TANNAKIAN RECONSTRUCTION

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In the following we fix primes p and ℓ such that $\ell \nmid p$, an algebraically closed field k of characteristic p, a split reductive group G with split maximal torus T, $\Lambda = \mathbb{Z}/\ell^c\mathbb{Z}$ for some $c \leq 1$ and a finite set I.

The goal of these notes are to construct a flat Λ-group scheme $\widehat{G}$ such that the category of $\widehat{G}$-representations in flat Λ-modules is equivalent to the Satake category $\mathrm{Sat}_G(\Lambda)$.

1. Tannakian duality

The goal of this section is to establish the general theory that allows us to construct a flat Λ-group scheme such that its category of representations is equivalent to the Satake category. The main ingredient of the construction will be the Barr-Beck monadacity theorem.

Definition 1.1. A symmetric monoidal category is called *rigid* if every object is dualizable.

Example 1.2. The following symmetric monoidal categories are rigid.

(i) The category of vector bundles over a scheme X, the dual of a vector bundle $\mathcal{F}$, seen as a locally free $\mathcal{O}_X$-module, is given by $\mathscr{H}om(\mathcal{F}, \mathcal{O}_X)$.

(ii) The category of perfect complexes over a ring A, the dual of a perfect complex P is given by $R\underline{\mathrm{Hom}}(P, A)$.

(iii) The category $\mathrm{Sat}^I_G(\Lambda)$, the dual of an object $A \in \mathrm{Sat}^I_G(\Lambda)$ is given by $sw^*\mathbb{D}(A)$, where $\mathbb{D}(A)$ denotes the Verdier dual of A and $sw\colon \mathrm{Hck}^I_G \to \mathrm{Hck}^I_G$ is the isomorphism that switches the two G-torsors on the points of the Hecke stack.

Definition 1.3. Let $F\colon \mathcal{C} \to \mathcal{A}$ be a functor. A parallel pair $f, g\colon A \to B$ in $\mathcal{C}$ is called *F-split* if it induces a coequalizer of the following form

$$F(A) \underset{F(g)}{\overset{F(f)}{\rightrightarrows}} F(B) \xrightarrow{h} C, \qquad t\colon F(B) \to F(A), \quad s\colon C \to F(B),$$

where t resp. s is a section of $F(f)$ resp. h and $F(g) \circ t = s \circ h$.

Remark 1.4. Assume $F\colon \mathcal{C} \to \mathcal{A}$ is an additive functor of additive categories and $f, g\colon A \to B$ is an parallel pair in $\mathcal{C}$. Further assume that $\ker(F(f-g))$ exists and that f, g is F-split, then the following sequence is split exact

$$0 \longrightarrow \ker(F(f-g)) \longrightarrow F(A) \xrightarrow{F(f-g)} F(B) \longrightarrow 0.$$

Example 1.5. Consider the functor $F^I_G\colon \mathrm{Sat}^I_G(\Lambda) \to \mathrm{Mod}^{\text{fin.proj.}}_\Lambda$. Let $f, g\colon A \to B$ be an F^I_G-split parallel pair ins $\mathrm{Sat}^I_G(\Lambda)$. Then $\mathrm{Coker}(f-g)$ exists by the above remark and and talk 9, and so $\mathrm{Coeq}(f, g) = \mathrm{Coker}(f-g)$ exists.

Definition 1.6. An endofunctor $T\colon \mathcal{A} \to \mathcal{A}$ is called a *monad* if there exists natural transformations $\varepsilon\colon \mathrm{id}_\mathcal{A} \to T$ and $\mu\colon T^2 \to T$ such that the following diagrams commute

$$\begin{array}{ccc} T^3 & \xrightarrow{\mu T} & T \\ {\scriptstyle T\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T, \end{array} \qquad \begin{array}{ccccc} T & \xrightarrow{\varepsilon T} & T & \xleftarrow{T\varepsilon} & T \\ & {\scriptstyle =}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle =} & \\ & & T. & & \end{array}$$

