TALK 3: UNIVERSALLY LOCALLY ACYCLIC SHEAVES

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In these notes we introduce universally locally acyclic sheaves following [HS21] (and [LZ20] and [FS21]). In [LZ20] ULA-sheaves are characterised as dualisable objects in a certain symmetric monoidal 2-category. We use this approach to rederive many useful properties of ULA-sheaves. As an application, we use ULA-sheaves to study the nearby cycles functor. The nearby cycles functor is used critically in [HS21] to show that their relative perverse t-structure has favourable properties.

These are notes for a talk from a workshop on the Geometric Satake equivalence in Clermont-Ferrand in January 2022. We closely follow [HS21, Sections 3 and 4] while trying to provide more details at various points.

Setup. We fix a prime ℓ. Let Λ be a finite ℓ-torsion ring (for example $\Lambda = \mathbb{Z}/\ell^n$)[1]. Further, we assume that all schemes are qcqs, and live over $\mathbb{Z}[\frac{1}{\ell}]$. For simplicity, we moreover assume that schemes have finite ℓ-cohomological dimension[2]. This assumption is satisfied for all schemes of finite type over a finite field or an algebraically closed field[3]; this includes the situation relevant to the proof of the geometric Satake equivalence.

Let us recall some of the facts from the previous talk, for precise references for the claims below compare the notes of talk 2. We denote by $D(X) = D(X_{\text{ét}}, \Lambda)$ the derived category of sheaves of Λ-modules on the étale site of X[4]. Furthermore, we denote by $D_{\text{cons}}(X) \subseteq D(X)$ the full subcategory of perfect-constructible complexes. Due to our finiteness assumption, $D(X)$ is left-complete and compactly generated with compact objects $D_{\text{cons}}(X)$. Recall moreover, that $D(X)$ supports a six-functor formalism. In other words, we have an adjoint pair of endofunctors $-\otimes^{\mathbb{L}}_{\Lambda} -$, $\mathrm{R}\mathcal{H}\mathrm{om}_{\Lambda}(-,-)$, for any morphism of schemes $f\colon X \to Y$ an adjunction

$$D(Y) \underset{Rf_*}{\overset{f^*}{\rightleftarrows}} D(X)$$

and finally, when f is separated and of finite type we have an adjunction

$$D(X) \underset{Rf^!}{\overset{Rf_!}{\rightleftarrows}} D(Y).$$

These functors satisfy the usual properties, for example proper (respectively smooth) base change or projection formulas. Moreover, $\otimes^{\mathbb{L}}_{\Lambda}$, f^* and $Rf_!$ preserve perfect-constructible objects by the usual finiteness results. However, their respective right adjoints may not preserve perfect-constructibles in general.

[1] [HS21] also consider more general coefficients Λ, for example $\Lambda = \mathbb{Z}_\ell$ or $\Lambda = \mathbb{Q}_\ell$ using the notion of constructible sheaves of [HRS21].

[2] This assumption is purely to simplify the presentation. In order to remove it, we would need to pass to the left-completion of the derived category. All the results presented here remain true.

[3] More generally, by a result of Gabber all affine schemes $X \to S$ of finite type over an affine scheme S all of whose connected components are spectra of absolutely integrally closed valuation rings have finite ℓ-cohomological dimension. This can be useful to reduce to the case of finite ℓ-cohomological dimension in general.

[4] For our purposes it suffices to consider $D(X)$ as a triangulated category. For the descent results in [HS21] (which we do not discuss in detail here) it is however necessary to view $D(X)$ as ∞-categories.

1. The classical approach to universal local acyclicity

Roughly speaking, ULA sheaves relative to a map $f\colon X \to S$ of schemes should be constructible sheaves on X such that their cohomology on all geometric fibres are isomorphic. We follow the presentation of [Sch20, Lecture 18].

For a scheme X and a geometric point $\bar{x}$ of X we denote by $X_{\bar{x}}$ the strict Henselisation of X at $\bar{x}$. We say that a geometric point $\bar{x}'$ of X is a *generisation* of $\bar{x}$ if $\bar{x}' \to X$ factors through $X_{\bar{x}}$.

Definition 1.1. Let $f\colon X \to S$ be a separated map of finite presentation. We say a complex $A \in D_{\mathrm{cons}}(X)$ is *f-locally acyclic (in the classical sense)* if for all geometric points $\bar{s} \to S$, geometric points $\bar{x} \to X$ lying over $\bar{s}$ and generisations $\bar{t}$ of $\bar{s}$

$$\begin{array}{ccccc} & & \bar{x} & \longrightarrow & X \\ & & \downarrow & & \downarrow \\ \bar{t} & \rightsquigarrow & \bar{s} & \longrightarrow & S \end{array}$$

the natural map

$$A_{\bar{x}} = R\Gamma(X_{\bar{x}}, A) \to R\Gamma(X_{\bar{x}} \times_{S_{\bar{s}}} \bar{t}, A)$$

is an isomorphism.

Moreover, we say that that A is *f-universally locally acyclic (in the classical sense)* (or *f-ULA*) if it is locally acyclic after any base change $S' \to S$.

While in general being locally acyclic may not be stable under base change, by a theorem of Gabber [LZ17, Corollary 6.6], when S is noetherian, A is locally acyclic if and only if A is universally locally acyclic.

Remark 1.2.

(i) By [Ill06, Corollary 3.5], A is universally locally acyclic if and only if after any base change $S' \to S$ the natural map

$$A_{\bar{x}} \to R\Gamma(X_{\bar{x}} \times_{S_{\bar{s}}} S_{\bar{t}}, A)$$

is an isomorphism. This characterisation allows for a generalisation to the world of diamonds and is thus used in [FS21].

(ii) These conditions clearly also imply that after any base change along $\mathrm{Spec} V \to S$ for any rank 1 valuation ring V with algebraically closed fraction field K and any geometric point $\overline{x} \to X$ mapping to the special point of $\mathrm{Spec} V$, the map

$$A|_{\overline{x}} = R\Gamma(X_{\overline{x}}, A) \to R\Gamma(X_{\overline{x}} \times_{\mathrm{Spec} V} \mathrm{Spec} K, A)$$

is an isomorphism. We show in Theorem 4.4 below that this condition is also sufficient.

Remark 1.3. When $S = X$ and $f\colon X \to X$ is the identity, for a generisation $\bar{y} \rightsquigarrow \bar{x}$ of a geometric point $\bar{x} \to X$ the map

$$A_{\bar{x}} = R\Gamma(X_{\bar{x}}, A) \to R\Gamma(X_{\bar{y}}, A) = A_{\bar{y}}$$

is the specialisation map. As perfect-constructible complexes are locally constant if and only if all specialisation maps are isomorphisms, compare [Sta, Tag 0GKC], it follows that a perfect-constructible complex on X is id_X-universally acyclic if and only if it is locally constant.

2. The symmetric monoidal 2-category of cohomological correspondences

We now present the approach of [LZ20] and [HS21] to define universally local acyclic sheaves, characterising ULA sheaves as dualisable objects in a certain symmetric monoidal 2-category of cohomological correspondences. [LZ20] and [HS21] consider slightly different categories, however the resulting notion of ULA-sheaves agrees. We follow the approach of [HS21].

We start by recalling the definition and basic properties of dualisable obejcts.

Definition 2.1. Let $(\mathcal{D}, \otimes, \mathbf{1})$ be a symmetric monoidal 2-category, where $\otimes$ denotes the tensor product on $\mathcal{D}$ and $\mathbf{1}$ its tensor unit. An object X of $\mathcal{D}$ is called *dualisable* if there exists an object $X^\vee$) together with maps $\varepsilon\colon X \otimes X^\vee \to \mathbf{1}$ and $\eta\colon \mathbf{1} \to X \otimes X^\vee$, called *counit* and *unit*, respectively, such that the composites

$$X \to X \otimes X^\vee \otimes X \to X, \qquad X^\vee \to X^\vee \otimes X \otimes X^\vee \to X^\vee$$

are isomorphic to the respective identities. The object $X^\vee$ is called the *dual* of X. It is unique if it exists, compare Remark 2.2 below, justifying the notation.

Remark 2.2 (c.f. [LZ20, Remark 1.3])**.** We collect some direct consequences. Let X be a dualisable object in a symmetric monoidal 2-category $(\mathcal{D}, \otimes, \mathbf{1})$ as defined above.

(i) The unit and counit of X show that $- \otimes X^\vee$ is both left and right adjoint to $- \otimes X$. Thus, internal Hom objects $\underline{\mathrm{Hom}}_{\mathcal{D}}(X, Y)$ exist for all objects Y of $\mathcal{D}$ and are given by $\underline{\mathrm{Hom}}_{\mathcal{D}}(X, Y) = Y \otimes X^\vee$. In particular, $X^\vee \cong \underline{\mathrm{Hom}}_{\mathcal{D}}(X, \mathbf{1})$.
(ii) The dual $X^\vee$ is dualisable with dual X. In particular, the biduality map $X \to (X^\vee)^\vee$ is an isomorphism.
(iii) For another dualisable object Y of $\mathcal{D}$ also the tensor product $X \otimes Y$ is dualisable with dual $X^\vee \otimes Y^\vee$. Unit and counit are given by the product of the respective maps for X and Y.

Lemma 2.3 ([LZ20, Lemma 1.4])**.** *Let $(\mathcal{D}, \otimes, \mathbf{1})$ be a symmetric monoidal 2-category. Then X is dualisable in $\mathcal{D}$ if and only if $\underline{\mathrm{Hom}}_{\mathcal{D}}(X, \mathbf{1})$ and $\underline{\mathrm{Hom}}_{\mathcal{D}}(X, X)$ exist and the natural map*

$$m\colon X \otimes \underline{\mathrm{Hom}}_{\mathcal{D}}(X, \mathbf{1}) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(X, X)$$

is an isomorphism.

The map m is constructed as follows: For all objects X of $\mathcal{D}$ we have a natural map $\varepsilon_X : X^\vee \otimes X \to \mathbf{1}$ given by the counit of the tensor-hom adjunction. The map m is then the adjoint of $\mathrm{id}_X \otimes \varepsilon_X$.

Proof. One direction follows directly from the previous remark. Let us now assume that $X \otimes \underline{\mathrm{Hom}}_{\mathcal{D}}(X, \mathbf{1}) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(X, X)$ is an isomorphism. We set $X^\vee = \underline{\mathrm{Hom}}_{\mathcal{D}}(X, \mathbf{1})$ and define the unit $\eta\colon X^\vee \otimes X \to \mathbf{1}$ as the evaluation, i.e. the adjoint map of the identity $X^\vee \to X^\vee$ and the counit as the composition $\mathbf{1} \to \underline{\mathrm{Hom}}_{\mathcal{D}}(X, X) \xrightarrow{m^{-1}} X \otimes X^\vee$, where the first map is the adjoint of the identity $X \to X$. We can now check that the maps satisfy the unit-counit equations. □

We now present the 2-categories [HS21] use to define universally locally acyclic sheaves. For the remainder of this section let us fix a base scheme S. Recall that by our standing assumption S is qcqs and ℓ is invertible on S.

