CONSTANT TERM FUNCTORS AND TENSOR STRUCTURES

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These are notes for a talk given by the author during a workshop on Geometric Satake in Clermont-Ferrand, in January 2022. Due to time constraints, a lot of details, proofs and lemmas were omitted in this talk, which have been added here. Almost all of the arguments below come from [FS21, Chapter VI], although translated into the setting of schemes. In particular, any mistakes in this translation are entirely due to the author.

Throughout these notes, we fix an algebraically closed field k, a reductive k-group G along with a maximal torus T and Borel B, a coefficient ring $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ for some prime $\ell \neq \mathrm{char}(k)$, and the projective line $X = \mathbb{P}^1_k$. Our main goal is to give the Satake category $\mathrm{Sat}^I_{G,\Lambda}$ a Tannakian-type structure. (Recall that this Satake category is the full subcategory of flat relatively perverse ULA sheaves in $\mathrm{D}_{\text{ét}}(\mathrm{Hk}_{G,I},\Lambda)^{\mathrm{bd}}$.) In particular, we want to equip it with the structure of a symmetric monoidal category, and find a symmetric monoidal fibre functor. The main tools we will use are the constant term functors, to reduce to the easier case of tori.

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1. Constant term functors

Let I be a finite index set, and $S \to X^I$ a morphism of schemes. For a regular dominant cocharacter $\lambda \in X_*(T)^+$, for which the associated parabolic is just the Borel, there is a $\mathbb{G}_m$-action on the affine Grassmannian $\mathrm{Gr}_{G,I}$, where $\mathbb{G}_m$ acts through $L^+_I G$, using the cocharacter λ. Since the fixed points of this $\mathbb{G}_m$-action are exactly $\mathrm{Gr}_{T,I}$, and sheaves on $\mathrm{Hk}_{G,I}$ are automatically $L^+_I G$-equivariant, and in particular $\mathbb{G}_m$-equivariant, hyperbolic localization gives an exact functor

$$\mathrm{CT}^I_{B,S} : \mathrm{D}_{\text{ét}}(\mathrm{Hk}_{G,I} \times_{X^I} S, \Lambda)^{\mathrm{bd}} \to \mathrm{D}_{\text{ét}}(\mathrm{Gr}_{T,I} \times_{X^I} S, \Lambda),$$

which we call the *constant term functor*. Aside from preserving both limits and colimits, as after forgetting the equivariance it can be described by either a left or a right adjoint, it has some other very useful properties:

Lemma 1.1. *The constant term functor $\mathrm{CT}^I_{B,S}$ is conservative.*

Proof. As we already know $\mathrm{CT}^I_{B,S}$ is exact, it is enough to show that for $A \in \mathrm{D}_{\text{ét}}(\mathrm{Hk}_{G,I} \times_{X^I} S, \Lambda)^{\mathrm{bd}}$, if $\mathrm{CT}^I_{B,S}(A) = 0$, then already $A = 0$. This can be checked on geometric fibres, so we may assume that S is a geometric point; here we use that hyperbolic localization is compatible with base change. Reducing the index set I if necessary, we may also assume that the points $(x_i)_{i\in I}$ determined by $S \to X$ are distinct. In this case, $\mathrm{Hk}_{G,I} \times_{X^I} S$ admits a stratification, indexed by $(X_*(T)^+)^I$, with the stratum corresponding to $(\mu_i)_i$ being isomorphic to $[S/(\prod_{i\in I}(L^+G)_{\mu_i} \times_X S)]$, where $(L^+G)_{\mu_i}$ is the stabilizer of μ_i inside L^+G. If $A \neq 0$, then boundedness of A implies that there is some maximal stratum on which it is nonzero, say $(\mu_i)_i$. Let $-\overline{\mu}$ be the anti-dominant representative of $(\mu_i)_i$, with corresponding semi-infinite orbit $S_{-\overline{\mu}}$. Then, using that $\mathrm{Gr}_{G,I,(\mu_i)_i} \cap S_{-\overline{\mu}} \cong S$, one sees that the restriction of $\mathrm{CT}^I_{B,S}(A)$ to $[-\overline{\mu}] \in \mathrm{Gr}_T$ is just the pullback of $A_{|[S/(\prod_{i\in I}(L^+G)_{\mu_i} \times_X S)]}$ to S, which is nonzero. □

Another useful property of the constant term functors, is that they preserve and reflect universal local acyclicity and perversity (up to a shift), as the following two propositions show. For a reductive group G, let $\pi_G : \mathrm{Gr}_{G,I} \to X^I$, or more generally $\pi_{G,S} : \mathrm{Gr}_{G,I} \times_{X^I} S \to S$, be the natural projection.