Definition 1.7. Let $T\colon \mathcal{A} \to \mathcal{A}$ be a monad. A *T-algebra in $\mathcal{A}$* is a pair (A, a), where A is an object in $\mathcal{A}$ and $a\colon TA \to A$ is a morphism in $\mathcal{A}$ such that the following diagrams commute

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon A} & TA \\ & \overset{\mathrm{id}_A}{\searrow} & \downarrow{\scriptstyle a} \\ & & A, \end{array} \qquad\qquad \begin{array}{ccc} T^2A & \xrightarrow{Ta} & TA \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle a} \\ TA & \xrightarrow{a} & A. \end{array}$$

Example 1.8. Let $\mathcal{A}$ be the category of flat Λ-modules and let us define the endofunctor $T\colon M \mapsto M \otimes_\Lambda A$ on $\mathcal{A}$, where A some flat Λ-module. If T is a monad, then we have natural transformations $\varepsilon\colon \mathrm{id}_\mathcal{A} \to T$ and $\mu\colon T^2 \to T$ such that the diagrams in the definition of a monad commute. Evaluating T at Λ gives us in particular Λ-linear maps $\varepsilon\colon \Lambda \to A$ and $\mu\colon A \otimes_\Lambda A \to A$ (here we abuse notation). The morphism ε gives us a distinguished element $1 \in A$ and μ a map $A \times A \to A$ that is compatible with the abelian group structure on A. Further, the compatibilities of μ and ε in the definition of a monad show that this endows A with a structure of an associated Λ-algebra. Going through the definitions, we also see that T is a monad if A is an associative Λ-algebra.

Going through the definitions of a T-algebra one also sees similarly that giving a T-algebra is equivalent to giving an A-module.

Example 1.9. Let $F\colon \mathcal{C} \to \mathcal{A}$ be a functor with left adjoint $L\colon \mathcal{A} \to \mathcal{C}$. Then $T \coloneqq F \circ L$ is a monad.

Indeed, the unit $\varepsilon\colon \mathrm{id}_\mathcal{A} \to T$ and counit $\eta\colon L \circ F \to \mathrm{id}_\mathcal{C}$ give the desired monad structure by defining $\mu = F \circ \eta \circ L\colon T^2 \to T$.

Theorem 1.10. *Let $F\colon \mathcal{C} \to \mathcal{A}$ be a functor. Then F satisfies the following assumptions*

(1) *F has a left adjoint L,*
(2) *F is conservative, and*
(3) *$\mathcal{C}$ has coequalizers by F-split parallel pairs and F preserves these*

if, and only if $\mathcal{C}$ is equivalent to the category of modules over the monad $F \circ L$.

Proof. [BW05, Thm. 3.14]. ☐

Definition 1.11. Let $\mathcal{A}$ and $\mathcal{C}$ be symmetric monoidal categories. Then $\mathcal{C}$ has a *tensored action of $\mathcal{A}$* if there is a functor $-\otimes-\colon \mathcal{A}\times\mathcal{C} \to \mathcal{C}$ such that we have natural isomorphisms $\alpha_{AA'X}\colon (A\otimes A')\otimes X \xrightarrow{\sim} A \otimes (A' \otimes X)$ for $A, A' \in \mathcal{A}$ and $X \in \mathcal{C}$, a natural isomorphism $\lambda_X\colon 1 \otimes X \to X$ for $X \in \mathcal{C}$, such that the usual diagram commute, i.e. associativity and right/left neutrality.

Example 1.12. The Satake category $\mathrm{Sat}_G^I(\Lambda)$ is tensored over $\mathrm{Mod}_\Lambda^{\text{fin.proj.}}$ as we can see every finite projective Λ-module as a flat perverse ULA sheaf over Hck_G^I concentrated in degree 0.