Definition 2.4. We define a symmetric monoidal 2-category $\mathcal{C}_S$ as follows:

- *Objects* are schemes $f : X \to S$ separated and of finite presentation over S.
- *Morphisms* are given by $\mathrm{Fun}_{\mathcal{C}_S}(X, Y) := D(X \times_S Y)$ with composition given by convolution
$$D(X \times_S Y) \times D(Y \times_S Z) \to D(X \times_S Z)$$
$$(A, B) \mapsto A \star B = R\pi_{XZ!}(\pi_{XY}^* A \otimes^{\mathbb{L}}_{\Lambda} \pi_{YZ}^* B),$$
where $\pi_{XY}, \pi_{XZ}, \pi_{YZ}$ are the obvious projections defined on $X \times_S Y \times_S Z$, and identities are given by
$$R\Delta_{X/S!}\Lambda = R\Delta_{X/S*}\Lambda,$$
where $\Delta_{X/S} : X \to X \times_S X$ is the diagonal. As we assumed X/S to be separated and finitely presented, its diagonal is a finitely presented closed immersion.
- The *symmetric monoidal structure* on $\mathcal{C}_S$ is given on objects by $X \boxtimes Y = X \times_S Y$, and similarly on morphisms by exterior tensor products. The tensor unit is given by S.

Note that as $D(X \times_S Y)$ is symmetric in X and Y, $\mathcal{C}_S$ is canonically isomorphic to its opposite category $\mathcal{C}_S^{\mathrm{op}}$, where the direction of 1-morphisms (but not 2-morphisms) is reversed.

We check that this construction is well-defined. For associativity of the composition we see for $A \in D(W \times_S X), B \in D(X \times_S Y)$ and $C \in D(Y \times_S Z)$ using proper base change and the projection formula that

$$\begin{aligned}(A \star B) \star C &\cong R\pi_{WZ!}(\pi_{WY}^*(R\pi_{WY!}(\pi_{WX}^* A \otimes_\Lambda^{\mathbb{L}} \pi_{XY}^* B)) \otimes_\Lambda^{\mathbb{L}} \pi_{YZ}^* C) \\ &\cong R\pi_{WZ!}(R\pi_{WYZ!}(\pi_{WXY}^*(\pi_{WX}^* A \otimes_\Lambda^{\mathbb{L}} \pi_{XY}^* B)) \otimes_\Lambda^{\mathbb{L}} \pi_{YZ}^* C) \\ &\cong R\pi_{WZ!}(R\pi_{WYZ!}(\pi_{WX}^* A \otimes_\Lambda^{\mathbb{L}} \pi_{XY}^* B) \otimes_\Lambda^{\mathbb{L}} \pi_{YZ}^* C) \\ &\cong R\pi_{WZ!}(R\pi_{WYZ!}(\pi_{WX}^* A \otimes_\Lambda^{\mathbb{L}} \pi_{XY}^* B \otimes_\Lambda^{\mathbb{L}} \pi_{YZ}^* C)) \\ &\cong R\pi_{WZ!}(\pi_{WX}^* A \otimes_\Lambda^{\mathbb{L}} \pi_{XY}^* B \otimes_\Lambda^{\mathbb{L}} \pi_{YZ}^* C).\end{aligned}$$

Here (and in the following), we use a slight abuse of notation and denote for example by π_{WZ} the projection to $W \times_S Z$ from both $W \times_S Y \times_S Z$ and $W \times_S X \times_S Y \times_S Z$. It is always clear from context which projection exactly is meant.

Similarly, we see that $A \star (B \star C) \cong R\pi_{WZ!}(\pi_{WX}^* A \otimes_\Lambda^{\mathbb{L}} \pi_{XY}^* B \otimes_\Lambda^{\mathbb{L}} \pi_{YZ}^* C)$, showing associativity. Moreover, in order to check that the identities defined above are indeed identities, we have to check that for $A \in D(X \times_S Y)$ we have

$$(R\Delta_{X/S!}\Lambda) \star A = R\pi_{X^{(1)}Y!}(\pi_{X^{(1)}X^{(2)}}^* R\Delta_{X/S!}\Lambda \otimes_\Lambda^{\mathbb{L}} \pi_{X^{(2)}Y}^* A) \cong A,$$

where we denote by $X^{(1)}$ (respectively $X^{(2)}$) the first (respectively second) factor of $X \times_S X \times_S Y = X^{(1)} \times_S X^{(2)} \times_S Y$. For this, we note that using the adjunctions we get for any $B \in D(X \times_S Y)$:

$$\begin{aligned}&\mathrm{Hom}(R\pi_{X^{(1)}Y!}(\pi_{X^{(1)}X^{(2)}}^* R\Delta_{X/S!}\Lambda \otimes_\Lambda^{\mathbb{L}} \pi_{X^{(2)}Y}^* A), B) \\ &= \mathrm{Hom}(\pi_{X^{(1)}X^{(2)}}^* R\Delta_{X/S!}\Lambda \otimes_\Lambda^{\mathbb{L}} \pi_{X^{(2)}Y}^* A, \pi_{X^{(1)}Y}^! B) \\ &= \mathrm{Hom}(R(\Delta_{X/S} \times \mathrm{id}_Y)_! \pi_X^* \Lambda, \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_{X^{(2)}Y}^* A, \pi_{X^{(1)}Y}^! B)) \\ &= \mathrm{Hom}(\Lambda, (\Delta_{X/S} \times \mathrm{id}_Y)^! \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_{X^{(2)}Y}^* A, \pi_{X^{(1)}Y}^! B)) \\ &= \mathrm{Hom}(\Lambda, \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(A, B)) \\ &= \mathrm{Hom}(A, B).\end{aligned}$$

In particular, the endofunctor $R\pi_{X^{(1)}Y!}(\pi_{X^{(1)}X^{(2)}}^* R\Delta_{X/S!}\Lambda \otimes_\Lambda^{\mathbb{L}} \pi_{X^{(2)}Y}^* -)$ on $D(X \times_S Y)$ is left adjoint to the identity functor. Hence, we have a natural isomorphism

$$R\pi_{X^{(1)}Y!}(\pi_{X^{(1)}X^{(2)}}^* R\Delta_{X/S!}\Lambda \otimes_\Lambda^{\mathbb{L}} \pi_{X^{(2)}Y}^* A) \to A.$$

Lemma 2.5. *All objects of $\mathcal{C}_S$ are self-dual in this sense (that is, all objects are dualisable and the dual of an object X is X itself) with unit $S \to X \times_S X$ and counit $X \times_S X \to S$ both given by $R\Delta_{X/S!}\Lambda \in D(X \times_S X)$.*

Proof. We have to check that the unit-counit equations are satisfied, in other words that the induced map $X \to X \times X \times X \to X$ is the identity on X. Plugging in the definitions we see that the map is given by

$$\begin{aligned}&(R\Delta_{X/S!}\Lambda \boxtimes \mathrm{id}_{X \times X}) \star (\mathrm{id}_{X \times X} \boxtimes R\Delta_{X/S!}\Lambda) \\ &\cong R\pi_{15!}(\pi_{1-4}^*(\pi_{12}^* R\Delta_{X/S!}\Lambda \otimes_\Lambda^{\mathbb{L}} \pi_{34}^* R\Delta_{X/S!}\Lambda) \otimes_\Lambda^{\mathbb{L}} \pi_{2-5}^*(\pi_{23}^* R\Delta_{X/S!}\Lambda \otimes_\Lambda^{\mathbb{L}} \pi_{45}^* R\Delta_{X/S!}\Lambda)) \\ &\cong R\pi_{15!}(\pi_{12}^* R\Delta_{X/S!}\Lambda \otimes_\Lambda^{\mathbb{L}} \pi_{23}^* R\Delta_{X/S!}\Lambda \otimes_\Lambda^{\mathbb{L}} \pi_{34}^* R\Delta_{X/S!}\Lambda \otimes_\Lambda^{\mathbb{L}} \pi_{45}^* R\Delta_{X/S!}\Lambda) \\ &\cong \mathrm{id}_X \star \mathrm{id}_X \star \mathrm{id}_X \star \mathrm{id}_X \\ &\cong \mathrm{id}_X,\end{aligned}$$

where we denote by π_- the projection to the corresponding factors of $X \times_S X \times_S X \times_S X \times_S X$. In the second to last step we used the formula in the proof of associativity of convolution above. □

In particular, internal Hom's exist in $\mathcal{C}_S$ and are given by $\underline{\mathrm{Hom}}_{\mathcal{C}_S}(X, Y) = X \times_S Y$.

Definition 2.6. We also consider the lax co-slice 2-category $\mathcal{C}'_S =_{S\backslash} \mathcal{C}_S$ (or equivalently the colax slice category). It is more explicitly given as follows:

- *Objects* of $\mathcal{C}'_S$ are pairs (X, A), where $f : X \to S$ is a separated map of finite presentation, i.e. an object of $\mathcal{C}_S$, and $A \in D(X) = \mathrm{Fun}_{\mathcal{C}_S}(S, X)$ a morphism from S to X in $\mathcal{C}_S$.
- *Morphisms* in $\mathcal{C}'_S$ between objects (X, A) and (Y, B) are given by a morphism C from X to Y in $\mathcal{C}_S$ together with a 2-morphism $A \star C \to B$, i.e. by $C \in D(X \times_S Y) = \mathrm{Fun}_{\mathcal{C}_S}(X, Y)$ together with a map
$$R\pi_{Y!}(\pi_X^* A \otimes_\Lambda^{\mathbb{L}} C) \to B,$$
where π_X, π_Y are the natural projections on $X \times_S Y$.

The symmetric mononoidal structure on $\mathcal{C}_S$ induces a symmetric monoidal structure on $\mathcal{C}'_S$, which is given by
$$(X, A) \boxtimes (Y, B) = (X \times_S Y, A \boxtimes B),$$
with tensor unit given by (S, Λ).