Proposition 1.2. *Let $A \in \mathrm{D}_{\text{ét}}(\mathrm{Hk}_{G,I} \times_{X^I} S, \Lambda)^{\mathrm{bd}}$. Then the following are equivalent:*

(1) *A is ULA over S,*
(2) *$\mathrm{CT}^I_{B,S}(A) \in \mathrm{D}_{\text{ét}}(\mathrm{Gr}_{G,I} \times_{X^I} S, \Lambda)$ is ULA over S,*
(3) *$R\pi_{T,S,*} \mathrm{CT}^I_{B,S}(A) \in \mathrm{D}_{\text{ét}}(S, \Lambda)$ is locally constant with perfect fibres.*

Proof. As hyperbolic localization preserves universal local acyclicity in general, we immediately get (1) ⇒ (2). For the converse (2) ⇒ (1), it is enough to show that if $\mathrm{CT}^I_{B,S}(A)$ is ULA over S, then the natural map $p^*_{G,1} R\underline{\mathrm{Hom}}(A, R\pi^!_{G,S}\Lambda) \otimes^L p^*_{G,2} A \to R\underline{\mathrm{Hom}}(p^*_{G,1} A, Rp^!_{G,2} A)$ is an isomorphism. By conservativity, it is enough to prove this after applying the constant term functor associated to the Borel $B^- \times B \subseteq G \times G$. But then we can use a similar isomorphism for $\mathrm{CT}^I_{B,S}(A)$, the fact that hyperbolic localization preserves exterior tensor products and inner Hom's, and some properties of the six-functor formalism to get a sequence of isomorphisms

$$\begin{aligned}
\mathrm{CT}^I_{B^- \times B,S}(p^*_{G,1} R\underline{\mathrm{Hom}}(A, R\pi^!_{G,S}\Lambda) \otimes^L p^*_{G,2} A) &\cong p^*_{T,1} \mathrm{CT}^I_{B^-,S}(R\underline{\mathrm{Hom}}(A, R\pi^!_{G,S}\Lambda)) \otimes^L p^*_{T,2} \mathrm{CT}^I_{B,S}(A) \\
&\cong p^*_{T,1} q^+_* R(p^+)^!(R\underline{\mathrm{Hom}}(A, R\pi^!_{G,S}\Lambda)) \otimes^L p^*_{T,2} \mathrm{CT}^I_{B,S}(A) \\
&\cong p^*_{T,1} q^+_* R\underline{\mathrm{Hom}}((p^+)^*(A), (p^+)^! R\pi^!_{G,S}\Lambda) \otimes^L p^*_{T,2} \mathrm{CT}^I_{B,S}(A) \\
&\cong p^*_{T,1} q^+_* R\underline{\mathrm{Hom}}((p^+)^*(A), (q^+)^! R\pi^!_{T,S}\Lambda) \otimes^L p^*_{T,2} \mathrm{CT}^I_{B,S}(A) \\
&\cong p^*_{T,1} R\underline{\mathrm{Hom}}(q^+_!(p^+)^*(A), R\pi^!_{T,S}\Lambda) \otimes^L p^*_{T,2} \mathrm{CT}^I_{B,S}(A) \\
&\cong p^*_{T,1} R\underline{\mathrm{Hom}}(\mathrm{CT}^I_{B,S}(A), R\pi^!_{T,S}\Lambda) \otimes^L p^*_{T,2} \mathrm{CT}^I_{B,S}(A) \\
&\cong R\underline{\mathrm{Hom}}(p^*_{T,1} \mathrm{CT}^I_{B,S}(A), Rp^!_{T,2} \mathrm{CT}^I_{B,S}(A)) \\
&\cong \mathrm{CT}^I_{B^- \times B,S}(R\underline{\mathrm{Hom}}(p^*_{G,1} A, Rp^!_{G,2} A),
\end{aligned}$$

where $p^+ : \mathrm{Gr}_B \to \mathrm{Gr}_G$ and $q^+ : \mathrm{Gr}_B \to \mathrm{Gr}_T$ are the maps coming from hyperbolic localization, $p_{G,1}, p_{G,2}$ are the two projections $G \times G \to G$, and similarly for $p_{T,1}$ and $p_{T,2}$.

Finally, the equivalence between (2) and (3) follows from the fact that, up to reductions, which does not change the category of étale sheaves, $\mathrm{Gr}_{T,I} \times_{X^I} S$ is the disjoint union of copies of S. □

Consider the locally constant map

$$\deg : \mathrm{Gr}_{T,I} \to X_*(T) \xrightarrow{\langle 2\rho, - \rangle} \mathbb{Z},$$

where the first map is given by summing the relative positions.

Proposition 1.3. *The constant term functor $\mathrm{CT}^I_{B,S}$ is t-exact for the perverse t-structure on the source, and the standard t-structure on the target (which is the same as the perverse t-structure). In particular, for some $A \in \mathrm{D}_{\text{ét}}(\mathrm{Hk}_{G,I} \times_{X^I} S, \Lambda)^{\mathrm{bd}}$, we have*

$$A \in {}^{p/S}\mathrm{D}^{\leqslant 0}(\mathrm{Hk}_{G,I} \times_{X^I} S, \Lambda)^{\mathrm{bd}} \iff \mathrm{CT}^I_{B,S}(A) \in \mathrm{D}^{\leqslant 0}(\mathrm{Gr}_{T,I} \times_{X^I} S, \Lambda).$$

Remark 1.4. Before we give the proof, let us note that this proposition is the analogue of [FS21, Proposition VI.7.4]. However, at that point in their manuscript, Fargues and Scholze have not yet proved that relative perversity is preserved by base change, or that the ${}^{p/S}\mathrm{D}^{\geqslant 0}$ parts of the t-structure can be determined on geometric fibres; in fact they use this proposition to show these properties. On the other hand, since we have access to the machinery of [HS21], we do already know these properties hold in this situation, by Theorem 6.1 of loc. cit.