Proposition 1.13 ([FS21, Prop. VI.10.2])**.** *Let $\mathcal{A}$ be a rigid symmetric monoidal category and let $\mathcal{C}$ be a rigid symmetric monoidal category tensored over $\mathcal{A}$. Moreover, let*

$$F\colon \mathcal{C} \to \mathcal{A}$$

be a symmetric monoidal $\mathcal{A}$-linear conservative functor, such that $\mathcal{C}$ admits and F reflects coequalizers by F-split parallel pairs. Assume that $\mathcal{C}$ is the filtered union of subcategories $\mathcal{C}_i$ stable under coequalizers of F-split parallel pairs and the $\mathcal{A}$-action, such that $F_i \coloneqq F_{|\mathcal{C}_i}$ is representable by $X_i \in \mathcal{C}$. Then

$$\mathcal{H} \coloneqq \varinjlim_i F(X_i)^\vee \in \mathrm{Ind}(\mathcal{A})$$

admits naturally the structure of a Hopf-algebra and $\mathcal{C}$ is naturally equivalent to the category of $\mathcal{H}$-comodules in $\mathcal{A}$.

Proof. The functor F_i has a left adjoint given by $A \mapsto A \otimes X_i$, as

$$\mathrm{Hom}(A \otimes X_i, Y) = \mathrm{Hom}(X_i, A^\vee \otimes Y) = F(A^\vee \otimes Y) = A^\vee \otimes F(Y) = \mathrm{Hom}(A, F(Y)).$$

Therefore, the functor $T\colon A \mapsto A \otimes F(X_i)$ is a Monad (see Example 1.9), which is equivalent to an associtive algebra structure on $F(X_i)$ (see Example 1.8) and the category of modules over $F(X_i)$ in $\mathcal{A}$ is by the Barr-Beck theorem (see Theorem 1.10) equivalent to $\mathcal{C}_i$. Dually, the category of comodules over the coalgebra $F(X_i)^\vee$ in $\mathcal{A}$ is equivalent to $\mathcal{C}_i$. Then, we claim that

$$\mathcal{H} \coloneqq \varinjlim_i F(X_i)^\vee \in \mathrm{Ind}(\mathcal{A})$$

has the structure of a Hopf-algebra and $\mathcal{C}$ is equivalent to the category of comodules over $\mathcal{H}$ in $\mathcal{A}$.

Indeed, for any $X \in \mathcal{C}$ there is an i such that $X \in \mathcal{C}_i$ so by adjunction, we have

$$F(F(X)^\vee \otimes X) = \mathrm{Hom}(X_i, F(X)^\vee \otimes X) = \mathrm{Hom}(F(X) \otimes X_i, X)$$

and thus get a map $F(X) \otimes X_i \to X$. Using $\mathcal{A}$-linearity and and that F is monoidal, this yields a map

$$F(X) \otimes F(X_i) = F(F(X) \otimes X_i) \to F(X)$$

and dually, we get the map

$$F(X) \to F(X) \otimes F(X_i)^\vee \to F(X) \otimes \mathcal{H}.$$

Thus, we get a functor from $\mathcal{C}$ to $\mathcal{H}$-comodules in $\mathrm{Ind}(\mathcal{A})$. Then we claim that the category of $\mathcal{H}$-comodules in $\mathrm{Ind}(\mathcal{A})$ is equivalent to the filtered colimit of $F(X_i)^\vee$-comodules in $\mathcal{A}$ which by the above is equivalent to $\mathcal{C}$.

To see this let M be an $\mathcal{H}$-comodule in $\mathrm{Ind}(\mathcal{A})$ (here we view $\mathrm{Ind}(\mathcal{A})$ as a symmetric monoidal category via colimit of termwise tensor product). In particular, we can write $M = \varinjlim_i M_i$, where $M_i \in \mathcal{A}$. The $\mathcal{H}$-module structure is given by a map

$$\varinjlim_i M_i \to \varinjlim_i M_i \otimes F(X_i)^\vee.$$

Hence, we see that every M_i has an $F(X_i)^\vee$-comodule structure and the $\mathcal{H}$-comodule structure on M is determined by those. This shows the equivalence and that the category of $\mathcal{H}$-comodules in $\mathrm{Ind}(\mathcal{A})$ is equivalent to $\mathcal{C}$.

Further we claim that for any i, j there is a k such that $\mathcal{C}_i \otimes \mathcal{C}_j \subseteq \mathcal{C}_k$.