Lemma 2.7 ([LZ20, Lemma 2.8])**.** *The symmetric monoidal structure on $\mathcal{C}'_S$ is closed with internal Hom's given by*
$$\underline{\mathrm{Hom}}_{\mathcal{C}'_S}((X, A), (Y, B)) = (X \times_S Y, \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_X^* A, R\pi_Y^! B)).$$

Proof. We construct an equivalence of categories
$$\mathrm{Fun}_{\mathcal{C}'_S}((X, A) \boxtimes (Y, B), (Z, C)) \to \mathrm{Fun}_{\mathcal{C}'_S}((X, A), \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_Y^* B, R\pi_Z^! C)).$$
Recall that objects on the left hand side are given by
$$(D, g\colon (A \boxtimes B) \star D \to C)$$
with $D \in D(X \times_S Y \times_S Z)$ while objects on the right hand side are given by
$$(D, g\colon A \star D \to \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_Y^* B, R\pi_Z^! C))$$
with $D \in D(X \times_S Y \times_S Z)$. The equivalence of categories is then given by
$$\begin{aligned}
\mathrm{Hom}((A \boxtimes B) \star D, C) &= \mathrm{Hom}(R\pi_{Z!}(\pi_X^* A \otimes_\Lambda^{\mathbb{L}} \pi_Y^* B \otimes_\Lambda^{\mathbb{L}} D), C) \\
&= \mathrm{Hom}(\pi_X^* A \otimes_\Lambda^{\mathbb{L}} \pi_Y^* B \otimes_\Lambda^{\mathbb{L}} D, \pi_Z^! C) \\
&= \mathrm{Hom}(\pi_X^* A \otimes_\Lambda^{\mathbb{L}} D, \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_Y^* B, \pi_Z^! C)) \\
&\overset{(*)}{=} \mathrm{Hom}(\pi_X^* A \otimes_\Lambda^{\mathbb{L}} D, \pi_{YZ}^! \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_Y^* B, \pi_Z^! C)) \\
&= \mathrm{Hom}(R\pi_{YZ!}(\pi_X^* A \otimes_\Lambda^{\mathbb{L}} D), \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_Y^* B, \pi_Z^! C)) \\
&= \mathrm{Hom}(A \star D, \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_Y^* B, \pi_Z^! C)),
\end{aligned}$$
which is clearly functorial. For $(*)$ note that the $\mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(-)$ in the third line is an object of $D(X \times Y \times Z)$ (and the projections are maps $\pi_Y\colon X \times Y \times Z \to Y$, and similarly for Z), while in the fourth line the $\mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(-)$ is then an object of $D(Y \times Z)$ (with the projections also only defined on $Y \times Z$). ☐

We can now give the definition of ULA-sheaves of [HS21]:

Definition 2.8. Let $f : X \to S$ be a separated map of finite presentation and $A \in D(X)$. Then A is called *f-universally locally acyclic* if $(X, A) \in \mathcal{C}'_S$ is dualisable.

We denote by $D^{\mathrm{ULA}}(X/S) \subseteq D(X)$ the subcategory of f-universally locally acyclic sheaves.

Remark 2.9. There are various (equivalent) variants of this definition of universally locally acyclic sheaves.

(i) We could also define the categories $\mathcal{C}_S$ (and thus $\mathcal{C}'_S$) using only constructible complexes as functor categories in $\mathcal{C}_S$. [HS21] call this *setting (B)*, while the definition we presented above is *setting (A)*. The inclusion $D_{\mathrm{cons}}(X) \subseteq D(X)$ induces a symmetric monoidal functor on the level of the categories $\mathcal{C}'_S$. Thus, as symmetric monoidal functors preserve dualisable objects, any dualisable object is also dualisable in the full category $\mathcal{C}'_S$. That any dualisable object in $\mathcal{C}'_S$ is dualisable in the corresponding perfect-constructible category essentially comes down to the fact that universally locally acyclic sheaves are automatically perfect-constructible by Proposition 3.7. We show that the two notions of universally locally acyclic sheaves agree in Remark 3.8 below. Note however, that in the perfect-constructible setting the analogue of category $\mathcal{C}'_S$ does not have internal Hom objects, in particular the internal Hom objects of $\mathcal{C}'_S$ may fail to be perfect-constructible in general.

(ii) In [FS21, Theorem IV.2.23], ULA sheaves are characterised as those sheaves $A \in D(X)$ such that A considered as a map $S \to X$ in $\mathcal{C}_S$ is a left adjoint. Recall that in a 2-category $\mathcal{D}$ a map $f\colon X \to Y$ is a left adjoint of $g\colon Y \to X$ if there are 2-morphisms $\alpha\colon \mathrm{id}_X \to gf$ and $\beta\colon fg \to \mathrm{id}_Y$ such that the composites

$$f \xrightarrow{f\alpha} fgf \xrightarrow{\beta g} f \qquad \text{and} \qquad g \xrightarrow{\alpha g} gfg \xrightarrow{g\beta} g$$

are the identities.

That this notion agrees with our definition is formal and shown in [HS21, Proposition 3.1].

(iii) In [LZ20], ULA-sheaves are defined as dualisable objects in a symmetric monoidal 2-category of cohomological correspondences (in place of $\mathcal{C}'_S$). However, there is a natural symmetric monoidal functor to $\mathcal{C}'_S$, and moreover, the internal Hom objects in both categories agree. In particular, by the characterisation of dualisable objects in Lemma 2.3, the dualisable objects in both categories agree.

In the following, by ULA sheaves we always mean ULA-sheaves in the sense of Definition 2.8 and refer to ULA-sheaves *in the classical sense* to perfect-constructible complexes satisfying Definition 1.1. We show in Theorem 4.4 below that the two notions of ULA-sheaves actually agree.

3. Properties of ULA sheaves

We can now collect properties of ULA-sheaves. Using the description of ULA-sheaves as certain dualisable objects, we are able to show for example that they satisfy Verdier biduality and a Künneth-type formula. The results in this section are essentially due to [LZ20]. We closely follow the presentation of [HS21, Section 3]. In this section $f\colon X \to S$ will always denote a separated map of finite presentation. First, we give the following description of duals of ULA-sheaves.

Remark 3.1. Let $A \in D(X)$ be f-universally locally acyclic. Using the formula of Remark 2.2 together with the description of internal Hom's in $\mathcal{C}'_S$ from Lemma 2.7 we see that the dual of (X, A) in $\mathcal{C}'_S$ is given by

$$(X, A)^\vee = \underline{\mathrm{Hom}}_{\mathcal{C}'_S}((X, A), (S, \Lambda)) = (X, \mathbb{D}_{X/S}(A)),$$

where we denote by $\mathbb{D}_{X/S}(A) = \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(A, f^!\Lambda)$ the relative Verdier dual of A.

The following properties of ULA-sheaves follow directly from the general discussion of dualisable objects above (compare Remark 2.2 and Lemma 2.3) together with the description of internal Hom's in $\mathcal{C}'_S$ from Lemma 2.7 (and the above description of duals).

Proposition 3.2 ([HS21, Proposition 3.4 (ii)]). *Let $A \in D(X)$ be f-universally locally acyclic. Then $\mathbb{D}_{X/S}(A)$ is f-universally locally acyclic and the biduality map*

$$A \to \mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A))$$

is an isomorphism.

Lemma 3.3 ([HS21, Proposition 3.4 (iv)]). *Let $g\colon Y \to S$ be separated and of finite presentation. Let $A \in D(X)$ be f-universally locally acyclic and $B \in D(Y)$. Then the natural maps*

$$\mathbb{D}_{X/S}(A) \boxtimes B = \pi_X^*\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} \pi_Y^* B \to \mathrm{R}\mathcal{H}\mathrm{om}_{\Lambda}(\pi_X^* A, R\pi_Y^! B)$$

and

$$\pi_X^* A \otimes^{\mathbb{L}}_{\Lambda} \pi_Y^* B \to \mathrm{R}\mathcal{H}\mathrm{om}_{\Lambda}(\pi_X^*\mathbb{D}_{X/S}(A), R\pi_Y^! B)$$

are isomorphisms in $D(X \times_S Y)$.

Proposition 3.4 ([HS21, Proposition 3.3]). *Let $f : X \to S$ be a separated map of finite presentation and $A \in D(X)$. Then A is f-universally locally acyclic if and only if the map*

$$\pi_1^*\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} \pi_2^* A \to \mathrm{R}\mathcal{H}\mathrm{om}_{\Lambda}(\pi_1^* A, R\pi_2^! A)$$

is an isomorphism in $D(X \times_S X)$.

Note that this also directly implies that the id_S-universally locally acyclic sheaves are exactly the dualisable objects in $D(S)$, in other words the locally constant sheaves with perfect values by [CD16, Remark 6.3.27].

We use these formulas to show the following permanence properties of universally locally acyclic sheaves.

Proposition 3.5. *Let $A \in D(X)$ be f-universally locally acyclic. Let $g\colon Y \to S$ be a separated map of finite presentation.*

(i) [LZ20, Proposition 2.23] *Let $h\colon X \to Y$ be a proper map of S-schemes. Then Rh_*A is g-universally locally acyclic.*
(ii) *Let $h\colon X \to Y$ be a smooth and separated map of S-schemes and let $B \in D(Y)$ be g-universally locally acyclic. Then h^*B is f-universally locally acyclic.*
(iii) *Let $j\colon S \to S'$ be smooth. Then A is $j \circ f$-universally locally acyclic.*
(iv) [HS21, Proposition 3.4 (i)] *Let $S' \to S$ be any map of schemes, and $f' : X' = X \times_S S' \to S'$ the base change of f, and $A' \in D(X')$ the pullback of A. Then A' is f'-universally locally acyclic.*
In particular, Verdier duality commutes with base change along $S' \to S$ in the sense that $h^*\mathbb{D}_{X/S}(A) \cong \mathbb{D}_{X'/S'}(A')$, where we denote by $h\colon X' \to X$ the projection.
(v) *Let $A \in D(X)$ and $B \in D(Y)$ be two universally locally acyclic sheaves. Then $A \boxtimes B \in D(X \times_S Y)$ is (f, g)-universally locally acyclic.*
(vi) *Let $B \in D(X)$ be a retract of A, that means there are maps $B \to A$ and $A \to B$ in $D(X)$ such that their composition $B \to A \to B$ is the identity of B. Then B is f-universally locally acyclic.*

Proof. (i) We follow the proof of [LZ20, Proposition 2.23]. It clearly suffices to show that the map

$$\mathbb{D}_{Y/S}(Rh_*A) \boxtimes B = \pi_Y^*\mathbb{D}_{Y/S}(Rh_*A) \otimes^{\mathbb{L}}_{\Lambda} \pi_Z^* B \to \mathrm{R}\mathcal{H}\mathrm{om}_{\Lambda}(\pi_Y^* Rh_*A, R\pi_Z^! B)$$

is an isomorphism for all $g\colon Z \to S$ separated and of finite presentation and $B \in D(Z)$. As A is f-universally locally acyclic, we see

$$\pi_X^*\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} \pi_Z^* B \to \mathrm{R}\mathcal{H}\mathrm{om}_{\Lambda}(\pi_X^* A, R\pi_Z^! B)$$

is an isomorphism by Lemma 3.3. Again, recall that by our convention π_Z denotes either of the projections $X \times_S Z \to Z$ or $Y \times_S Z \to Z$ depending on the context. We apply $R(h \times \mathrm{id}_Z)_*$ and obtain on the one hand using the projection formula and proper base change

$$\begin{aligned} R(h \times \mathrm{id}_Z)_*(\pi_X^*\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} \pi_Z^* B) &\cong \big(R(h \times \mathrm{id}_Z)_*\pi_X^*\mathbb{D}_{X/S}(A)\big) \otimes^{\mathbb{L}}_{\Lambda} \pi_Z^* B \\ &\cong \big(\pi_X^* Rh_*\mathbb{D}_{X/S}(A)\big) \otimes^{\mathbb{L}}_{\Lambda} \pi_Z^* B \\ &\cong \pi_X^*\mathbb{D}_{Y/S}(Rh_*A) \otimes^{\mathbb{L}}_{\Lambda} \pi_Z^* B, \end{aligned}$$

and on the other hand

$$\begin{aligned} R(h\times \mathrm{id}_Z)_*\mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_X^*A, R\pi_Z^!B) &\cong \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(R(h\times\mathrm{id}_Z)_*\pi_X^*A, R\pi_Z^!B)\\ &\cong \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_Y^*Rh_*A, R\pi_Z^!B)\end{aligned}$$

again using a version of relative Poincaré duality and proper base change. This shows the claim.

