CATEGORIES OF ÉTALE SHEAVES AND THE SIX FUNCTOR FORMALISM

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These notes correspond to the first of four talks covering the necessary sheaf-theoretic background of the workshop on Geometric Satake, January 2022, in Clermont Ferrand and Darmstadt. The aim is to introduce the setting of étale sheaves that will be worked with throughout the workshop and to recall properties of the six functor formalism that will appear in some proofs of the following talks.

Notation: Throughout the talk, we assume every scheme X to be a quasi-separated and quasi-compact scheme and Λ to be a (unital and commutative) ring. Further restrictions on X and Λ adapted to our purposes will be stated at appropriate places. Moreover, we will mostly consider triangulated categories and not stable ∞-categories in the context of derived categories.

1. Introduction

Recall the *derived category $D(\Lambda)$* of the ring Λ which is defined as the derived category of the abelian category Mod_Λ of modules over Λ. This is a classical showcase example of a derived category and one can think about its objects as chain complexes of Λ-modules. Complexes which are quasi-isomorphic to bounded complexes of finitely generated projective Λ-modules are called *perfect*. They have nice properties like being closed under triangles and the tensor product and form a full triangulated subcategory $\mathrm{D}_{\mathrm{perf}}(\Lambda) \subset \mathrm{D}(\Lambda)$ [StP, 066R, 0GM0, 0ATI]. They are exactly the *compact* objects in the derived category $\mathrm{D}(\Lambda)$ [StP, 07LT], i.e. those objects K such that the map

$$\bigoplus_{i \in I} \mathrm{Hom}_{\mathrm{D}(\Lambda)}(K, E_i) \to \mathrm{Hom}_{\mathrm{D}(\Lambda)}(K, \bigoplus_{i \in I} E_i)$$

is bijective for any set I and objects $E_i \in \mathrm{D}(\Lambda)$ parametrized by I [StP, 07LS].

In light of the previous stated triangulated categories, we want to introduce a new derived category being of fundamental relevance for the workshop:

Main Definition. Derived category of étale sheaves
For a scheme X, let $X_{\text{ét}}$ denote the site of étale schemes over X. Then, the category $\mathcal{S}\mathrm{h}(X_{\text{ét}}, \Lambda)$ of sheaves of Λ-modules on $X_{\text{ét}}$ is an abelian category.
We define

$$\mathrm{D}(X, \Lambda) := \mathrm{D}(\mathcal{S}\mathrm{h}(X_{\text{ét}}, \Lambda))$$

as the *(derived) category of étale sheaves* on X.

The category $\mathrm{D}(X, \Lambda)$ naturally carries a t-structure as derived category of an abelian category.

Remark 1.1. In [GL18, 4.1], one can find a definition of this category (for a quasi-projective k-scheme) as derived category in the ∞-world, i.e. it is defined as a stable ∞-category and its homotopy category recovers the corresponding triangulated version. By [GL18, 4.1.14], one has in this setting an equivalence

$$\mathrm{D}(X, \Lambda) \cong \mathcal{S}\mathrm{h}(X_{\text{ét}}, \mathrm{D}(\Lambda))$$

of ∞-categories where the right hand category denotes the ∞-category of étale ∞-sheaves of Λ-complexes.

Of particular importance there will be the subcategory $\mathrm{D_{cons}}(X, \Lambda) \subset \mathrm{D}(X, \Lambda)$ of so called *perfect constructible complexes* which, as we will examine, has under some additional assumptions on the scheme X and the ring Λ nice (closedness) properties, especially with respect to the six functor formalism. In a nutshell, a perfect constructible complex is a complex of étale Λ-sheaves that is locally constant with perfect values along a stratification. More precisely, this means that the complex is locally isomorphic to a complex in the image of $\mathrm{D_{perf}}(\Lambda)$ under the following functor:

Remark 1.2. Constant complex
To every complex $K \in \mathrm{D}(\Lambda)$ one can assign *a constant complex* $\underline{K}$ in $\mathrm{D}(X, \Lambda)$ given as the image under the pullback functor $\mathrm{D}(\Lambda) \to \mathrm{D}(X, \Lambda)$. More precisely, this functor is the induced derived functor corresponding to the constant sheaf functor $\mathrm{Mod}_\Lambda \to \mathcal{S}\mathrm{h}(X_{\text{ét}}, \Lambda)$ sending a Λ-module M to the sheafification of the constant functor

$$X_{\text{ét}}^{\text{op}} \longrightarrow \mathrm{Mod}_\Lambda, \ S \longmapsto M.$$

Referring to Remark 1.1, one can also think of the constant complex $\underline{K}$ as the constant étale sheaf of value K. The functor $\mathrm{D}(\Lambda) \to \mathrm{D}(X, \Lambda)$ has a right adjoint, that can be thought of as a global sections functor as it is given as the derived version of the global sections functor on the level of sheaves.

Example 1.3. If $X = *$ is a geometric point, then one has

$$\mathrm{D}(X, \Lambda) = \mathrm{D}(\Lambda) \text{ and } \mathrm{D_{cons}}(X, \Lambda) = \mathrm{D_{perf}}(\Lambda).$$

2. Functoriality

We have the following functorial properties of $\mathrm{D}(X, \Lambda)$:

- $\mathrm{D}(X, \Lambda)$ is a tensor triangulated category with *symmetric* monoidal structure, i.e. the tensor product
$$- \otimes - : \mathrm{D}(X, \Lambda) \times \mathrm{D}(X, \Lambda) \to \mathrm{D}(X, \Lambda)$$
(induced by the tensor product on sheaves) is symmetric (and associative).
- For all $N \in \mathrm{D}(X, \Lambda)$, the functor
$$- \otimes N : \mathrm{D}(X, \Lambda) \to \mathrm{D}(X, \Lambda)$$
has a right adjoint
$$\underline{\mathrm{Hom}}(N, -) : \mathrm{D}(X, \Lambda) \to \mathrm{D}(X, \Lambda)$$
called *interal hom*. This means, that $\mathrm{D}(X, \Lambda)$ is a *closed* symmetric monoidal category.
- [GL18, 4.1.9]: Let $f : X \to Y$ be a morphism of schemes. Then, f determines a base change functor
$$Y_{\text{ét}} \to X_{\text{ét}}, \ U \longmapsto U \times_Y X.$$
The ordinary pushforward of sheaves along f (given by composition with the base change) and its left adjoint pullback functor induce an adjunction
$$f^* : \mathrm{D}(Y, \Lambda) \rightleftarrows \mathrm{D}(X, \Lambda) : f_*.$$
- [GL18, 4.1.10]: Let $f : X \to Y$ be an étale morphism, then composition with f induces a functor
$$u : X_{\text{ét}} \to Y_{\text{ét}}.$$
Here, the pullback f^* can be seen as a composition with u and it preserves limits. Hence, it admits a left adjoint $f_!$.
[GL18, 4.2.8]: If f is moreover an open immersion, the functor $f_!$ is called *extension by zero along* f. Its image is spanned by those complexes $K \in \mathrm{D}(Y, \Lambda)$ with $K_{Y \setminus X} \simeq 0$, where $K_{Y \setminus X}$ denotes the pullback along $g : Y \setminus X \to Y$. Here we have $f^* f_! = \mathrm{id}$, which indeed means that $f_!$ is fully faithful.

- More generally, if $f\colon X \to Y$ is a locally closed immersion, then f factors as a closed immersion followed by an open immersion by definition. As we consider quasi-compact and quasi-separated schemes, a factorization is given by

$$X \xrightarrow{j} \overline{X} \xrightarrow{i} Y$$

with j open immersion into the scheme-theoretic closure $\overline{X}$ and i closed immersion (See [StP, 01QV] and note that a morphism of qcqs schemes is quasi-compact by [GW10, Proposition 10.3. and Remark 10.4.]). In this case, one defines

$$f_! \coloneqq i_* \circ j_! \colon \mathrm{D}(X,\Lambda) \to \mathrm{D}(Y,\Lambda)$$

as *extension by zero from X to Y*.