Proof. By conservativity, the first statement implies the second. By Remark 1.4 and the fact that hyperbolic localization commutes with base change, we may assume that S is a geometric point. We will also only show that $\mathrm{CT}^I_{B,S}(A)[\deg]$ preserves the $\leqslant 0$ part, the similar assertion for the $\geqslant 0$ part can be proven in the same way, by replacing the $*$'s by !'s and vice versa, and using the semi-infinite orbits for the opposite Borel instead of the usual semi-infinite orbits.

Reducing I, we may assume $S \to X^I$ maps into the open locus where $x_i \neq x_{i'}$ for $i \neq i'$. In this case, there is again a stratification $j_{(\mu_i)_i} : \mathrm{Hk}_{G,I,(\mu_i)_i} \times_{X^I} S \hookrightarrow \mathrm{Hk}_{G,I} \times_{X^I} S$, indexed by $(X_*(T)^+)^I$.

Note that we have $A \in {}^{p/S}\mathrm{D}^{\leqslant 0}$ if and only if $j^*_{(\mu_i)_i} A \in D^{\leqslant -d_{(\mu_i)_i}}$ for all $(\mu_i)_i \in (X_*(T)_+)^I$, where $d_{(\mu_i)_i} = \sum_{i \in I} \langle 2\rho, \mu_i \rangle$. In particular, using excision triangles, we may assume that $A = j_{(\mu_i)_i,!} A_{(\mu_i)_i}$ for some $A_{(\mu_i)_i} \in \mathrm{D}^{\leqslant -d_{(\mu_i)_i}}_{\text{ét}}(\mathrm{Hk}_{G,I,(\mu_i)_i} \times_{X^I} S, \Lambda)$. Moreover, using the truncation functors we get a filtration of $A_{(\mu_i)_i}$ into complexes concentrated in a single degree; in particular, we may assume $A_{(\mu_i)_i}$ is concentrated in degree $-d_{(\mu_i)_i}$.

Now, recall that $\mathrm{Hk}_{G,I,(\mu_i)_i} \times_{X^I} S = [S/(\prod_{i \in I} (L^+G)_{\mu_i} \times_X S)]$, and that pullback along the corresponding quotient map is fully faithful for complexes concentrated in a single degree. And since this quotient map is a morphism over S, we may assume $A_{(\mu_i)_i}$ is the pullback of some $C \in \mathrm{D}_{\text{ét}}(S, \Lambda)$, also concentrated in degree $-d_{(\mu_i)_i}$. By a dévissage argument, we may assume that $\Lambda = \mathbb{F}_\ell$ and $C = \mathbb{F}_\ell[d_{(\mu_i)_i}]$, and by the Künneth formula that $I = \{*\}$ contains only a single element. In particular, we are reduced to showing that if $A = j_{\mu,!}\mathbb{F}_\ell[\langle 2\rho, \mu \rangle]$, then $\mathrm{CT}_{B,S}(A)[\deg]$ lies in degrees $\leqslant 0$.

To show this last part, let $\nu \in X_*(T)$ be a cocharacter, with corresponding semi-infinite orbit $S_\nu \times_X S$. Recall that $\dim((S_\nu \times_X S) \cap (\mathrm{Gr}_{G,\{*\},\mu} \times_X S)) \leqslant \langle \rho, \mu + \nu \rangle$. In particular, the restriction of $\mathrm{CT}_{B,S}(j_{\mu,!}\mathbb{F}_\ell[\langle 2\rho, \mu \rangle])$ to $[\nu] \in \mathrm{Gr}_{T,\{*\}} \times_X S$, which is just

$$R\Gamma_c((S_\nu \times_{X^I} S) \cap (\mathrm{Gr}_{G,\{*\},\mu} \times_{X^I} S), \mathbb{F}_\ell)[\langle 2\rho, \mu \rangle],$$

sits in degrees $\leqslant 2\langle \rho, \mu + \nu \rangle - \langle 2\rho, \mu \rangle = \langle 2\rho, \nu \rangle$. Shifting by the degree map, we see that the result indeed lies in degrees $\leqslant 0$, and this is what we wanted. □

From this point on, we will only use the case where $S = X^I$. So even though most results will hold in more generality, we restrict ourselves to this case for simplicity.

The previous proposition gives the following corollary, which we will use quite a few times later on.

Corollary 1.5. *A complex $A \in \mathrm{D}^{\mathrm{ULA}}_{\text{ét}}(\mathrm{Hk}_{G,I}, \Lambda)^{\mathrm{bd}}$ is flat perverse if and only if $R\pi_{T,*}\,\mathrm{CT}^I_B(A)[\deg] \in \mathrm{D}_{\text{ét}}(X^I, \Lambda)$ is isomorphic to a finite projective Λ-module, concentrated in degree 0.*

Proof. By Proposition 1.2, $R\pi_{T,*}\,\mathrm{CT}^I_B(A)[\deg]$ is locally constant with perfect fibres. By Proposition 1.3, we have that A is relatively perverse if and only if $R\pi_{T,*}\,\mathrm{CT}^I_B(A)[\deg]$ is concentrated in degree 0. Finally, A being flat perverse is equivalent to $R\pi_{T,*}\,\mathrm{CT}^I_B(A)[\deg]$ having Tor-amplitude in $[0,0]$. And since we are working over the simply connected base X^I, this is in turn equivalent to $R\pi_{T,*}\,\mathrm{CT}^I_B(A)[\deg]$ being isomorphic (not just locally) to a finite projective Λ-module in degree 0. □