Indeed, first we claim that $\mathcal{C}_i$ (resp. $\mathcal{C}_j$) is generated by X_i (resp. X_j) under tensors with $\mathcal{A}$ and coequalizers of F-split parallel paris, so $\mathcal{C}_i \otimes \mathcal{C}_j$ is generated under these operations by $X_i \otimes X_j$. Therefore, for any k such that $X_i \otimes X_j \in \mathcal{C}_k$, we have $\mathcal{C}_i \otimes \mathcal{C}_j \subseteq \mathcal{C}_k$.

To show the claim let $X \in \mathcal{C}_i$ and let us look at the counit $\varepsilon(X)\colon X_i \otimes F(X) \to X$ of the adjunction in the beginning. Let us denote the left adjoint of F_i by L_i. Then we have the following diagram

$$L_i F L_i F(X) \underset{\varepsilon L_i F X}{\overset{L_i F \varepsilon(X)}{\rightrightarrows}} L_i F(X) \xrightarrow{\varepsilon(X)} X.$$

Applying F to the diagram yields an F-split coequliazer by [BW05, Cor. 3.9] and since F reflects those, we see that the above diagram is a coequalizer. Now note that we have

$$L_i F L_i F(X) = X_i \otimes F(X_i) \otimes F(X), \quad L_i F(X) = X_i \otimes F(X)$$

proving the claim.

Let X_k respresent $F_{|\mathcal{C}_k}$, then we have a natural map $X_k \to X_i \otimes X_j$, that is adjoint to $1 \to F(X_i \otimes X_j) = F(X_i) \otimes F(X_j)$ which is the tensor product of the natural unit maps $1 \to F(X_i)$ and $1 \to F(X_j)$. Therefore, we get a map

$$\mathcal{H} \otimes \mathcal{H} = \varinjlim_{i,j} F(X_i)^\vee \otimes F(X_j)^\vee \cong \varinjlim_{i,j} F(X_i \otimes X_j)^\vee \to \varinjlim_k F(X_k)^\vee = \mathcal{H}.$$

We also have a unit map $1 \to \mathcal{H}$ that is induced by $1 = F(1) \to F(X_i)^\vee$ (note that $F(1) = \mathrm{Hom}(X_i, 1)$ and so applying F and then dualizing the map $X_i \to 1$ corresponding to $1 = F(1)$

gives the desired map). This makes $\mathcal{H}$ into a commutative algebra object in $\mathcal{A}$. To make $\mathcal{H}$ into a Hopf-algebra note that we have

$$F(X_i)^\vee = F(X_i^\vee) = \underline{\mathrm{Hom}}(X_i, X_i^\vee) = \underline{\mathrm{Hom}}(X_i \otimes X_i, 1).$$

Now switching of the components defines the antipode (that is actually an involution) of the comultiplication on $\mathcal{H}$. □

2. $I = \{*\}$ case

Let $W \subseteq X_*(T)^+$ be a finite subset closed under the dominance order. We can define $\mathrm{Hck}_{G,W}$ as the union of all open Schubert cells $\mathrm{Hck}_{G,\mu}$ for $\mu \in W$. This is a closed substack and similarly, we can define the full subcategory $\mathrm{Sat}_{G,W}(\Lambda)$. We will not be able to construct left adjoint for the fiber functors from $\mathrm{Sat}_G(\Lambda) \to \mathrm{Mod}_\Lambda^{\mathrm{fin.proj}}$ but if we can construct them for finite subsets $X_*(T)^+$ (closed under the dominance order), we can still use Proposition 1.13, as we can write the Satake category as a filtered colimit over the Satake categories corresponding to the closures of the Schubert cells of the Hecke stack.

We will first do this for the case where I is a singleton and later use this to construct the left adjoint for general I.