The definition of $f_!$ can be made even more generally - still under some restrictions on f as we will specify in the next section. This leads to a third adjoint pair of functors

$$f_!\colon \mathrm{D}(X,\Lambda) \rightleftarrows \mathrm{D}(Y,\Lambda) : f_!.$$

The six mentioned functors $(\otimes, \underline{\mathrm{Hom}}, f^*, f_*, f_!, f^!)$ also exist in other settings and they underlie certain compatibility conditions. These are described by **Grothendiecks six-functor formalism**.

3. The six-functor formalism

Here, we want to recall the (general) six-functor formalism by Grothendieck. The following listing tries to merge the most important properties of the six-functor formalism. Have a look at [CD19, A.5] for a more extensive version and at Gallauer's Introduction [Ga21] as well as at [KW01, Chapter II.7, e.g. Corollary 7.5], [Hö17, Introduction] for additional insights and proofs. Afterwards, we want to survey the existence of this formalism for perfect constructible complexes.

Let $\mathcal{C}$ be a suitable category of schemes (e.g. qcqs schemes of finite type over a field) and assume that for all $X \in \mathcal{C}$ there is given a (closed tensor) triangulated category $C(X)$ (e.g. derived category of 'sheaves' over X). This setting satisfies the Grothendieck six-functor formalism if the following conditions hold:

(1) There exist **3 pairs of adjoint (exact/triangulated) functors**

$$f^*\colon C(Y) \rightleftarrows C(X) : f_* \qquad \text{for any } f\colon X \to Y$$

$$f_!\colon C(X) \rightleftarrows C(Y) : f^! \qquad \text{for any } f\colon X \to Y \text{ separated, finite type}$$

$\otimes$ $\qquad$ $\underline{\mathrm{Hom}}$ $\qquad$ symmetric closed monoidal structure, s.t.

(i) The *pullback/inverse image* f^* is symmetric monoidal, i.e. there exists a natural isomorphism

$$f^*(- \otimes_{C(Y)} -) \xrightarrow{\sim} f^*(-) \otimes_{C(X)} f^*(-).$$

(ii) The *direct image with compact/proper support* $f_!$ is
  - for f open immersion the left adjoint of f^* (which implies $f^* = f^!$)
  - for f separated of finite type defined as

$$f_! \coloneqq p_* \circ j_!$$

where j is an open immersion and p a proper morphism given by the Nagata compactification

$$f\colon X \xrightarrow{j} \overline{X} \xrightarrow{p} Y.$$

Remark 3.1. For f smooth, the functor f^* admits a left adjoint $f_\#$ that is in general not given by $f_!$ (as it is for open immersions by (1)(ii)). But $f_!$ and $f_\#$ are related to each other, see [Ga21, 1.36] or [CD19, A.5.1.4.].

And one has the following compatibility conditions:

(2) The adjunctions are functorial, i.e. there exist covariant (resp. contravariant) 2-functors
$$f \longmapsto f_*,\ f \longmapsto f_! \text{ (resp. } f \longmapsto f^*,\ f \longmapsto f^!).$$
(3) There is a natural transformation
$$\alpha_f \colon f_! \to f_*$$
which is invertible when f is proper. Moreover, α is a morphism of 2-functors.

(4) **(Base change formula)**
For any cartesian square
$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & D \end{array}$$
there is a canonical (Beck-Chevalley-)transformation
$$g^* f_* \to k_* k^* g^* f_* \cong k_* h^* f^* f_* \to k_* h^*$$
given by using unit and counit of the $*$-adjunctions. It is an isomorphism if

(a) **(Proper base change)** f is proper, or
(b) **(Smooth base change)** g is smooth.

Remark 3.2. Sometimes, one can find a different base change formulation in terms of $f_!$ and $f^!$ (e.g. see [CD19, A.5.1.5.] or [Hö17, Introduction]):
Let f be separated of finite type, then there exist natural isomorphisms
(a) $g^* f_! \cong k_! h^*$
(b) $h_* k^! \cong f^! g_*$.