We mentioned earlier that we want to equip the Satake category with the fibre functor. This is the following result:

Corollary 1.6. *Taking direct sums of cohomology induces a functor*

$$F^I = \bigoplus_{m \in \mathbb{Z}} R^m \pi_{G,*} : \mathrm{Sat}^I_{G,\Lambda} \to \mathrm{LocSys}(X^I, \Lambda).$$

This functor is exact, conservative and faithful, and has the property that if $f : A \to B$ is a morphism in $\mathrm{Sat}^I_{G,\Lambda}$ such that $\ker(F^I(f))$ is a direct summand of $F^I(A)$, then f has a kernel in $\mathrm{Sat}^I_{G,\Lambda}$. A similar assertion also holds for cokernels.

Proof. We will show in Lemma 1.7 below that $F^I \cong \mathcal{H}^0(R\pi_{T,*}\,\mathrm{CT}^I_B[\deg])$, so that F^I takes values in local systems by Corollary 1.5. Moreover, F^I is exact by Proposition 1.3, using the fact that, up to reductions, π_T is the disjoint union of isomorphisms. Similarly, we see that F^I is conservative, using Lemma 1.1. Now, let $f : A \to B$ be a morphism in $\mathrm{Sat}^I_{G,\Lambda}$ such that $\ker(F^I(f))$ is a direct summand of $F^I(A)$, and consider its kernel in $\mathrm{Perv}(\mathrm{Hk}_{G,I}, \Lambda)$; we have to show this kernel is ULA and flat perverse. But this follows by Proposition 1.2 and Corollary 1.5, as $\ker(F^I(f))$ is a direct summand of $F^I(A)$. The similar assertion for cokernels can be proven in a similar way. Finally, to prove faithfulness, let $f, g : A \to B$ map to the same morphism in $\mathrm{LocSys}(X^I, \Lambda)$, and consider the morphism $f - g$ in $\mathrm{Sat}^I_{G,\Lambda}$. As this maps to the zero morphism, the kernel of $F^I(f - g)$ is a direct summand, so that $f-g$ has a kernel in $\mathrm{Sat}^I_{G,\Lambda}$. But the natural inclusion of this kernel into A is an isomorphism after F^I by exactness, so that faithfulness follows from conservativity. □

Lemma 1.7. *For $A \in \mathrm{Sat}^I_{G,\Lambda}$, we have*

$$\bigoplus_{i\in\mathbb{Z}} \mathcal{H}^i(R\pi_{G,*}(A)) \cong \mathcal{H}^0(R\pi_{T,*}\,\mathrm{CT}^I_B(A)[\deg]).$$

Proof. Using the stratification into semi-infinite orbits $(\mathrm{Gr}_{B,I})_{\mathrm{red}} = \coprod_{\nu\in X_*(T)} S_\nu$, we see that

$$\mathrm{CT}_B(A) = \bigoplus_{\nu\in X_*(T)} R(q_\nu)_!(A_{|S_\nu}), \tag{1.1}$$

where $q_\nu : S_\nu \to X \to \mathrm{Gr}_{T,I}$ is the restriction of q^+. On the other hand, the same decomposition into semi-infinite orbits, gives a filtration on the complex $R\pi_{G,*}(A)$, with associated graded $\bigoplus_\nu R(q_\nu)_!(A_{|S_\nu})$. Now, we can decompose $\mathrm{Gr}_{G,I}$ into the unions of Schubert cells $\mathrm{Gr}_{G,I,(\mu_i)_{i\in I}}$, according to the parity of $\sum_{i\in I}\langle 2\rho, \mu_i\rangle$; as the Bruhat ordering can only compare cocharacters with the same such parity, this is a decomposition into clopen subsets. Finally, we note that when restricted to these clopen subsets, $\bigoplus_\nu R(q_\nu)_!(A_{|S_\nu})$ is concentrated in either even or odd degrees, so that the spectral sequence associated to our filtered complex degenerates. After shifting by the degree map, this, along with (1.1), gives the desired isomorphism. □

Remark 1.8. Although the (shorter) spectral sequence argument we have given only gives a non-canonical isomorphism, it is possible to construct a canonical isomorphism, by adapting [BR18, Theorem 5.9] to the setting of Beilinson-Drinfeld affine Grassmannians. We omit details.

As a final result in this section, we want to show that constant term functors induce functors on the Satake categories. While for CT^I_B this follows from previous results, the assertion actually holds more generally, replacing the Borel B by certain parabolics $P \subseteq G$.