Theorem 2.1. *The functor*

$$F_{G,W} := \bigoplus_m \mathcal{H}^m(R\pi_{G*})\colon\ \mathrm{Sat}_{G,W}(\Lambda) \to \mathrm{Mod}_\Lambda^{\mathrm{fin.proj}}$$

has a left adjoint L_W. Moreover, it is the restriction of the left adjoint to

$$F' := \bigoplus_m \mathcal{H}^m(R\pi_{G*})\colon\ \mathrm{Perv}(\mathrm{Hck}_{G,W}, \Lambda) \to \mathrm{Shv}_{\text{ét}}(X, \Lambda).$$

Proof. First note that the base change functor naturally has a left adjoint. Since all the categories involved are symmetric monoidal and $F_{G,W}$ is symmetric monoidal and linear with respect to finite projective Λ-modules seen as complexes concentrated in degree 0, the left adjoint, if it exists, is uniquely determined by its value of $\Lambda \in \mathrm{Mod}_\Lambda^{\mathrm{fin.proj}}$ as then $L_W(V) = L_W(\Lambda) \otimes V$ for any $V \in \mathrm{Mod}_\Lambda^{\mathrm{fin.proj}}$ (note that any finite projective Λ-module can be seen as an object in $D(\mathrm{Hck}_{G,W}, \Lambda)^{\mathrm{bd}}$ and using the characterization of flat perverse ULA sheaves via the constant term functor, we see that V can be seen as an object in the Satake category).

Note that F' admits a left adjoint L' by the adjoint functor theorem **(?)**. Therefore, if $P_W := L'_W(\Lambda)$ is ULA and flat perverse the claim is shown.

By the characterization of flat perverse and ULA sheaves via the constant term functor it is enough to show that $F'(P_W)$ is equivalent to a finite projective Λ module concentrated in degree 0. We will show this by induction on W.

Let $\mu \in W$ be a character of dimension $d_\mu = \langle 2\rho, \mu\rangle$ and let us look at the inclusion of the Schubert cell $j_\mu\colon \mathrm{Hck}_{G,\mu} \hookrightarrow \mathrm{Hck}_{G,W}$. By definition, we have

$$\mathrm{Hom}(P_W, {}^{\mathfrak{p}}Rj_{\mu*}\Lambda[d_\mu]) = F'(\mathcal{H}^0(Rj_{\mu*}\Lambda[d_\mu]))$$

and by [FS21, Prop.VI.7.9], this is a finite free Λ-module. Further by adjunction, we have that $\mathrm{Hom}({}^{\mathfrak{p}}j_\mu^* P_W, \Lambda[d_\mu]))$ is a finite free Λ-module. Now ${}^{\mathfrak{p}}j_\mu^* P_W$ is concentrated on an open Schubert cell $\mathrm{Hck}_{G,\mu}$, so by definition of the perverse t-structure it is equivalent to $M[d_\mu]$, where M is a Λ-module. Note that Λ is Gorenstein, so its dualizing complex is given by $\Lambda[0]$. By the above, we know that $\mathrm{Hom}(M, \Lambda)$ is finite free and by [Sta22, 0A7C], we have that $M \cong \mathrm{Hom}(\mathrm{Hom}(M, \Lambda), \Lambda)$ and the functor $R\underline{\mathrm{Hom}}(-, \Lambda[0])$ induces an antiequivalence between the bounded derived category of Λ-modules with coherent cohomology. Thus, M is finitely generated and therefore isomorphic to a direct sum of a finite free Λ-module and torsion Λ-module. The torsion part has to vanish since otherwise $\mathrm{Hom}(M, \Lambda)$ would have torsion which cannot happen as it is finite free (note that $\Lambda = \mathbb{Z}/\ell^c\mathbb{Z}$). Therefore, M is a finite free Λ-module.

Now let us take a maximal element $\mu \in W$ and set $W^\circ = W \setminus \{\mu\}$. Let us look at the excat sequence

$$0 \longrightarrow K \longrightarrow {}^{\mathfrak{p}}j_{\mu!}j_\mu^* P_W \longrightarrow P_W \longrightarrow Q \longrightarrow 0$$

in $\mathrm{Perv}(\mathrm{Hck}_{G,W}, \Lambda)$, where K is the kernel and Q is the cokernel of ${}^{\mathfrak{p}}j_{\mu!}j_\mu^* P_W \to P_W$. By (see [BBD82, Prop. 1.3.17]), we have