(ii) We proceed as in (i): As B is g-universally locally acyclic, we get from Lemma 3.3 that

$$\pi_Y^*\mathbb{D}_{Y/S}(B)\otimes_\Lambda^{\mathbb{L}}\pi_Z^*C \to \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_Y^*B, R\pi_Z^!C)$$

is an isomorphism for all $g\colon Z\to S$ separated and of finite presentation and $C\in D(Z)$. Working locally on Y, we may assume that h is smooth of constant relative dimension d. In particular, $Rh^! = h^*[d]$. This time, we apply $(h\times\mathrm{id}_Z)^*$ to the isomorphism and obtain

$$\begin{aligned}\pi_X^*h^*\mathbb{D}_{Y/S}(B)\otimes_\Lambda^{\mathbb{L}}\pi_Z^*C &\cong \pi_X^*\mathbb{D}_{X/S}(Rh^!B)\otimes_\Lambda^{\mathbb{L}}\pi_Z^*C\\ &\cong \pi_X^*\mathbb{D}_{X/S}(h^*B[d])\otimes_\Lambda^{\mathbb{L}}\pi_Z^*C\\ &\cong (\pi_X^*\mathbb{D}_{X/S}(h^*B)\otimes_\Lambda^{\mathbb{L}}\pi_Z^*C)[-d],\end{aligned}$$

and

$$\begin{aligned}(h\times\mathrm{id}_Z)^*\mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_Y^*B, R\pi_Z^!C) &\cong R(h\times\mathrm{id}_Z)^!\mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_Y^*B, R\pi_Z^!C)[-d]\\ &\cong \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda((h\times\mathrm{id}_Z)^*\pi_Y^*B, R(h\times\mathrm{id}_Z)^!R\pi_Z^!C)[-d]\\ &\cong \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_Y^*h^*B, R\pi_Z^!C)[-d].\end{aligned}$$

Thus, the natural map

$$\pi_X^*\mathbb{D}_{Y/S}(h^*A)\otimes_\Lambda^{\mathbb{L}}\pi_Z^*B \to \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_Y^*h^*A, R\pi_Z^!B)$$

is an isomorphism. This shows the claim.

(iii) Working locally, we assume that j is smooth of constant relative dimension d. We denote by $Y = X\times_{S'}S$ and by $B = \tilde{\pi}_X^*A$ the pullback of A to Y, where $\tilde{\pi}_X\colon Y\to X$ is the projection. In particular, $\tilde{\pi}_X$ is smooth of relative dimension d. Then $X\times_S Y = X\times_{S'}X$ and the projections $\pi_i\colon X\times_{S'}X\to X$ are identified with $\pi_1=\pi_X$ and $\pi_2 = \tilde{\pi}_X\circ\pi_X$ for the projections π_X,π_Y on $X\times_S Y$. By the second part of Lemma 3.3, we find

$$\begin{aligned}\pi_1^*\mathbb{D}_{X/S'}(A)\otimes_\Lambda^{\mathbb{L}}\pi_2^*A &= \pi_X^*\mathbb{D}_{X/S}(A)[d]\otimes_\Lambda^{\mathbb{L}}\pi_Y^*B\\ &\xrightarrow{\cong} \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_X^*A, R\pi_Y^!B)[d]\\ &= \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_1^*A, R\pi_2^!A)\end{aligned}$$

is an isomorphism. Hence, A is $j\circ f$-universally locally acyclic.

(iv) We claim that for a map $s\colon S'\to S$ the pullback functor $\mathcal{C}_S\to\mathcal{C}_{S'}: X\mapsto X\times_S S'$ is symmetric monoidal. In order to see this, we note that $(X\times_S Y)_{S'} = X_{S'}\times_{S'}Y_{S'}$, and on functor categories for a pair of complexes $A\in D(X)$ and $B\in D(Y)$ we have

$$s_{X\times_S Y}^*(A\boxtimes B) = s_{X\times_S Y}^*(\pi_X^*A\otimes_\Lambda^{\mathbb{L}}\pi_Y*B)\cong \pi_{X_{S'}}^*s_X^*A\otimes_\Lambda^{\mathbb{L}}\pi_{Y_{S'}}^*s_Y^*B = s_X^*A\boxtimes s_Y^*B,$$

where we denote by $s_{X\times_S Y}, s_X, s_Y$ the respective base changes of s. This also shows that the induced pullback functor $\mathcal{C}'_S\to\mathcal{C}'_{S'}$ is symmetric monoidal and hence preserves dualisable objects.

More precisely, the pullback functor maps dual pairs to dual pairs. This shows the claim for Verdier duals using Remark 3.1.

(v) This follows from Remark 2.2: By assumption (X,A) and (Y,B) are dualisable in $\mathcal{C}'_S$, hence also $(X,A)\boxtimes(Y,B) = (X\times_S Y, A\boxtimes B)$ is dualisable.

(vi) Since B is a retract of A, the natural map $\mathbb{D}_{X/S}(B)\boxtimes B\to\mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_1^*B, R\pi_2^!B)$ is a retract of $\mathbb{D}_{X/S}(A)\boxtimes A\to\mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_1^*A, R\pi_2^!A)$. But retracts of isomorphisms are again isomorphisms.

□

Lemma 3.6. *Let $g\colon X \to Y$ be a separated map of finite type between schemes of finite ℓ-cohomological dimension. Then $Rg^!$ commutes with direct sums.*

[LZ17, Lemma 2.13] show the result under the assumption that X and Y both are finite dimensional Noetherian. In our setting it is a special case of a classical general fact: If a left adjoint functor between compactly generated triangulated categories preserves compact objects, its right adjoint commutes with direct sums, see [Nee96, Theorem 5.1] (or also [FS21, Lemma IV.2.20]).

Proof. As $D(X)$ is compactly generated with compact objects given by $D_{\mathrm{cons}}(X)$, it suffices to check that for all $A \in D_{\mathrm{cons}}(X)$ and $B_i \in D(X)$ the canonical map

$$\mathrm{Hom}(A, Rg^!(\bigoplus_i B_i)) \cong \mathrm{Hom}(A, \bigoplus_i Rg^! B_i)$$

is an isomorphism. But by compactness of A and the fact that $Rg_!$ preserves perfect-constructible objects we have

$$\begin{aligned}\mathrm{Hom}\left(A, Rg^!(\bigoplus_i B_i)\right) &\cong \mathrm{Hom}\left(Rg_! A, \bigoplus_i B_i\right) \cong \bigoplus_i \mathrm{Hom}\left(Rg_! A, B_i\right)\\ &\cong \bigoplus_i \mathrm{Hom}\left(A, Rg^! B_i\right) \cong \mathrm{Hom}\left(A, \bigoplus_i Rg^! B_i\right).\end{aligned}$$

□

We can now show that ULA-sheaves in our sense are indeed already perfect-constructible.

Proposition 3.7 ([HS21, Proposition 3.4 (iii)])**.** *Let $A \in D(X)$ be f-universally locally acyclic. Then A is perfect-constructible.*

Proof. By our assumption X has finite ℓ-cohomological dimension[5] and thus perfect-constructibility is equivalent to compactness in $D(X)$. We thus have to show that $R\mathrm{Hom}_{D(X)}(A, -)$ commutes with direct sums. By Lemma 3.3 applied to (X, B) for any $B \in D(X)$ we get that

$$\pi_1^* \mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} \pi_2^* B \cong \mathrm{R}\mathcal{H}\mathrm{om}_{\Lambda}(\pi_1^* A, R\pi_2^! B).$$

Taking $R\Gamma \circ R\Delta^!_{X/S}$, we find that

$$\begin{aligned} R\mathrm{Hom}_{D(X)}(A, B) &\cong R\Gamma(X, \mathrm{R}\mathcal{H}\mathrm{om}_{\Lambda}(R\Delta^*_{X/S}\pi_1^* A, R\Delta^!_{X/S} R\pi_2^! B))\\ &\cong R\Gamma(X, R\Delta^!_{X/S}(\pi_1^* \mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} \pi_2^* B)).\end{aligned}$$

Now the functor on the right commutes with all direct sums in B. For pullback and $\otimes^{\mathbb{L}}_{\Lambda}$ this is clear as they are left-adjoints, for $R\Gamma$ this is a standard result ([Sta, Tag 0F11]), and for $R\Delta^!_{X/S}$ this follows from the previous Lemma 3.6. Thus, A is compact, hence perfect-constructible. □

Remark 3.8. Recall from Remark 2.9 that we could also define universally locally acyclic sheaves using versions of the categories $\mathcal{C}_S$ and $\mathcal{C}'_S$ using only the categories of perfect-constructible complexes as functor categories in $\mathcal{C}_S$. Let us denote by $\tilde{\mathcal{C}}_S$ and $\tilde{\mathcal{C}}'_S$ the corresponding perfect-constructible versions. We already argued in Remark 2.9 that dualisable objects in $\tilde{\mathcal{C}}'_S$ are also dualisable in $\mathcal{C}'_S$.

