszul duality for Kac–Moody groups*, Represent. Theory **17** (2013), 1–98.

[Bo] A. Borel, *Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup*, Invent. Math. **35** (1976), 233–259.

[CG] N. Chriss and V. Ginzburg, *Representation theory and complex geometry* Birkhäuser Boston, Inc., Boston, MA, 1997.

[CW] J. Ciappara and G. Williamson, *Lectures on the geometry and modular representation theory of algebraic groups*, J. Aust. Math. Soc. **110** (2021), no. 1, 1–47.

[FM] M. Finkelberg and I. Mirković, *Semi-infinite flags I. Case of global curve* $\mathbb{P}^1$, in *Differential topology, infinite-dimensional Lie algebras, and applications*, 81–112, Amer. Math. Soc. Transl. Ser. 2, vol. 194, Amer. Math. Soc., 1999.

[Ga] D. Gaitsgory, *Construction of central elements in the affine Hecke algebra via nearby cycles*, Invent. Math. **144** (2001), no. 2, 253–280.

[IM] N. Iwahori and H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of* $\mathfrak{p}$*-adic Chevalley groups*, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 5–48.

[KL] D. Kazhdan and G. Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math. **87** (1987), no. 1, 153–215.

[Lu2] G. Lusztig, *Bases in equivariant K-theory*, Represent. Theory 2 (1998), 298–369.

[MV] I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. **166** (2007), 95–143.

[Wi] G. Williamson, *Algebraic representations and constructible sheaves*, Jpn. J. Math. **12** (2017), no. 2, 211–259.

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