$$Q = {}^{\mathfrak{p}}Ri_{\mu*}Li_\mu^* P_W,$$

where i_μ is the inclusion of the complement of the open Schubert cell with reduced structure . As μ was chosen maximal, we know that $(\mathrm{Hck}_{G,W} \setminus \mathrm{Hck}_{G,\mu})_{\mathrm{red}} = \mathrm{Hck}_{G,W^\circ}$. As ${}^{\mathfrak{p}}Ri_{\mu*}Li_\mu^*$ is a left adjoint (again use [BBD82, Prop. 1.3.17]), we see that ${}^{\mathfrak{p}}Ri_{\mu*}Li_\mu^* L'_W$ must be a left adjoint for F'_{W° and therefore $Q \simeq P_{W^\circ}$. Also, note that $F'({}^{\mathfrak{p}}j_{\mu!}j_\mu^* P_W)$ is finite projective, as ${}^{\mathfrak{p}}j^* P_W$ is finite projective module sitting in one degree, so by [FS21, Prop. VI.7.5], we have that $F'({}^{\mathfrak{p}}j_{\mu!}j_\mu^* P_W)$ is finite projective. Therefore, $F'(Q)$ is a finite projective Λ-module. Thus, by induction it is enough to show that $K = 0$. For this note that K lies in the kernel of

$${}^{\mathfrak{p}}j_{\mu!}j_\mu^* P_W \to {}^{\mathfrak{p}}Rj_{\mu*}j_\mu^* P_W.$$

Again, as ${}^{\mathfrak{p}}j_\mu^* P_W$ is finite free, we can use [FS21, Prop. VI.7.5] to see that there is an a such that $\ell^a K = 0$. Let K' denote the kernel of the above cunstruction for $\Lambda' = \mathbb{Z}/\ell^{c+a}\mathbb{Z}$. Using that everything is functorial, we get a map from K' to K by the map $\Lambda' \to \Lambda$. Since ${}^{\mathfrak{p}}j_{\mu!}j_\mu^* P_W$ is in the Satake category, so in particular flat over Λ, we see that the image of K' in K lies in $\ell^a K = 0$ (note that $\Lambda = \Lambda'/(\ell^a\mathbb{Z}/\ell^{a+c}\mathbb{Z})$ and so the base change of K' lies in the image of the base change $K' \otimes_{\Lambda'} \Lambda$, which is $\ell^a K'$). But as all constructions are compatible with base change, we see that $K' \to K$ has to be surjective **(?)**, so $K = 0$. □

3. The general case

Let $W_i \subseteq X_*(T)^+$ be a finite family of finite subsets stable under the dominance order indexed by I. As in the absolute case, we can define $\mathrm{Hck}^I_{G,(W_i)_i}$ and $\mathrm{Sat}^I_{G,(W_i)_i}(\Lambda)$.

We use the case for I singleton to construct a left adjoint for the fiber functor $F_{G,(W_i)_i}\colon \mathrm{Sat}^I_{G,(W_i)_i}(\Lambda) \to \mathrm{Mod}_\Lambda^{\mathrm{fin.proj}}$. We can already do this on perverse sheaves since the compatibility of F with exterior products shows that on perverse sheaves the left adjoint is given by exterior products. We then have to show that the image of the unit under the left adjoint lies in the Satake category, which follows again from the I singleton case.

Theorem 3.1. *The functor*

$$F_{G,(W_i)_i} \coloneqq \bigoplus_m \mathcal{H}^m(R\pi_{G*})\colon \mathrm{Sat}^I_{G,(W_i)_i}(\Lambda) \to \mathrm{Mod}_\Lambda^{\mathrm{fin.proj}}$$

has a left adjoint $L_{(W_i)_i}$ satisfiyng

(i) $L_{(W_i)_i}(V) = L_{(W_i)_i}(1) \otimes V$, *and*
(ii) $L_{(W_i)_i}(1) = *_{i\in I} L_{W_i}(1)$.