We are now in a position to show that dualisable objects of $\mathcal{C}_S$ are already dualisable objects of $\tilde{\mathcal{C}}'_S$. Let $A \in D(X)$ be f-universally locally acyclic. By Proposition 3.2 also its relative Verdier dual $\mathbb{D}_{X/S}(A)$ is f-universally locally acyclic. Following Proposition 3.7, both A and $\mathbb{D}_{X/S}(A)$ are then perfect-constructible and hence both (X, A) and $(X, \mathbb{D}_{X/S}(A))$ are objects of $\tilde{\mathcal{C}}'_S$. We claim that they are dual to each other also in $\tilde{\mathcal{C}}'_S$. We want to show that the unit and counit witnessing the duality in $\mathcal{C}'_S$ also are already maps $(S, \Lambda) \to (X, A) \boxtimes (X, \mathbb{D}_{X/S}(A))$ in $\tilde{\mathcal{C}}'_S$. But both the unit and counit are given by $A \boxtimes \mathbb{D}_{X/S}(A)$, which is perfect-constructible, hence a map in $\tilde{\mathcal{C}}'_S$.

[5] In general we can reduce to this case by arguing v-locally on S.

This finishes the proof that the two notions of universally locally acyclic sheaves using $\mathcal{C}'_S$ and $\tilde{\mathcal{C}}'_S$, respectively, agree.

We will need the following calculation.

Lemma 3.9 ([HS21, proof of Proposition 3.4 (v)])**.** *Let $g\colon Y \to S$ be a quasi-compact and quasi-separated map. Let $A \in D(X)$ be f-universally locally acyclic. Then the natural map*

$$A \otimes^{\mathbb{L}}_{\Lambda} f^* Rg_* \Lambda \to R\pi_{X*} A|_{X \times_S Y}$$

is an isomorphism.

Proof. We first consider the case that $g : Y \to S$ is separated and of finite presentation. Applying $R\pi_{X*}$ to the formula from Lemma 3.3 for $B = \Lambda$ yields

$$\begin{aligned} R\pi_{X*}(A|_{X\times_S Y}) &\cong R\pi_{X*}(\pi_X^* A \otimes \Lambda) \\ &\cong R\pi_{X*} \mathrm{R}\mathcal{H}\mathrm{om}_{\Lambda}(\pi_X^* \mathbb{D}_{X/S}(A), R\pi_Y^! \Lambda) \\ &\cong \mathrm{R}\mathcal{H}\mathrm{om}_{\Lambda}(\mathbb{D}_{X/S}(A), R\pi_{X*} R\pi_Y^! \Lambda) \\ &\cong \mathrm{R}\mathcal{H}\mathrm{om}_{\Lambda}(\mathbb{D}_{X/S}(A), Rf^! Rg_* \Lambda). \end{aligned}$$

Again applying Lemma 3.3 for $(S, Rg_*\Lambda)$, we obtain an isomorphism

$$A \otimes^{\mathbb{L}}_{\Lambda} f^* Rg_* \Lambda \to \mathrm{R}\mathcal{H}\mathrm{om}_{\Lambda}(\mathbb{D}_{X/S}(A), Rf^! Rg_* \Lambda).$$

Thus, the natural map

$$A \otimes^{\mathbb{L}}_{\Lambda} f^* Rg_* \Lambda \to R\pi_{X*} A|_{X \times_S Y}$$

is an isomorphism.

We extend this isomorphism to all maps $g\colon Y \to S$ (note that g is automatically qcqs as S, Y are assumed to be qcqs) by writing $Y = \lim_{i \in I} Y_i$ for a directed system $g_i\colon Y_i \to S$ of separated S-schemes of finite presentation. □

As a next step we show that ULA-sheaves are ULA in the classical sense.

Proposition 3.10 ([HS21, Proposition 3.4 (v)])**.** *Let $A \in D(X)$ be f-universally locally acyclic. For any geometric point $\overline{x} \to X$ with image $\overline{s} \to S$, and any generisation $\overline{t}$ of $\overline{s}$, the maps*

$$A_{\overline{x}} = R\Gamma(X_{\overline{x}}, A) \to R\Gamma(X_{\overline{x}} \times_{S_{\overline{s}}} S_{\overline{t}}, A) \to R\Gamma(X_{\overline{x}} \times_{S_{\overline{s}}} \overline{t}, A)$$

are isomorphisms. In particular, A is f-universally locally acyclic in the classical sense.

Proof. The last sentence of the claim follows from the first as A is perfect-constructible by Proposition 3.7 and as being ULA is stable under base change by Proposition 3.5 (i).

Let $\bar{x} \to X$ be a a geometric point of X lying over $\bar{s} \to S$. To show the claim, we may base change to $S_{\bar{s}}$. For any generisation $\bar{t} \rightsquigarrow \bar{s}$ the maps $Y = S_{\bar{t}} \to S_{\bar{s}}$ or $Y = \bar{t} \to S_{\bar{s}}$ are in particular qcqs. We compute the stalks of at $\overline{x} \to X$ over $\overline{s} \to S$ and obtain

$$(A \otimes^{\mathbb{L}}_{\Lambda} f^* Rg_* \Lambda)_{\bar{x}} = A_{\bar{x}} \otimes^{\mathbb{L}}_{\Lambda} (Rg_*\Lambda)_{\bar{s}} = A_{\bar{x}} \otimes^{\mathbb{L}}_{\Lambda} R\Gamma(Y, \Lambda) = A_{\bar{x}}$$

and

$$(R\pi_{X*} A|_{X \times_{S_{\bar{s}}} Y})_{\bar{x}} = R\Gamma(X_{\bar{x}} \times_{S_{\bar{s}}} Y, A)$$

using the formulas for stalks at geometric points of tensor products and derived direct images (we use the bounded below case in [Sta, Tag 03Q9] together with the left completeness of $D(X)$). □

We mention the following arc-descent properties of universal locally acyclic sheaves. Recall that a sheaf $\mathcal{F}$ on qcqs R-schemes with values in some ∞-category with small limits and filtered colimits is called *finitary* if for every system $\{Y_i\}_{i \in I}$ indexed over some cofiltered partially ordered set I with affine transition maps, we have $\varinjlim_{i \in I} \mathcal{F}(Y_i) \xrightarrow{\cong} \mathcal{F}(\varprojlim_{i \in I} Y_i)$, compare for example [BM21, Definition 3.1].

Proposition 3.11 ([HS21, Proposition 3.7])**.** *Consider the functor taking any S' over S to the ∞-category $\mathcal{D}^{\mathrm{ULA}}(X'/S') \subset \mathcal{D}(X')$ of universally locally acyclic sheaves on $X' = X \times_S S'$ over S'. This defines a finitary arc-sheaf of ∞-categories.*

In particular, if $A \in D(X)$ and $S' \to S$ is an arc-cover such that $A|_{X'}$ is universally locally acyclic over S', then A is universally locally acyclic over S.

Proof. In our setting, by [BM21, Theorem 5.4, Theorem 5.13] the functor $X \mapsto \mathcal{D}_{\mathrm{cons}}(X)$ defines a finitary arc-sheaf of ∞-categories.[6] It thus suffices to show that universal local acyclicity can be checked arc-locally on the base in order to conclude that it defines an arc-sheaf as well. Let therefore $A \in D(X)$ and let $S' \to S$ be an arc-cover such that $A|_{X'}$ is universally locally acyclic over S'. By Propositions 3.2 and 3.7 its relative Verdier dual $\mathbb{D}_{X'/S'}(A|_{X'})$ is perfect-constructible over X'. Hence, by arc-descent for perfect-constructible sheaves, $\mathbb{D}_{X'/S'}(A|_{X'})$ descends to a perfect-constructible sheaf B over X. Note that it may not be a priori clear that $B = \mathbb{D}_{X/S}(A)$. In order to check that (X, B) is a dual of (X, A) in $\mathcal{C}'_S$, we claim that a unit and counit map are given by $A \boxtimes B \in D_{\mathrm{cons}}(X \times_S X)$. We may argue arc-locally to show that unit and counit satisfy the required identities. But over X', $(A \boxtimes B)|_{X'} = A|_{X'} \boxtimes \mathbb{D}_{X'/S'}(A)$ satisfies these identities by assumption. Hence, A is f-universally locally acyclic.

It remains to check that $S' \mapsto \mathcal{D}^{\mathrm{ULA}}(X'/S')$ is finitary. We again denote by $\tilde{\mathcal{C}}'_S$ the perfect-constructible version of $\mathcal{C}'_S$ from Remark 3.8. Recall that the dualisable objects of $\tilde{\mathcal{C}}'_S$ and $\mathcal{C}'_S$ agree. As the functor $X \mapsto \mathcal{D}_{\mathrm{cons}}(X)$ is finitary, the formation of $\tilde{\mathcal{C}}'_S$ takes cofiltered limits of affine schemes S to filtered colimits of symmetric monoidal 2-categories. Hence, the same is true for dualisable objects. ☐

In fact, one can check universal local acyclicity after pullback to absolutely integrally closed, rank 1 valuation rings (note that the rank of a valuation ring is its Krull dimension). Recall that a valuation ring is absolutely integrally closed if and only if its fraction field is algebraically closed by [Sta, Tag 0DCQ] and [Sta, Tag 00IC].

Corollary 3.12 ([HS21, Corollary 3.9])**.** *Let $A \in D_{\mathrm{cons}}(X)$. Then A is f-universally locally acyclic if and only if for all rank 1 valuation rings V with algebraically closed fraction field K and all maps* $\mathrm{Spec} V \to S$*, the restriction $A|_{X_V} \in D(X_V)$ to $X_V = X \times_S \mathrm{Spec} V$ is universally locally acyclic over V.*

Proof. Note that the condition is clearly necessary by Proposition 3.5 (iv). The converse follows from results for finitary arc-sheaves (compare [BM21]): We may first assume that all connected components of S are spectra of valuation rings by [BM21, Proposition 3.30]. By finitaryness we may hence assume that S is the spectrum of a finite rank valuation ring by [BM21, Lemma 2.22]. By arc-descent we may then reduce to the case that S is the spectrum of a rank 1 valuation ring, but in this case A is ULA by assumption. ☐

4. Nearby cycles

We use our study of ULA-sheaves to study nearby cycles over absolutely integrally closed valuation rings. Nearby cycles will be used critically in the next talk to show that the relative perverse t-structure has favourable properties. We consider the following setup: Let V be an absolutely integrally closed valuation ring. We denote by $\eta \in \mathrm{Spec} V$ its generic point and by $s \in \mathrm{Spec} V$ its special point. Nearby cycles are used to compare the cohomologies of the generic and special fibres of schemes over S. For a scheme X over S we denote by $j\colon X_\eta \to X$ the inclusion of the generic and by $i\colon X_s \to X$ the inclusion of the special fibre. The nearby cycles functor $R\Psi$ on étale sheaves is then defined as first pushing forward along j and then taking the pullback along i. In other words,

$$R\Psi := i^* Rj_*\colon D(X_\eta) \to D(X_s).$$

We use our study of universally locally acyclic sheaves to obtain results for the nearby cycles functor. Classically, the nearby cycles functor was studied over henselian discrete valuation rings [Del77, Th.