Fix a cocharacter $\lambda \in X_*(T)$, inducing a parabolic $P \subseteq G$, the opposite parabolic P^-, and the Levi subgroup $M = P \cap P^- \subseteq G$. Moreover, let $\overline{M}$ be the maximal torus quotient of M, i.e., the cocenter. As this is a torus, there is a natural locally constant function $\mathrm{Gr}_{\overline{M},I} \to X_*(\overline{M})$, obtained by summing the relative positions. In particular, composing this with the map $\mathrm{Gr}_{M,I} \to \mathrm{Gr}_{\overline{M},I}$ induced by the projection, and pairing with $2\rho_G - 2\rho_M$, we get a degree map

$$\deg_P : \mathrm{Gr}_{M,I} \to \mathrm{Gr}_{\overline{M},I} \to X_*(\overline{M}) \to \mathbb{Z}.$$

On the other hand, λ induces a $\mathbb{G}_m$-action on $\mathrm{Gr}_{G,I}$, where $\mathbb{G}_m$ acts through $L^+_I\lambda$. So similarly to CT^I_B, hyperbolic localization gives a functor

$$\mathrm{CT}^I_P : \mathrm{D}_{\text{ét}}(\mathrm{Hk}_{G,I},\Lambda)^{\mathrm{bd}} \to \mathrm{D}_{\text{ét}}(\mathrm{Hk}_{M,I},\Lambda)^{\mathrm{bd}},$$

by dividing out $L^+_I M$ from the diagram

$$\mathrm{Gr}_{G,I} \leftarrow \mathrm{Gr}_{P,I} \to \mathrm{Gr}_{M,I},$$

and first pulling back along $\mathrm{Gr}_{G,I} \to L^+_I M\backslash \mathrm{Gr}_{G,I}$.

Lemma 1.9. *The functor CT^I_P induces a functor*

$$\mathrm{CT}^I_P[\deg_P] : \mathrm{Sat}^I_{G,\Lambda} \to \mathrm{Sat}^I_{M,\Lambda}.$$

Proof. As hyperbolic localization preserves ULA sheaves, we have to check that $\mathrm{CT}^I_P[\deg_P]$ preserves flat perverse sheaves. For this, note that if $P' \subseteq P$ is another parabolic of G with image $Q \subseteq M$, then the functors

$$\mathrm{CT}^I_{P'}[\deg_{P'}] \quad \text{and} \quad \mathrm{CT}^I_Q[\deg_Q] \circ \mathrm{CT}^I_P[\deg_P]$$

are naturally isomorphic; this follows from $(-)_!$-$(-)^*$- base change for the fibre product $\mathrm{Gr}_{P',I} \cong \mathrm{Gr}_{P,I} \times_{\mathrm{Gr}_{M,I}} \mathrm{Gr}_{Q,I}$, cf. Lemma 1.10. Now Corollary 1.5 tells us that $\mathrm{CT}^I_P[\deg_P]$ preserves flat perverse sheaves. □

The following lemma was used in the proof:

Lemma 1.10. *Let K, L, M be smooth affine group schemes, and $K \to M$ and $L \to M$ group homomorphisms, with $L \to M$ surjective. Assume that $K \times_M L$ is also smooth. Then there is a natural isomorphism $\mathrm{Gr}_{K\times_M L,I} \cong \mathrm{Gr}_{K,I} \times_{\mathrm{Gr}_{M,I}} \mathrm{Gr}_{L,I}$ for any finite set I.*

Proof. First, there is a morphism $\psi\colon \mathrm{Gr}_{K\times_M L,I} \to \mathrm{Gr}_{K,I} \times_{\mathrm{Gr}_{M,I}} \mathrm{Gr}_{L,I}$ by the universal property of the fibre product. Conversely, let $x\colon R \to X^I$ be any scheme. Let us denote by $\mathcal{E}_{?,0}$ the trivial torsor. An element of $(\mathrm{Gr}_{K,I} \times_{\mathrm{Gr}_{M,I}} \mathrm{Gr}_{L,I})(R)$ can be represented by a pair $(\mathcal{E}_K, \beta_K)$, with $\mathcal{E}_K$ an K_R-torsor on X_R and $\beta_K : \mathcal{E}_{K|X_R-\Gamma_x} \cong \mathcal{E}_{K,0|X_R-\Gamma_x}$, a similar pair $(\mathcal{E}_L, \beta_L)$ for L, and an isomorphism $\alpha\colon \mathcal{E}_K \times^K M \cong \mathcal{E}_L \times^L M$, commuting with β_K and β_L under the natural identifications $\mathcal{E}_{0,K} \times^K M \cong \mathcal{E}_{0,M} \cong \mathcal{E}_{0,L} \times^L M$. Let us denote $\mathcal{E}_K \times^K M \cong \mathcal{E}_L \times^L M$ by $\mathcal{E}_M$.

Using the natural morphisms $\mathcal{E}_K \cong \mathcal{E}_K \times^K K \to \mathcal{E}_K \times^K M \cong \mathcal{E}_M$ and $\mathcal{E}_L \to \mathcal{E}_M$, we can consider the fibre product $\mathcal{E}_K \times_{\mathcal{E}_M} \mathcal{E}_L$. We claim this is an $K \times_M L$-torsor. Indeed, we can choose a cover U of X_R trivializing both $\mathcal{E}_K$ and $\mathcal{E}_L$, and hence also $\mathcal{E}_M$. More specifically, we want to choose equivariant isomorphisms $\mathcal{E}_K \times_{X_R} U \cong K \times U$ and $\mathcal{E}_L \times_{X_R} U \cong L \times U$, which induce the same trivialization of $\mathcal{E}_M$; this is possible by surjectivity of $L \to M$. It is then clear that $\mathcal{E}_K \times_{\mathcal{E}_M} \mathcal{E}_L$ is a $K \times_M L$-torsor. Moreover, the isomorphisms β_K and β_L induce an isomorphism $\beta_{K\times_M L}\colon (\mathcal{E}_K \times_{\mathcal{E}_M} \mathcal{E}_L)_{|X_R-\Gamma_x} \cong (\mathcal{E}_{0,K\times_M L})_{|X_R-\Gamma_x}$.