Proof. Certainly, if we can show *(ii)*, then *(i)* uniquely characterizes the left adjoint using Theorem 2.1. First note that

$$F' \coloneqq \bigoplus_m \mathcal{H}^m(R\pi_{G*})\colon \mathrm{Perv}(\mathrm{Hck}^I_{G,(W_i)_i}, \Lambda) \to \mathrm{Shv}_{\text{ét}}(X^I, \Lambda)$$

has a left adjoint since we can decompose I into singletons, where we already know the existence of a left adjoint (see Theorem 2.1).

For $i \in I$ let L^i denote the left adjoint corresponding to F^i in Theorem 2.1. Then the exterior product defines a left adjoint for F' as the functor F' commutes with exterior products.

Let $P_{(W_i)_i}$ be the image of the unit under the left adjoint. We have that $F'(*_i P_{(W_i)_i}) \cong \boxtimes_{i\in I} F_{W_i}(P_{W_i})$ by compatibility of F with the fusion product. Let $P_{(W_i)_i} \to *_i P_{W_i}$ be the adjoint to the section of the above isomorphism. We can write F' using hyperbolic localization as a !-pullback followed by $*$-pushforward, so the left adjoint is given by ${}^{\mathfrak{p}}\mathcal{H}^0(q_!^- p^{-*}(-))$ (note that $q_!^- p^{-*}A \in {}^{\mathfrak{p}}D^{\geq 0}$ for any $A \in \mathrm{Shv}_{\text{ét}}(X^I, \Lambda)$ seen as a complex concentrated in degree 0 in $D(\mathrm{Gr}_T^I, \Lambda)$, see [BBD82, Prop. 1.3.17]). In particular, using the Künneth formula, we see that the left adjoint commutes with exterior products, so for the inclusion

$$j^*\colon \mathrm{Sat}^I_{G,(W_i)_i}(\Lambda) \to \mathrm{Sat}^{I,I_1,\dots,I_n}_{G,(W_i)_i}(\Lambda),$$

we have that $j^*P_{(W_i)_i} \simeq j^* *_i P_{W_i}$. Now using that $B \in \mathrm{Sat}^I_{G,(W_i)_i}(\Lambda)$ is equivalent to ${}^{\mathfrak{p}}\mathcal{H}(Rj*j^*(B))$, we see with Yoneda that the morphism $P_{(W_i)_i} \to *_i P_{W_i}$ is indeed an equivalence. □

4. The Satake group

We know that the reduced subscheme of Gr_G^I is given by $\bigcup_{\mu_\bullet \in (X_*(T)^+)^I} \mathrm{Gr}^I_{G,\leq \mu_\bullet}$. So, we can write the Satake category Sat_G^I as a filtered union over $(X_*(T)^+)^I$ of $\mathrm{Sat}^I_{G,(W_{\mu_i})_i}$, where for $\mu_\bullet \in (X_*(T)^+)^I$ we set $W_{\mu_i} = \{\nu \in X_*(T)^+ \mid \nu \leq \mu_i\}$.

Using Theorem 3.1 and Proposition 1.13 (note that by Talk 10 the functor F^I satisfies all of the assumptions in the proposition), we now get a Hopf algebra $\mathcal{H}_G^I$ corresponding to $\mathrm{Sat}_G^I(\Lambda)$. By construction and Theorem 3.1 (ii), we finally get

Proposition 4.1. *We have an isomorphism of Hopf-algebras*

$$\bigotimes_i \mathcal{H}_G^{\{i\}} \cong \mathcal{H}_G^I.$$

So the group scheme in the $I = \{*\}$ not only determines the Satake category $\mathrm{Sat}_G(\Lambda)$ but also $\mathrm{Sat}_G^I(\Lambda)$. This allows us to define $\widehat{G}$ as the group scheme associated to $\mathcal{H}_G^{\{*\}}$ and we have $\mathrm{Rep}_{\widehat{G}}(\Lambda) \simeq \mathrm{Sat}_G(\Lambda)$. Note that $\mathrm{Ind}(\mathrm{Mod}_\Lambda^{\text{fin.proj}})$ is equivalent to the category of flat Λ-modules. In particular $\widehat{G}$ is a flat group scheme over Λ.

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