[6] [HS21, Theorem 2.2] extends this result to more general coefficients.

finitude]. We rederive many of the classical results in our (non-noetherian) setting, in which they are essentially due to [LZ20, Section 3], while the fact that nearby cycles commute with Verdier duality in this setting already was observed by [Fuj97, proof of Lemma 1.5.1]. As before, we closely follow the presentation of [HS21, Section 4].

Theorem 4.1 ([HS21, Theorem 4.1, Corollary 4.2])**.** *Let $S = \mathrm{Spec} V$ be an absolutely integrally closed valuation ring V with fraction field K. Let X be a separated scheme of finite presentation over S with generic fibre X_η. The restriction functor*

$$j^*\colon D^{\mathrm{ULA}}(X/S) \to D_{\mathrm{cons}}(X_\eta)$$

is an equivalence, whose inverse is given by $Rj_* : D_{\mathrm{cons}}(X_\eta) \to D(X)$, where $j : X_\eta \to X$ is the inclusion. In particular, both the functors $Rj_*\colon D_{\mathrm{cons}}(X_\eta) \to D(X)$ and $R\Psi\colon D_{\mathrm{cons}}(X_\eta) \to D(X_s)$

(i) *preserve constructibility;*
(ii) *their formation commutes with any pullback along a map $S' = \mathrm{Spec} V' \to \mathrm{Spec} V$ where $V \to V'$ is a flat map in the case of Rj_*, respectively a faithfully flat map in the case of $R\Psi$, of absolutely integrally closed valuation rings;*
(iii) *commute with (relative) Verdier duality;*
(iv) *and satisfy a Künneth-type formula: if Y is another scheme of finite presentation over S, then the diagrams*

$$\begin{array}{ccc} D_{\mathrm{cons}}(X_\eta) \times D_{\mathrm{cons}}(Y_\eta) & \xrightarrow{\boxtimes} & D_{\mathrm{cons}}((X \times_S Y)_\eta) \\ \downarrow{\scriptstyle Rj_* \times Rj_*} & & \downarrow{\scriptstyle Rj_*} \\ D(X) \times D(Y) & \xrightarrow{\boxtimes} & D(X \times_S Y) \end{array}$$

respectively

$$\begin{array}{ccc} D_{\mathrm{cons}}(X_\eta) \times D_{\mathrm{cons}}(Y_\eta) & \xrightarrow{\boxtimes} & D_{\mathrm{cons}}((X \times_S Y)_\eta) \\ \downarrow{\scriptstyle R\Psi_* \times R\Psi_*} & & \downarrow{\scriptstyle R\Psi_*} \\ D(X_s) \times D(Y_s) & \xrightarrow{\boxtimes} & D((X \times_S Y)_s) \end{array}$$

commute.

Proof. We start by deducing properties (i)-(iv) from the first assertion.

(i) For Rj_* this is part of the first assertion (using that ULA-sheaves are in particular perfect-constructible by Proposition 3.7), for $R\Psi$ we moreover use that i^* preserves perfect-constructible complexes.
(ii) We first note that flatness (respectively faithfull flatness) of the map $V \to V'$ guarantees that the induced map $\mathrm{Spec} V' \to \mathrm{Spec} V$ maps the generic point η' of $\mathrm{Spec} V'$ to the generic point η of $\mathrm{Spec} V$ (respectively that moreover the special point s' of $\mathrm{Spec} V'$ maps to the special point s of $\mathrm{Spec} V$). We denote by $j'\colon X_{\eta'} \to X_{V'}$ the inclusion of the generic fibre of the base change of X to $\mathrm{Spec} V'$, and by $g\colon X_{\mathrm{Spec} V'} \to X$ the projection. By the discussion above we also have an induced map $g_\eta\colon X_{\eta'} \to X_\eta$ of the generic fibres, such that $j \circ g_\eta = g \circ j'$. We need to check that the natural map

$$(Rj_*A)|_{X_{V'}} = g^*(Rj_*A) \to Rj'_*(g_\eta^*A) = Rj'_*(A|_{X_{\eta'}})$$

is an isomorphism. For this we recall that by the first part of the theorem we have natural isomorphisms $j^*Rj_*A \to A$ and $B \to Rj_*j^*B$. Hence, we have

$$(j')^*g^*(Rj_*A) \cong g_\eta^*j^*Rj_*A \cong g_\eta^*A,$$

using again that Rj_*A is f-ULA and that Rj_* and j^* are quasi-inverse equivalences by the first part of the theorem. We apply Rj'_* to obtain the claim. For the respective result on nearby cycles we note that the fact that $V \to V'$ is faithfully flat gives rise to a map

$g_s\colon X_{s'} \to X_s$ of the special fibres. We denote by $R\Psi' = (i')^* \circ Rj'_*\colon D(X_{\eta'}) \to D(X_{s'})$ the nearby cycles functor over SpecV'. This implies together with the above that

$$(R\Psi A)|_{X'_s} = g_{s'}^* i^* Rj_* A \cong (i')^* g^* Rj_* A \cong (i')^* Rj'_* g_\eta A = R\Psi'(A|_{X_{\eta'}}).$$

(iii) We recall that relative Verdier duality preserves universally locally acyclic sheaves by Proposition 3.2 and that Verdier duality for universally locally acyclic sheaves commutes with arbitrary base change by Proposition 3.5 (iv). We thus obtain for $A \in D_{\mathrm{cons}}(X_\eta)$ that

$$j^*\mathbb{D}_{X/S}(Rj_*A) \cong \mathbb{D}_{X_\eta/\eta}(j^*Rj_*A) \cong \mathbb{D}_{X_\eta/\eta}(A),$$

giving the claim for Rj_* after applying Rj_*. For nearby cycles it then follows that

$$\mathbb{D}_{X_s/s} R\Psi A = i^*\mathbb{D}_{X/S}(Rj_*A) = i^*Rj_*\mathbb{D}_{X_\eta/\eta}A = R\Psi(\mathbb{D}_{X_\eta/\eta}(A)).$$

(iv) This follows from preservation of universal local acyclicity under exterior tensor products by Proposition 3.5 (v). Namely, in a similar fashion as before we see that

$$\begin{aligned} j^*_{X\times_S Y}(Rj_{X*}A \boxtimes Rj_{Y*}B) &\cong j^*_{X\times_S Y}(\pi_X^* Rj_{X*}A \otimes^{\mathbb{L}}_\Lambda \pi_Y^* Rj_{Y*}B) \\ &\cong \pi_{X_\eta} A \otimes^{\mathbb{L}}_\Lambda \pi_{Y_\eta} B \cong A \boxtimes B \end{aligned}$$

and

$$\begin{aligned} R\Psi_X A \boxtimes R\Psi_Y B &\cong i^*_{X\times_S Y}(Rj_{X*}A \boxtimes Rj_{Y*}B) \\ &\cong i^*_{X\times_S Y} Rj_{X\times_S Y}(A \boxtimes B) \cong R\Psi_{X\times_S Y}(A\boxtimes B). \end{aligned}$$

Now we prove the first part of the Theorem:

Fully faithfulness of $j^*\colon D^{\mathrm{ULA}}(X/S) \to D_{\mathrm{cons}}(X_\eta)$: We show that for any $A \in D^{\mathrm{ULA}}(X/S)$, the natural map $A \to Rj_*j^*A$ is an isomorphism. In order to see that the map is an isomorphism, we apply Lemma 3.9 to $Y = \mathrm{Spec}K \to \mathrm{Spec}V$ and get an isomorphism $A \otimes^{\mathbb{L}}_\Lambda f^*Rj_*\Lambda \xrightarrow{\cong} Rj_*j^*A$. It thus suffices to show that $Rj_*\Lambda = \Lambda$. We compute the stalks of Rj_* at a geometric point $s \in S$ using again [Sta, Tag 03Q9] and obtain

$$(Rj_*\Lambda)_s = R\Gamma(\eta \times_S S_s, \Lambda) = R\Gamma(\eta, \Lambda) = \Lambda.$$

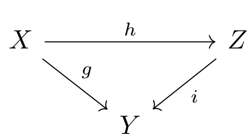
Hence, $R\Gamma(\eta \times_S S_s, \Lambda)$ is locally constant and thus indeed constant as V is absolutely integrally closed. Thus, $A \to Rj_*j^*A$ is an isomorphism and j^* is fully faithful.

It remains to show that

$$j^* : D^{\mathrm{ULA}}(X/S) \hookrightarrow D_{\mathrm{cons}}(X_\eta)$$

is *essentially surjective*: Indeed, we have just seen that the inverse functor is necessarily given by Rj_*. It thus suffices to check that Rj_*A is f-universally locally acyclic for all $A \in D_{\mathrm{cons}}(X_\eta)$. We adapt Deligne's proof of the constructibility of nearby cycles, [Del77, Th. finitude, Théorème 3.2]. We will argue by induction on the (relative) dimension d of X. We start by proving the following reduction steps.

Step 1: Let $g\colon X \to Y$ be a quasi-finite map of S-schemes. We show that if the claim holds for Y, it also holds for X. By Zariski's Main Theorem (compare [Sta, Tag 05K0]) we can factor $g = i \circ h$, where $h\colon X \to Z$ is a quasi-compact open immersion and $i\colon Z \to Y$ is finite.



It thus suffices to show the reduction in the case that g is a quasi-compact open immersion and in the case that g is finite.

We first consider the case that g is an open immersion: We consider the cartesian square of open immersions:

$$\begin{array}{ccc} X_\eta & \xrightarrow{g_\eta} & Y_\eta \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j_Y} \\ X & \xrightarrow{g} & Z \end{array}$$

Let us assume that we know the assertion for $f'\colon Y \to S$. Let $A \in D_{\mathrm{cons}}(X_\eta)$. Then using that g is an open immersion (and hence $g^* R g_! \cong \mathrm{id}$ is an isomorphism by [BS13, Lemma 6.1.12]), we find also using smooth base change

$$Rj_* A \cong Rj_* g_\eta^* R g_{\eta,!} A \cong g^* R j_{Y*} R g_{\eta,!} A$$

is the pullback along a smooth map of the f'-universally locally acyclic sheaf $Rj_{Y*} Rg_{\eta,!} A$. (Note that $Rg_{\eta,!} A \in D(Y_\eta)$ is again constructible). Hence, it is f-universally locally acyclic by Proposition 3.5 (ii).

