TALK $8\frac{1}{2}$: SHEAVES ON STACKS

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The aim of this talk is to explain a construction of categories of étale sheaves on (pre-)stacks using Kan extensions in the framework of ∞-categories.

Fix a prime number $\ell \in \mathbb{N}$. Denote by Λ a finite ℓ-torsion ring. By definition, all rings and schemes in this talk are over $\mathbb{Z}[\ell^{-1}]$. For a scheme X, we denote by $\mathrm{D}(X) := \mathrm{D}_{\text{ét}}(X, \Lambda)$ the left completion of $\mathrm{D}(X_{\text{ét}}, \Lambda)$, see [BS15, Definition 3.3.1].

Remark 0.1. If X has finite Λ-cohomological dimension, then $\mathrm{D}(X_{\text{ét}}, \Lambda)$ is left complete, that is, the map $A \to \lim_{n \geqslant 0} \tau^{\geqslant -n} A$ is an isomorphism for all $A \in \mathrm{D}(X_{\text{ét}}, \Lambda)$, see [BS15, Lemma 6.4.3].

1. Limit extended sheaf theories

Let Pr^{L} be the category with objects the presentable ∞-categories and with maps the colimit preserving functors. We denote by $\mathrm{Pr}^{\mathrm{St}}$ the full subcategory of stable objects (so the homotopy category is triangulated). Both categories are bicomplete and the inclusion $\mathrm{Pr}^{\mathrm{St}} \subset \mathrm{Pr}^{\mathrm{L}}$ preserves both limits and colimits (reference).

There is a natural enrichment $\mathcal{D}(X) \in \mathrm{Pr}^{\mathrm{St}}$ such that $\mathrm{D}(X) = \mathrm{ho}(\mathcal{D}(X))$ on homotopy categories. We consider the Yoneda embedding

$$\mathrm{AffSch} \to \mathrm{PreStk} := \mathrm{Fun}(\mathrm{AffSch}^{\mathrm{op}}, \mathrm{Ani}), \tag{1.1}$$

where AffSch is the category of affine schemes (over $\mathbb{Z}[\ell^{-1}]$, by definition) and Ani the ∞-category of anima (also called spaces, Kan complexes or ∞-groupoids).

Definition 1.1. The functor

$$\mathcal{D}\colon \mathrm{PreStk}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{St}} \tag{1.2}$$

is the right Kan extension of $\mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{St}}, X \mapsto \mathcal{D}, f \mapsto f^*$. For each $X \in \mathrm{PreStk}$, one denotes $\mathrm{D}(X) := \mathrm{ho}(\mathcal{D}(X))$ the homotopy category.

The right Kan extension exists because $\mathrm{Pr}^{\mathrm{St}}$ is complete.

Properties 1.2. (1) *The functor (1.2) is limit preserving. In particular, if $X \in \mathrm{PreStk}$, $X = \mathrm{colim}_{T \to X} T$ with $T \in \mathrm{AffSch}$, then $\mathcal{D}(X) = \lim_{T \to X} \mathcal{D}(T)$ with transition maps given by $*$-pullback. Here we note that the (non-full) inclusion $\mathrm{Pr}^{\mathrm{St}} \subset \mathrm{Cat}_\infty$ preserves limits so that the limit can equivalently be computed in Cat_∞.*

(2) *For every $f\colon Y \to X$ in PreStk, one has an adjunction*

$$f^* : \mathcal{D}(X) \leftrightarrows \mathcal{D}(Y) : Rf_*, \tag{1.3}$$

where f^* exists by construction and Rf_* is defined as its right adjoint using (refernce). Note that Rf_* is in general not colimit preserving, so it is only a functor in Cat_∞.