This gives a map $\mathrm{Gr}_K \times_{\mathrm{Gr}_M} \mathrm{Gr}_L \to \mathrm{Gr}_{K\times_M L}$, which is readily seen to be inverse to ψ. □

2. Convolution

In this section, we show that the convolution product preserves the Satake category. Recall that it was given as

$$\star_I = Rm_* q^*(- \boxtimes -) : \mathrm{D}_{\text{ét}}(\mathrm{Hk}_{G,I}, \Lambda)^{\mathrm{bd}} \times \mathrm{D}_{\text{ét}}(\mathrm{Hk}_{G,I}, \Lambda)^{\mathrm{bd}} \to \mathrm{D}_{\text{ét}}(\mathrm{Hk}_{G,I}, \Lambda)^{\mathrm{bd}},$$

for the maps

$$\mathrm{Hk}_{G,I} \times_{X^I} \mathrm{Hk}_{G,I} \xleftarrow{q} \mathrm{Hk}_{G,I} \tilde{\times} \mathrm{Hk}_{G,I} \xrightarrow{m} \mathrm{Hk}_{G,I}.$$

Consider a k-algebra R and points $(x_i)_i \in X^I(R)$. Then, as usual for the definition of Beilinson-Drinfeld Grassmannians and Hecke stacks, we denote by $\mathbb{D}_{x_I,R}$ the ring of regular functions of the formal affine scheme obtained by completing X^I along Γ_{x_I}, where $\Gamma_{x_I} = \cup_{i\in I} \Gamma_{x_i}$,

Proposition 2.1. *The convolution product induces a functor*

$$\star_I : \mathrm{Sat}^I_{G,\Lambda} \times \mathrm{Sat}^I_{G,\Lambda} \to \mathrm{Sat}^I_{G,\Lambda}.$$

Proof. First, we note that m is proper and that the exterior product of ULA sheaves remains ULA, as follows by the definition of ULA sheaves as dualizable objects in a certain category. In particular, the convolution product preserves ULA sheaves. Let $A_1, A_2 \in \mathrm{Sat}^I_{G,\Lambda}$; we have to show $A_1 \star_I A_2$ is flat perverse. Since A_1 and A_2 are already ULA, we can reduce to the case $I = \{*\}$ by using the Künneth formula.

Over X^2, consider the stack $\tilde{\mathrm{Hk}}$, parametrizing the following data, for a k-algebra R:

- Two points $(x_1, x_2) \in X^2(R)$,
- Three G-torsors $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2$ on $\mathbb{D}_{x_{\{1,2\}},R}$,
- An isomorphism $\mathcal{E}_0 \cong \mathcal{E}_1$ away from Γ_{x_1}, and
- An isomorphism $\mathcal{E}_1 \cong \mathcal{E}_2$ away from Γ_{x_2}.

Note that away from the diagonal $\Delta \subseteq X^2$, this torsor corresponds to the usual hecke stack:

$$\tilde{\mathrm{Hk}} \times_{X^2} (X^2 \backslash \Delta) \cong \mathrm{Hk}_{G,2} \times_{X^2} (X^2 \backslash \Delta),$$

where for some integer $m \in \mathbb{Z}_{\geqslant 0}$, we denote by $\mathrm{Hk}_{G,m}$ the Hecke stack associated to an unnamed index set of cardinality m. On the other hand, over the diagonal $\Delta \cong X$, the stack $\tilde{\mathrm{Hk}}$ agrees with the convolution Hecke stack $\mathrm{Hk}_{G,1} \tilde{\times} \mathrm{Hk}_{G,1}$ used to define the convolution product. Moreover, there are natural maps

$$p_1, p_2 : \tilde{\mathrm{Hk}} \to \mathrm{Hk}_{G,1}$$

and

$$m : \tilde{\mathrm{Hk}} \to \mathrm{Hk}_{G,2},$$

respectively forgetting the torsors $\mathcal{E}_2$, $\mathcal{E}_0$ and $\mathcal{E}_1$. Note that the torsors classified by $\tilde{\mathrm{Hk}}$ and $\mathrm{Hk}_{G,1}$ do not live on the same disk. However, there are always maps between those disks, so pulling back the torsors gives the desired morphism of stacks.

Now, consider the complex $C := Rm_*(p_1^* A_1 \otimes^L p_2^* A_2)$ on $\mathrm{Hk}_{G,2}$, which is ULA as both A_1 and A_2 are. Moreover, away from the diagonal, m is an isomorphism, so that C is just the exterior product of A_1 and A_2, which is hence concentrated in degree 0. As C is ULA, $R\pi_{T,*}\,\mathrm{CT}^2_B(C)[\deg]$ is locally constant with perfect fibres. But since away from the diagonal, this is concentrated in degree 0, the complement of the diagonal being open dense implies that $R\pi_{T,*}\,\mathrm{CT}^2_B(C)$ is concentrated in degree 0 over the whole X^2. In particular, the restriction of C to the diagonal is also isomorphic to a finite projective Λ-module concentrated in degree 0. But this restriction is exactly $A_1 \star A_2$, so we conclude by Corollary 1.5. □

3. Fusion

We would like to use the convolution product to make $\mathrm{Sat}^I_{G,\Lambda}$ into a symmetric monoidal category. To construct the commutativity constraint, we will define a more general *fusion* product on $\mathrm{Sat}^I_{G,\Lambda}$, which comes with natural commutativity and associativity constraints (although we will need to modify them), and show that this fusion product specializes to the convolution product.