It remains to show the claim for finite maps g. We use the criterion of Proposition 3.4 and show that the map

$$\pi_1^* \mathbb{D}_{X/S}(Rj_* A) \otimes_\Lambda^{\mathbb{L}} \pi_2^* R j_* A \to \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_1^* R j_* A, R\pi_2^! R j_* A)$$

is an isomorphism in $D(X \times_S X)$. For this we note that by the arguments of the proof of Proposition 3.5 (i) the pushforward along g of the map yields the corresponding map for $Rg_* Rj_* A = Rj_* Rg_{\eta *} A$ in $D(Y)$. As pushforward along g is conservative (see Lemma 4.2 below), this shows the claim.

Step 2: We show that the theorem implies the similar statement for valuation rings V whose fraction field K is not necessarily algebraically closed, but such that the absolute Galois group is pro-p where p is the residue characteristic of V (if positive). We choose an algebraic closure $\bar{K}$ of K and choose a valuation ring $\bar{V}$ that dominates V in $\bar{K}$. Then $\bar{V}$ clearly satisfies the hypothesis of the Theorem. Note that any finite extension K' of K inside $\bar{K}$ has p-power degree. Namely we can factor K'/K in a separable and a purely inseparable part. The normal closure of any separable extension has p-power degree by the assumption on the absolute Galois group of K. If the purely inseparable extension is non-trivial (this can only happen if $\mathrm{char}(K) = p$), it of course also has p-power degree. We can write $\bar{V}$ as the colimit of its finitely presented sub-V-algebras $V \to V'$. Note that in particular any such V' is an integral domain, hence V-torsion free and thus flat over V. This implies that $V \to V'$ is finite free of p-power degree (as the extension of corresponding fraction fields has p-power degree).

More precisely, we show that if we have that for all $B \in D_{\mathrm{cons}}(X_{\bar{K}})$ we have that $R\bar{j}_* B$ is universally locally acyclic over $\bar{V}$, we also get that for all $A \in D_{\mathrm{cons}}(X_\eta)$ its pushforward $Rj_* A$ is universally locally acyclic over V. We denote by $g\colon X_{\bar{V}} \to X$, respectively $g_\eta\colon X_{\bar{K}} \to X_K = X_\eta$, the projections.

$$\begin{array}{ccc} X_{\bar{K}} & \xrightarrow{\bar{j}} & X_{\bar{V}} \\ \downarrow{\scriptstyle g_\eta} & & \downarrow{\scriptstyle g} \\ X_K & \xrightarrow{j} & X \end{array}$$

Let $A \in D_{\mathrm{cons}}(X_\eta)$. By our assumption and smooth base change we see that $R\bar{j}_* g_\eta^* A = g^* R j_* A$ is universally locally acyclic over $\bar{V}$. As the sheaf of ULA-sheaves is finitary by Proposition 3.11, we get that $g^* R j_* A$ arises as the base change of some ULA-sheaf over some finite extension V' of V. In particular, $Rj_* A$ is universally locally acyclic sheaf after base change to some finite extension V' of V of p-power degree.

Let us denote $g'\colon X_{V'} \to X$, by construction this map is finite. As pushforward along proper maps preserves universally locally acyclic sheaves, $Rg'_* R\bar{j}_* (g'_\eta)^* A = Rj_* R g'_{\eta *} (g'_\eta)^* A$ is universally locally acyclic over V. By the topological invariance of the étale site, we may assume that the field extension $K \to K'$ is finite étale. In this case we see that $Rg_{\eta,*}$ is both left and right adjoint to g_η^*. Moreover, the natural map

$$A \to R g_{\eta,*} g_\eta^* A \to A$$

is given by multiplication by the degree $[K' \colon K]$, which by construction is a power of p. As our coefficients are ℓ-torsion, this means that the map is an isomorphism. Thus, Rj_*A is a retract of the f-universally locally acyclic sheaf $Rg'_{\eta *}(g'_\eta)^*A$. Using Proposition 3.5 (vi), we obtain that Rj_*A is itself f-universally locally acyclic as well.

Step 3: We consider the case that $f \colon X \to S$ has relative dimension 0, in other words, the case that f is quasi-finite. This is the base case for our induction. By step 1, it suffices to show the claim for $f = \mathrm{id}_S \colon S \to S$. For this, we first note that a perfect-constructible complex $A \in D_{\mathrm{cons}}(\mathrm{Spec}(K))$ is automatically dualisable in $D(K)$, hence the natural map

$$\mathbb{D}_{K/K}(A) \otimes A \to \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(A, A)$$

is an isomorphism in $D(\mathrm{Spec}(K))$. We take the shriek pushforward along $j \colon \mathrm{Spec}(K) \to \mathrm{Spec}(V)$ and obtain, using that the natural map $j^*Rj_*A \to A$ is an isomorphism and the projection formula, that

$$Rj_!(\mathbb{D}_{K/K}(A) \otimes A) \cong Rj_!\mathbb{D}_{K/K}(A) \otimes Rj_*A \cong \mathbb{D}_{V/V}(Rj_*A) \otimes Rj_*A$$

and

$$Rj_!(\mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(A, A)) \cong \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(Rj_*A, Rj_*A).$$

Hence, Rj_*A is id_V-universally locally acyclic.

Step 4: Let now f have relative dimension at most d for some $d > 0$. The question is in particular étale-local on X, so it suffices to construct for each point of X an étale neighbourhood where Rj_*A is universally locally acyclic. Let x be a point of X. If the relative dimension of f at x is $d' < d$, there exists an open neighbourhood of x where f has relative dimension at most d'. Hence, Rj_*A is f-universally locally acyclic on this open neighbourhood of x by induction hypothesis.

Let us now assume that the relative dimension of f at x is d. By Zariski's Main Theorem [Sta, Tag 00QE] there is an open affine neighbourhood $X' \subseteq X$ of x such that $X' \to S$ factors over $X' \to \mathbb{A}^d_S \to S$, where $X' \to \mathbb{A}^d_S$ is quasi-finite. Hence, it suffices to show the claim for $X = \mathbb{A}^d_S$, or after choosing an open immersion $\mathbb{A}^d_S \to \mathbb{P}^d_S$ for $X = \mathbb{P}^d_S$ by Step 1. We now analyse the situation for $X = \mathbb{A}^d_S$ by considering projections to $\mathbb{A}^1_S$.

We denote by $\eta' = \mathrm{Spec}(k(t))$ the generic point of the special fibre of $\mathbb{A}^1_S = \mathrm{Spec}(V[t])$. We fix a geometric point $\bar{\eta}'$ lying over η' and denote by W the strict henselisation of $\mathbb{A}^1$ at $\bar{\eta}'$. We note that W is a valuation ring: its valuation is induced by the valuation on the fraction field $K(t)$ of $\mathbb{A}^1_V$ which associates to a polynomial the minimal valuations in K of its coefficients. Note that the associated valuation ring is the local ring of $\mathbb{A}^1_V$ at η' and the value group of $K(t)$ is the value group of K. In particular, the value group of W agrees with the value group of V and is hence divisible (as V is absolutely integrally closed). Moreover, W has a separably closed residue field. By standard results from Galois theory for valuation rings, any separable algebraic extension of the fraction field L of W is wildly ramified. In particular, this implies that L is algebraically closed if the residue characteristic of W is 0, respectively that the absolute Galois group of the fraction field of L is pro-p when the residue field of W has characteristic $p > 0$.

Let now U be any open neighbourhood of the generic point in the special fibre of $\mathbb{A}^1_S$ and let $Z = \mathbb{A}^1_S \backslash U$ denote its complement. Then the fibre Z_s of Z over any point $s \in S$ is finite, namely it is the complement of a non-empty open subset of $\mathbb{A}^1$ over a field. In particular, Z is quasi-finite over S.

For $i = 1, \ldots, d$ let us denote by $\pi_i \colon \mathbb{A}^d_S \to \mathbb{A}^1_S$ the projection to the i-th component. Let us fix some index i for now. As $\pi_{i,W} \colon \mathbb{A}^d_S \times_{\mathbb{A}^1_S} \mathrm{Spec}(W) \cong \mathbb{A}^{d-1}_W \to \mathrm{Spec} W$ has relative dimension $d-1$, we see that $Rj_{W*}(A|_{\mathbb{A}^{d-1}{}_{\tilde{\eta}}})$ is universally locally acyclic over $\mathrm{Spec}(W)$ by the inductive hypothesis together with step 2, where $\tilde{\eta}$ is the generic point of $\mathrm{Spec}(W)$ and $j_W \colon \mathbb{A}^{d-1}_{\tilde{\eta}} \to \mathbb{A}^{d-1}_W$ is the inclusion of the generic fibre over $\mathrm{Spec}(W)$. By smooth base change we see that $Rj_{W*}(A|_{\mathbb{A}^{d-1}_{\tilde{\eta}}}) = (Rj_*A)|_{\mathbb{A}^{d-1}_W}$. As the category of ULA-sheaves defines a finitary sheaf by Proposition 3.11, there is some étale map $U_i \to \mathbb{A}^1_S$ such that $Rj_*A|_{\mathbb{A}^{d-1}_{U_i}}$ is universally locally acyclic over U_i. But $U_i \to S$ is smooth, so it follows that $Rj_*A|_{X_{U_i}}$ is universally locally acyclic over S by Proposition 3.5 (iii).

Note that the image of U_i in $\mathbb{A}^1_S$ is open and its complement $Z_i = \mathbb{A}^1_S \backslash U_i$ is quasi-finite over S by the argument above. As schemes over S we have of course $\mathbb{A}^{d-1}_{U_i} \cong U_i \times \mathbb{A}^{d-1}_S$ with U_i sitting in the i-th component. Now the union of all such étale maps $\mathbb{A}^{d-1}_{U_i} \to \mathbb{A}^d_S$ for all indices i covers an open subset of X, whose complement is given by $Z_1 \times_S \ldots \times_S Z_d$, which is quasi-finite over S. As universal local acyclicity can in particular be checked étale-locally, we have shown that Rj_*A is universally locally acyclic on $\mathbb{A}^d_S \backslash (Z_1 \times_S \ldots \times_S Z_d)$.

Step 5: We show the claim for $X = \mathbb{P}^d_S$ by adapting the argument of [HS21, Lemma 4.3]. Let now $A \in D(\mathbb{P}^d_\eta)$. By Proposition 3.4 we have to show that the natural map

$$\pi_1^* \mathbb{D}_{X/S}(Rj_*A) \otimes^{\mathbb{L}}_\Lambda \pi_2^* Rj_*A \to \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_1^* Rj_*A, R\pi_2^! Rj_*A) \tag{4.1}$$

is an isomorphism.

We use the standard covering of $\mathbb{P}^d_S$ by copies of $\mathbb{A}^d_S$ and find for each copy of $\mathbb{A}^d_S$ by the previous step an open subscheme $U \to \mathbb{A}^d_S$ on which Rj_*A is universally locally acyclic over S. The complement Z of the union of these open subschemes in $\mathbb{P}^d_S$ is still quasi-finite over S by the previous step (Z is now even finite over S as Z is proper over S in this setting).