(3) *The functor $\mathcal{D}$ is an étale sheaf of ∞-categories: If $X \in \mathrm{PreStk}$ and $f\colon X \to X_{\text{ét}}$ the étale sheafification (or stackification), then*

$$f^*\colon \mathcal{D}(X_{\text{ét}}) \to \mathcal{D}(X) \tag{1.4}$$

is an equivalence. In other words, the functor $\mathcal{D}\colon \mathrm{PreStk}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{St}}$ factors through the sheafification functor $\mathrm{PreStk}^{\mathrm{op}} \to \mathrm{Stk}_{\text{ét}}, X \mapsto X_{\text{ét}}$ where, by definition, $\mathrm{Stk}_{\text{ét}}$ is the localization of PreStk at the maps $\mathrm{colim}\, S^{\bullet/T} \to T$ induced by the Čech nerves for all $S \to T$ in AffSch. By [HS21, Theorem], $\mathcal{D}$ is a sheaf for universal submersions ($\Longrightarrow$ v-sheaf $\Longrightarrow$ fpqc sheaf), but not an arc sheaf.

(4) *For* $X \in \mathrm{PreStk}$*, the category* $\mathcal{D}(X)$ *carries a t-structure* $(\mathcal{D}^{\leqslant 0}(X), \mathcal{D}^{\geqslant 0}(X))$ *such that* $f^*\colon \mathcal{D}(X) \to \mathcal{D}(Y)$ *is t-exact for all* $f\colon Y \to X$ *in* PreStk*.*
(5) *For* $X \in \mathrm{PreStk}$*, one has the full subcategory*

$$\mathcal{D}_{\mathrm{cons}}(X) \subset \mathcal{D}(X) \tag{1.5}$$

of perfect-constructible complexes compatibly with $*$*-pullbacks.*

2. Sheaves on the Hecke stack

Let k be an algebraically closed field, $X \to \mathrm{Spec}(k)$ a smooth, separated curve and G be a smooth, affine, connected k-group scheme.

Recall the definition of Hecke stacks from Talk 7: For a finite index set I and a point $x_I = (x_i)_{i\in I} \in X^I(R)$ for some k-algebra R, the union of the graphs $\Gamma_{x_I} = \cup_{i\in I}\Gamma_{x_i} \subset X_R$ defines a relative effective Cartier divisor over R. The formal completion $(X_R/\Gamma_{x_I})^\wedge$ is a formal affine scheme, say, equal to $\mathrm{Spf}(A_{x_I})$. We define the affine schemes

$$\mathbb{D}_{x_I} := \mathrm{Spec}(A_{x_I}), \quad \mathbb{D}^*_{x_I} := \mathbb{D}_{x_I}\backslash\Gamma_{x_I}. \tag{2.1}$$

If $J \subset I$, $x_J = (x_i)_{i\in J}$, then there is a natural map $\mathbb{D}_J \to \mathbb{D}_I$ compatible with the punctured discs.

Example 2.1. For $X = \mathbb{A}^1_k$ and $(x_1, x_2) \in X^2(k) = k^2$, one has $A_{(x_1,x_2)} = k[T]^\wedge_{(T-x_1)(T-x_2)}$.

The central objects are as follows:

Definition 2.2. For any finite index set I, there are the following functors $\mathrm{Alg}_k /X^I \to 1\text{-}\mathrm{Groupoids}$ of "Beilinson-Drinfeld type" given on a k-algebra R and a point $x_I \in X^I(R)$ as follows:

(1) The Hecke stack $\mathrm{Hk}_{G,I}(R)$ parametrizes two G-torsors $\mathcal{E}_1, \mathcal{E}_2$ on $\mathbb{D}_{x_I}$ and an isomorphism $\alpha\colon \mathcal{E}_1 \cong \mathcal{E}_2$ over $\mathbb{D}^*_{x_I}$.
(2) The affine Grassmannian $\mathrm{Gr}_{G,I}(R)$ parametrizes $\underline{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \alpha) \in \mathrm{Hk}_{G,I}(R)$ together with an isomorphism $\beta\colon \mathcal{E}_2 \cong \mathcal{E}_0$ on $\mathbb{D}_{x_I}$ where $\mathcal{E}_0$ is the trivial G-torsor.
(3) The loop group $L_I G(R)$ parametrizes $\gamma \in G(\mathbb{D}^*_{x_I})$. Its subfunctor $L^+_I G(R)$, called the positive loop group, parametrizes $\gamma \in G(\mathbb{D}_{x_I})$.