More precisely, for any surjective map $\chi : I \to J$ of finite sets, there are morphisms

$$\mathrm{Hk}_{G,I} \leftarrow \mathrm{Hk}_{G,I} \times_{X^I} X^J \to \mathrm{Hk}_{G,J}.$$

Here, the morphism $X^J \to X^I$ is defined on R-valued points as $(x_j)_j \mapsto (x_{\alpha(i)})_i$ whose base change to $\mathrm{Hk}_{G,I}$ is the left morphism. On the other hand, the right morphism is the closed immersion defined by sending a tuple $((x_i)_{i\in I}, \mathcal{E}_0, \mathcal{E}_1, \alpha)$, with $x_i = x'_i$ if $\chi(i) = \chi(i')$, to the tuple $((x_{\chi^{-1}(j)})_{j\in J}, \mathcal{E}_0, \mathcal{E}_1, \alpha)$ (using a slight abuse of notation, as again the torsors are defined over different disks, this time because the graphs appearing differ). In particular, as base change preserves relative notions such as ULA and relative perverse, and because pushforward along closed immersions also preserves these two properties, pull-push along this diagram defines a functor

$$\Psi_G : \mathrm{Sat}^I_{G,\Lambda} \to \mathrm{Sat}^J_{G,\Lambda}.$$

Lemma 3.1. *The functor Ψ_G is compatible with the fibre functor and constant term functors.*

Proof. For the constant term functors CT^I_P, we want to see that $\mathrm{CT}^J_P \circ \Psi_G$ and $\Psi_M \circ \mathrm{CT}^I_P$ are naturally isomorphic. But this follows from base change, using the $(-)_!(-)^*$-description of constant terms and the fact that $\mathrm{Hk}_{G,I} \times_{X^I} X^J \to \mathrm{Hk}_{G,J}$ is a closed immersion, so that *- and !-pushforward agree.

In particular, for the compatibility with the fibre functor, we are reduced to the case $G = T$ by Lemma 1.7. In that case, we are again done by base change, using that π_T is ind-proper. □

Now, for any decomposition $I = I_1 \sqcup \ldots \sqcup I_k$ of an index set I, let $j : X^{I;I_1,\ldots,I_k} \subseteq X^I$ be the open dense subset consisting of those points $(x_i)_i$ such that $x_i \neq x_{i'}$ whenever i and i' do not lie in the same I_j. Over this open dense subset, we have a variant of the Satake category:

Definition 3.2. The Satake category $\mathrm{Sat}^{I;I_1,\ldots,I_k}_{G,\Lambda}$ is the full subcategory of $\mathrm{D}_{\text{ét}}(\mathrm{Hk}_{G,I} \times_{X^I} X^{I;I_1,\ldots,I_k}, \Lambda)^{\mathrm{bd}}$ spanned by the flat relatively perverse and ULA objects.

In particular, the restriction j^* preserves Satake categories. It turns out this restriction is very well behaved:

Lemma 3.3. *The two restriction functors $j^* : \mathrm{Sat}^I_{G,\Lambda} \to \mathrm{Sat}^{I;I_1,\ldots,I_k}_{G,\Lambda}$ and $j^* : \mathrm{LocSys}(X^I, \Lambda) \to \mathrm{LocSys}(X^{I;I_1,\ldots,I_k}, \Lambda)$ are fully faithful.*

Proof. Note that the restriction $j^* : \mathrm{Perv}(\mathrm{Hk}_{G,I}) \to \mathrm{D}_{\text{ét}}(\mathrm{Hk}_{G,I} \times_{X^I} X^{I;I_1,\dots I_k}, \Lambda)^{\mathrm{bd}}$ admits a right adjoint ${}^p\mathcal{H}^0 Rj_*$. To show the restriction on Satake categories is fully faithful, it is enough to show that $A \to {}^p\mathcal{H}^0 Rj_* j^* A$ is an isomorphism for all $A \in \mathrm{Sat}^I_{G,\Lambda}$ (even though its right adjoint might not preserve ULA objects). For this, let $i : Z \subset X^I$ be the reduced closed complement of $X^{I;I_1,\dots,I_k}$, and choose some $A \in \mathrm{Sat}^I_{G,\Lambda}$. It suffices to show that $i_* i^! A \in {}^p\mathrm{D}^{\geqslant 2}_{\text{ét}}(\mathrm{Hk}_{G,I}, \Lambda)^{\mathrm{bd}}$. By (a suitable variant of) Corollary 1.5, we can check this after $R\pi_{T,*}\,\mathrm{CT}^I_B[\deg]$, where we get a local system of finite projective Λ-modules. We conclude by noting that $i_* i^! \Lambda \in \mathrm{D}^{\geqslant 2}_{\text{ét}}(X^I, \Lambda)$, as Z admits a stratification by smooth codimension $\geqslant 1$ strata, by pulling back the partial diagonals of X^I.