We first show that the map (4.1) is an isomorphism when restricted to $(\mathbb{P}^d_S \backslash Z) \times_S \mathbb{P}^d_S$ and $\mathbb{P}^d_S \times_S (\mathbb{P}^d_S \backslash Z)$: Its restriction to $(\mathbb{P}^d_S \backslash Z) \times_S \mathbb{P}^d_S$ is given by (using the compatibility of pullback with $\otimes^{\mathbb{L}}_\Lambda$ and $\mathrm{R}\mathcal{H}\mathrm{om}_\Lambda$)

$$\pi_1^* \mathbb{D}_{(\mathbb{P}^d_S \backslash Z)/S}(Rj_*A|_{\mathbb{P}^d_S \backslash Z}) \otimes^{\mathbb{L}}_\Lambda \pi_2^* Rj_*A \to \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_1^* Rj_*A|_{\mathbb{P}^d_S \backslash Z}, R\pi_2^! Rj_*A),$$

which is an isomorphism by Lemma 3.3 using that $Rj_*A|_{\mathbb{P}^d_S \backslash Z}$ is universally locally acyclic. In a similar fashion, we see that the restriction to $\mathbb{P}^d_S \times_S (\mathbb{P}^d_S \backslash Z)$ is given by

$$\pi_1^* \mathbb{D}_{\mathbb{P}^d_S/S}(Rj_*A) \otimes^{\mathbb{L}}_\Lambda \pi_2^* Rj_*A|_{\mathbb{P}^d_S \backslash Z} \to \mathrm{R}\mathcal{H}\mathrm{om}_\Lambda(\pi_1^* Rj_*A, R\pi_2^! Rj_*A|_{\mathbb{P}^d_S \backslash Z}),$$

which is again an isomorphism by Lemma 3.3 and Proposition 3.2. In particular, the cone C of the map (4.1) is supported on $Z \times_S Z$.

Moreover, as $f \colon \mathbb{P}^d_S \to S$ is proper, the pushforward of (4.1) to S gives by the proof of Proposition 3.5 (i) the corresponding equation for Rf_*Rj_*A. But Rf_*Rj_*A is id_S-universally locally acyclic by the remark at the end of step 2, hence the pushforward of (4.1) to S is an isomorphism. In particular, $R(f \times_S f)_*C$ vanishes. As C is supported on $Z \times Z$, we see that

$$R(f \times_S f)_*C \cong R(f \times_S f)_* R(i \times_S i)_*(i \times_S i)^*C \cong R(g \times_S g)_*(i \times_S i)^*C,$$

where $i \colon Z \to \mathbb{P}^d_S$ is the inclusion and $g \colon Z \to S$ is the structure map. As g is quasi-finite, $R(g \times_S g)_*$ is conservative by Lemma 4.2 below, hence C vanishes and (4.1) is an isomorphism. This shows the claim and finishes the proof of Theorem 4.1. □

We used that pushforward along quasi-finite maps is conservative, which we prove now.

Lemma 4.2. *Let $f \colon X \to S$ be a quasi-finite map. Then $Rf_* \colon D(X) \to D(S)$ is conservative.*

Proof. By Zariski's Main Theorem we may factor f as an open immersion followed by a finite map. Note that for open immersions f the pushforward is fully faithful, hence conservative. It thus suffices to consider the case that f is finite.

Let $\bar{s} \to S$ be a geometric point. Then $S_{\bar{s}}$ is a strictly henselian local ring. Thus, $X_{\bar{s}} := X \times_S S_{\bar{s}}$, which is finite over $S_{\bar{s}}$, is the spectrum of a finite product of strictly henselian local rings finite over $S_{\bar{s}}$. Hence,

$$(Rf_*A)_{\bar{s}} = R\Gamma(X_{\bar{s}}, A) = \bigoplus_{\bar{x}} R\Gamma(X_{\bar{x}}, A) = \bigoplus_{\bar{x}} A_{\bar{x}},$$

where the sum ranges over all geometric points $\bar{x}$ that lie over $\bar{s}$. Thus, a map $A \to B$ in $D(X)$ is an isomorphism at all geometric points $\bar{x} \to X$ if and only if the induced map $Rf_*A \to Rf_*B$ is an isomorphism at all geometric points $\bar{s} \to S$. Thus, Rf_* is conservative. □

As a corollary, perfect-constructible sheaves over schemes over a field are automatically ULA.

Corollary 4.3. *Let K be a field, and $f\colon X \to S = \operatorname{Spec}(K)$ a separated map of finite presentation. Then the inclusion $D^{\mathrm{ULA}}(X/S) \to D_{\mathrm{cons}}(X)$ is an equivalence.*

Proof. When K is algebraically closed, this is the previous theorem for $V = K$. For general K we have that for an algebraic closure K^{alg} of K the map $\operatorname{Spec}(K^{\mathrm{alg}}) \to \operatorname{Spec}(K)$ is an arc-cover. As universal local acyclicity can be checked arc-locally on the base, the assertion follows for general fields K. □

We can thus reformulate the first part of Theorem 4.1 in the following way. As when the base $S = \operatorname{Spec}(K)$ is a field, all perfect-constructible complexes are exactly the universally locally acyclic sheaves, the assertion of the theorem is that restriction to the generic fibre induces an equivalence

$$j^*\colon D^{\mathrm{ULA}}(X/S) \to D^{\mathrm{ULA}}(X_\eta/\eta)$$

between ULA-sheaves over X and ULA-sheaves over the generic fibre.

Using these results, we see that our definition of universal local acyclicity agrees with the usual definition. More precisely:

Theorem 4.4. *Let $f : X \to S$ be a separated map of finite presentation between qcqs schemes and let $A \in D_{\mathrm{cons}}(X)$ be a perfect-constructible complex. The following conditions are equivalent:*

(i) *A is f-universally acyclic. In other words, the pair (X, A) defines a dualisable object in the symmetric monoidal 2-category $\mathcal{C}'_S$.*

(ii) *The following condition holds after any base change in S. For any geometric point $\overline{x} \to X$ mapping to a geometric point $\overline{s} \to S$, and a generization $\overline{t} \to S$ of $\overline{s}$, the map*
$$A|_{\overline{x}} = R\Gamma(X_{\overline{x}}, A) \to R\Gamma(X_{\overline{x}} \times_{S_{\overline{s}}} S_{\overline{t}}, A)$$
is an isomorphism.

(iii) *The following condition holds after any base change in S. For any geometric point $\overline{x} \to X$ mapping to a geometric point $\overline{s} \to S$, and a generization $\overline{t} \to S$ of $\overline{s}$, the map*
$$A|_{\overline{x}} = R\Gamma(X_{\overline{x}}, A) \to R\Gamma(X_{\overline{x}} \times_{S_{\overline{s}}} \overline{t}, A)$$
is an isomorphism.

(iv) *After base change along $\operatorname{Spec} V \to S$ for any rank 1 valuation ring V with algebraically closed fraction field K and any geometric point $\overline{x} \to X$ mapping to the special point of $\operatorname{Spec} V$, the map*
$$A|_{\overline{x}} = R\Gamma(X_{\overline{x}}, A) \to R\Gamma(X_{\overline{x}} \times_{\operatorname{Spec} V} \operatorname{Spec} K, A)$$
is an isomorphism.

Proof. By Proposition 3.10, (i) implies (ii) and (iii), and each of them has (iv) as a special case. Thus, it remains to prove that (iv) implies (i). By Corollary 3.12, we can assume that $S = \operatorname{Spec} V$ is the spectrum of an absolutely integrally closed valuation ring of rank 1. Then Theorem 4.1 shows that being f-universally locally acyclic is equivalent to the map $A \to Rj_*j^*A$ being an isomorphism. By formal properties of adjunctions, it is clearly an isomorphism in the generic fibre, so one has to check that it is an isomorphism in the special fibre. We may check stalkwise that the map is an isomorphsim. But the maps on the stalks are exactly the maps in condition (iv), hence isomorphisms by assumption. □

References

[BM21] B. Bhatt and A. Mathew: *The arc-topology*, Duke Math. J. **170** (2021), no. 9, 1899-1988. 10, 11

[BS13] B. Bhatt, P. Scholze: *The pro-étale topology for schemes*, https://arxiv.org/abs/1309.1198. 14

[CD16] D. C. Cisinski, F. Déglise: *Étale motives*. Compos. Math., 152(3), 556-666, 2016. 7

[Del77] P. Deligne, *Cohomologie étale*, Lecture Notes in Mathematics, Vol. 569, Springer-Verlag, Berlin-New York, 1977, Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 1/2, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier. 12, 13

[FS21] L. Fargues, P. Scholze: *Geometrization of the local Langlands correspondence*, https://arxiv.org/abs/2102.13459, 2021. 1, 2, 6, 9

[Fuj97] K. Fujiwara: *Rigid geometry, Lefschetz-Verdier trace formula and Deligne's conjecture*, Invent. Math. **127** (1997), 489–533. 12

[HS21] D. Hansen, P. Scholze: *Relative perversity*, https://arxiv.org/abs/2109.06766, 2021. 1, 2, 3, 5, 6, 7, 9, 10, 11, 12, 16

[HRS21] T. Hemo, T. Richarz, and J. Scholbach, *Constructible sheaves on schemes and a categorical Künneth formula*, https://arxiv.org/abs/2012.02853v3, 2021. 1

[Ill94] Luc Illusie: *Autour du théorème de monodromie locale*, Astérisque (1994), no. 223, 9–57, Périodes p-adiques (Bures-sur-Yvette, 1988).

[Ill06] L. Illusie: *Vanishing cycles over general bases, after P. Deligne, O. Gabber, G. Laumon and F. Orgogozo*, `https://www.imo.universite-paris-saclay.fr/~illusie/vanishing1b.pdf`, 2006. 2

[LZ17] Q. Lu, W. Zheng: *Duality and nearby cycles*, Duke Math. J. **168** (2019), no. 16, 3135-3213. 2, 9

[LZ20] Q. Lu, W. Zheng: *Categorical traces and a relative Lefschetz–Verdier formula*, https://arxiv.org/abs/2005.08522. 1, 2, 3, 5, 6, 7, 12

[Nee96] A. Neeman: *The Grothendieck duality theorem via Bousfield's techniques and Brown representability*, J. Amer. Math. Soc. **9** (1996), no. 1, 205–236. 9

[Sch20] P. Scholze: *Geometrization of the local Langlands correspondence*, lecture notes and videos, https://www.math.uni-bonn.de/people/scholze/Geometrization/. 2

[Sta] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu`. 2, 9, 10, 11, 13, 15