(5) **(Projection formula/Künneth type formula)**
For all f separated of finite type, the canonical transformation
$$f_!(f^*(-) \otimes -) \to - \otimes f_!(-)$$
is an isomorphism.

(6) **(Relative Poincare Duality)**
For any separated morphism of finite type $f \colon X \to Y$, there exists a natural isomorphism
$$f_* \underline{\mathrm{Hom}}(-, f^!(-)) = \underline{\mathrm{Hom}}(f_!(-), -).$$
(7) **(Duality)**
For every scheme X, there exists a *dualizing complex* $K_X \in C(X)$ such that with the definition
$$D_X \colon C(X) \to C(X), M \longmapsto \underline{\mathrm{Hom}}_X(M, K_X)$$
one has:
(i) K_X is a dualizing object of $C(X)$, i.e.
$$M \xrightarrow{\sim} D_X(D_X(M))$$
or shortly
$$D \circ D = \mathrm{id}.$$

(ii) For any separated morphism $f\colon X \to Y$ of finite type, there exist natural isomorphisms
- $D_Y \circ f^* \cong f^! \circ D_X$
- $f^* \circ D_X \cong D_Y \circ f^!$
- $D_X \circ f_! \cong f_* \circ D_Y$
- $f_! \circ D_Y \cong D_X \circ f_*$

(iv) and the formulas

$$\underline{\mathrm{Hom}}_X(-,-) = D_X(- \otimes D_X(-))$$

and

$$f^! D(- \otimes D(-)) = D(f^*(-) \otimes D f^!(-))$$
$$\Leftrightarrow f^! \underline{\mathrm{Hom}}(-,-) = \underline{\mathrm{Hom}}(f^*(-), f^!(-))$$

hold.

4. Perfect constructible complexes

Now, we introduce perfect constructible complexes as it is done in [BS13, 6.3.]. See also [GL18, Chapter 4.2.] for corresponding results in an ∞-setting (of quasi-projective schemes over a field).

Definition 4.1. Perfect constructible complexes
A complex $K \in \mathrm{D}(X,\Lambda)$ is called *(perfect) constructible* if there exists a

- finite stratification $\{X_i \xrightarrow{k_i} X\}_{i\in\{1,\dots,n\}}$ by locally closed constructible subsets $X_i \subset X$, i.e. $X = \coprod_{i=1}^n X_i$ with each $X_i = U_i \cap V_i^c$ for some $U_i, V_i \subset X$ quasicompact open,

- s.t. $K_{|X_i} := k_i^* K$ is locally constant with perfect values on $X_{\text{ét}}$, i.e. there is an étale covering $\{U_{ij} \to X_i\}_{j\in J_i}$ such that $K_{|X_i|U_{ij}} \simeq \underline{L_{ij}}$ for some perfect complex $L_{ij} \in \mathrm{D}_{\mathrm{perf}}(\Lambda)$.

The subcategory of perfect constructible complexes is denoted by $\mathrm{D}_{\mathrm{cons}}(X,\Lambda)$.

Proposition 4.2. *The subcategory* $\mathrm{D}_{\mathrm{cons}}(X,\Lambda) \subset \mathrm{D}(X,\Lambda)$ *is a triangulated subcategory closed under tensor products.*

Proof. The statements can be found in [BS13, 6.3.5.] and [BS13, 6.3.9.]. The proofs given there are very brief, we will restate the idea here. As two stratifications given as in the definition of perfect constructible complexes have a common refinement, one can reduce the claims to cases where $K \in \mathrm{D}_{\mathrm{cons}}(X,\Lambda)$ is (locally) constant with perfect values. For these cases, one can show closedness under tensor products and triangles (with some additional preliminary considerations for the second statement, see [BS13, 6.3.6.]) as one has similar results for $\mathrm{D}_{\mathrm{perf}}(\Lambda)$. ☐

4.1. Compatibility with the six-functor formalism.
For the workshop, it is sufficient to make the following additional assumptions on the scheme X and the ring Λ:

- The ring Λ is a finite local l-torsion ring for some l prime number (e.g. $\Lambda = \mathbb{Z}/l^n\mathbb{Z}$).
- All schemes X have finite l-cohomological dimension, i.e. one has $H^i(X_{\text{ét}},\Lambda) = 0$ for all $i > d$ where d denotes the cohomological dimension (e.g. finite type schemes over an algebraically closed or finite field).