Note that $\mathrm{Gr}_{G,I}$ is (equivalent to) a set valued functor and that $L_I G$, $L^+_I G$ are group valued functors.

The forgetful morphism $\mathrm{Gr}_{G,I} \to \mathrm{Hk}_{G,I}, (\underline{\mathcal{E}}, \beta) \mapsto \underline{\mathcal{E}}$ is an $L^+_I G$-torsor and induces an equivalence $\mathrm{Hk}_{G,I} \cong \left[L^+_I G\backslash \mathrm{Gr}_{G,I}\right]_{\text{ét}}$. We choose a $L^+_I G$-stable filtered presentation $\mathrm{Gr}_{G,I} = \mathrm{colim} X_i$ by finite type k-schemes X_i with closed transition morphisms. Writing $L^+_I G = \lim_{i\geqslant 0} G_i$, $G_i = \mathrm{Res}_{\Gamma^{(i)}_{x_{\mathrm{univ}}}/X^I}(G)$ and possibly renumbering the X_i, we may assume that the $L^+_I G$-action on each X_i factors through G_i. In particular, we obtain as objects in PreStk:

$$\mathrm{Hk}_{G,I} = \mathrm{colim} \left[L^+_I G\backslash X_i\right]_{\text{ét}}, \quad [L^+_I G\backslash X_i]_{\text{ét}} = \lim_{j\geqslant i} [G_j\backslash X_i]_{\text{ét}}. \tag{2.2}$$

Definition 2.3. One defines the following full subcategories of $\mathrm{D}(\mathrm{Hk}_{G,I})$, respectively $\mathrm{D}(\mathrm{Gr}_{G,I})$ of sheaves with bounded supports:

(1) $\mathrm{D}(\mathrm{Hk}_{G,I}, \Lambda)^{\mathrm{bd}} = \mathrm{colim}\, \mathrm{D}([L^+_I G\backslash X_i]_{\text{ét}})$;
(2) $\mathrm{D}(\mathrm{Gr}_{G,I}, \Lambda)^{\mathrm{bd}} = \mathrm{colim}\, \mathrm{D}(X_i)$.

In both cases, the transition maps are given by $*$-push forward.

The following lemma allows to relate the above categories:

Lemma 2.4. *For each* G_i *acting on* X_i*, there are natural equivalences*

$$\mathcal{D}([L^+_I G\backslash X_i]_{\text{ét}}) \xrightarrow{\cong} \mathcal{D}([G_i\backslash X_i]_{\text{ét}}) \xrightarrow{\cong} \lim \left(\mathcal{D}(X_i) \underset{\mathrm{pr}^*}{\overset{\mathrm{act}^*}{\rightrightarrows}} \mathcal{D}(G_i \times X_i) \rightrightarrows \cdots \right). \tag{2.3}$$

Furthermore, the induced functor on the hearts of the standard t-structure

$$\mathrm{D}([L^+_I G\backslash X_i]_{\text{ét}})^\heartsuit \to \mathrm{D}(X_i)^\heartsuit \tag{2.4}$$

is fully faithful with essential image those objects $A \in \mathrm{D}(X_i)^\heartsuit$ *such that* $\mathrm{act}^* A \cong \mathrm{pr}^* A$*.*

Proof. The affine group scheme $L_I^+G \to X^I$ is strictly pro-algebraic in the sense of [RS20, Appendix A.2] with geometrically connected fibers and the kernel of $L_I^+G \to G_i$ is split pro-unipotent for every $i \geqslant 0$. So the first arrow in (2.3) being an equivalence follows from $\mathbb{A}^1$-invariance using that the coefficients Λ are of torsion invertible on $\mathrm{Spec}(k)$, the argument of [RS20, Proposition 2.2.11] translates to our context. By étale descent, we have an equivalence $\mathcal{D}([G_i\backslash X_i]_{\text{ét}}) \cong \mathcal{D}(G_i\backslash X_i)$. Using that $G_i\backslash X_i$ is the colimit of the Bar resolution, the second arrow in (2.3) is an equivalence because (1.2) is limit-preserving. For the fully faithfulness of (2.4) and the description of its essential image, we refer to Talk 9. □

References

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