In [BS13, 6.7.], one can find how constructibility is preserved by the six functor formalism in the pro-étale world. Under the assumptions above, one can apply these results to the étale world.
(See [GL18, Proposition 4.2.15] for corresponding statements in a setting of ∞-sheaves on a quasi-projective k-scheme.)
Upshot: Most of the six-functor formalism preserves constructibility if one restricts to finitely presented morphisms and (quasi)-excellent, l-coprime schemes, where l-coprime means that l is invertible in X, i.e. X is a scheme over $\mathbb{Z}[l^{-1}]$. (See also [HRS21, Remark 3.48]).

Indeed, we can summarize the situation as follows:

(i) For $f: X \to Y$ finitely presented and either f proper or Y l-coprime and quasi-excellent, the pushforward f_* preserves constructibility [BS13, 6.7.1].

(ii) The pullback f^* preserves constructibility in any case [BS13, 6.5.9].

(iii) The functor $f_!$ preserves constructibility if f is
- separated finitely presented [BS13, 6.7.7] or
- étale [BS13, 6.3.8] or
- a locally closed constructible immersion [BS13, 6.3.11].

(iv) If f is a separated finitely presented map of l-coprime and quasi-excellent schemes, then the functor $f^!$ exists on the level of constructible sheaves [BS13, 6.7.19].

(v) One has the
-**(Smooth base change)** if all schemes are l-coprime and $f: X \to Y$ is finitely presented and Y quasi-excellent [BS13, 6.7.4].
-**(Proper base change)** always for f_* [BS13, 6.7.5]. With respect to 3.2, there also exists a base change formula for $f_!$ if f is separated finitely presented [BS13, 6.7.10].

(vi) The **internal hom** of constructible complexes on X is constructible if X is quasi-excellent and l-coprime [BS13, 6.7.13].

(vii) The **(Projection formula)** holds for constructible complexes whenever f is separated finitely presented [BS13, 6.7.14].

(viii) There exists a constructible **dualizing complex** if X is excellent and l-coprime [BS13, 6.7.20].

Remark 4.3. (a) One may wonder why we have to consider finitely presented morphisms instead of only morphisms of finite type. The reason for this is, that we need the morphism f to preserve constructibility of topological spaces. By [EGA, IV 1.8.4.], the image of a constructible subset under a finitely presented morphism is constructible. For the same reason, one adds the condition of constructibility for locally closed immersions $f: X \to Y$ in (iii), i.e. the underlying topological space of the locally closed subscheme X has to be constructible. Then, the morphism f identifies constructible subsets of X with those of Y contained in X.

(b) If one considers schemes of finite type over an algebraically closed or finite field k such that $l \nmid \mathrm{char}(k)$, all the additional assumptions are automatically satisfied. Particularly, all schemes locally of finite type over a field are excellent [GW10, 12.51.] and all morphisms between schemes of finite type over a field are of finite type (which is equivalent to finitely presented in this case).

We end this section with some more properties of $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$.

Remark 4.4. (a) Under the cohomological finiteness assumption on X, i.e. all affine $U \in X_{\text{ét}}$ have finite Λ-cohomological dimension, the subcategory $\mathrm{D}_{\mathrm{cons}}(X, \Lambda) \subset \mathrm{D}(X, \Lambda)$ is exactly given by the full subcategory of compact objects [BS13, 6.4.8].

(b) For $\Lambda = \mathbb{F}_l$ regular, the derived category $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ carries a t-structure. It is given by restriction of the t-structure on $\mathrm{D}(X, \Lambda)$ as described in Section 3.6 of [HRS21], especially note [HRS21, 3.32].

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