This last argument also shows that the restriction on local systems is fully faithful. □

Now, over $X^{I;I_1,\dots,I_k}$ we have an isomorphism

$$\mathrm{Hk}_{G,I} \times_{X^I} X^{I;I_1,\dots,I_k} \cong \Big(\prod_j \mathrm{Hk}_{G,I_j}\Big) \times_{X^I} X^{I;I_1,\dots,I_k},$$

so that exterior products gives a functor

$$\Phi : \mathrm{Sat}^{I_1}_{G,\Lambda} \times \ldots \times \mathrm{Sat}^{I_k}_{G,\Lambda} \to \mathrm{Sat}^{I;I_1,\dots,I_k}_{G,\Lambda}.$$

Proposition 3.4. *The functor Φ takes values in the full subcategory $\mathrm{Sat}^I_{G,\Lambda} \subseteq \mathrm{Sat}^{I;I_1,\dots,I_k}_{G,\Lambda}$.*

Proof. Define the stack $\mathrm{Hk}_{G,I;I_1,\dots,I_k}$ over X^I, parametrizing the following data, for a k-algebra R:

- $(x_i)_{i\in I} \in X^I(R)$,
- G-torsors $\mathcal{E}_0, \dots, \mathcal{E}_k$ on $\mathbb{D}_{x_I,R}$,
- Isomorphisms $\mathcal{E}_{j-1} \cong \mathcal{E}_j$ away from $\bigcup_{h\in I_j} \Gamma_{x_h}$.

In particular, there are natural maps

$$p_j : \mathrm{Hk}_{G,I;I_1,\dots,I_k} \to \mathrm{Hk}_{G,I_j}$$

remembering only one isomorphism, and

$$m : \mathrm{Hk}_{G,I;I_1,\dots,I_k} \to \mathrm{Hk}_{G,I}$$

obtained by composing all isomorphisms. Similarly as in the proof of Proposition 2.1, note that the torsors classified by the different Hecke stacks don't lie on the same disks, but again there is always a map between those disks along which we can pull back the torsors.

Now, given objects $A_j \in \mathrm{Sat}^{I_j}_{G,\Lambda}$ for each j, consider the object $C = Rm_*(p_1^* A_1 \otimes^L \ldots \otimes^L p_k^* A_k)$ on $\mathrm{Hk}_{G,I}$, which is ULA, so that $R\pi_{T,*}\,\mathrm{CT}^I_B(C)[\deg]$ is locally constant with perfect fibres. Moreover, one readily checks that m is an isomorphism over $X^{I;I_1,\dots,I_k}$, so that over this open subset, C is just the exterior product. In particular, over $X^{I;I_1,\dots,I_k}$, $R\pi_{T,*}\,\mathrm{CT}^I_B(C)[\deg]$ is concentrated in degree 0. As $X^{I;I_1,\dots,I_k}$ is open dense in X^I, this implies that $R\pi_{T,*}\,\mathrm{CT}^I_B(C)[\deg]$ is concentrated in degree 0 everywhere, so that $C \in \mathrm{Sat}^I_{G,\Lambda}$ by Corollary 1.5. Finally, since C is exactly $\Phi(A_1, \dots, A_k)$, we see that $\Phi(A_1, \dots, A_k) \in \mathrm{Sat}^I_{G,\Lambda}$. □

Remark 3.5. The functor Φ above will be used to get a monoidal structure on $\mathrm{Sat}^I_{G,\Lambda}$. As its definition involves an exterior product, it is equipped with natural commutativity and associativity constraints. However, this naive commutativity constraints will lead to certain sign issues, so we modify this constraint by hand. Decompose the Hecke stack as $\mathrm{Hk}_{G,I} = \mathrm{Hk}^{\mathrm{even}}_{G,I} \coprod \mathrm{Hk}^{\mathrm{odd}}_{G,I}$, where a Schubert cell $\mathrm{Hk}_{G,I,(\mu_i)_i}$ is contained in the even or odd part according to whether $\sum_{i\in I}\langle 2\rho, \mu_i\rangle$ is even or odd; note that the dominance order can only compare elements with the same such parity, so that we actually get a decomposition into clopen substacks. Then, when commuting two complexes concentrated on $\mathrm{Hk}^{\mathrm{odd}}_{G,I}$, we change the naive commutativity constraint by adding a minus sign.

Another way to phrase this is that the diagram

$$\begin{array}{ccc} \mathrm{Sat}^{I_1}_{G,\Lambda} \times \ldots \times \mathrm{Sat}^{I_k}_{G,\Lambda} & \xrightarrow{\Phi} & \mathrm{Sat}^{I;I_1,\ldots,I_k}_{G,\Lambda} \\ \downarrow{\scriptstyle (F^{I_j})_j} & & \downarrow{\scriptstyle F^{I;I_1,\ldots,I_k}} \\ \mathrm{LocSys}(X^{I_1},\Lambda) \times \ldots \times \mathrm{LocSys}(X^{I_k},\Lambda) & \xrightarrow{\boxtimes} & \mathrm{LocSys}(X^{I;I_1,\ldots,I_k},\Lambda) \end{array} \tag{3.1}$$

naturally commutes, and functorially in $I_1, \ldots, I_k$. This follows from the implicit shifts in the fibre functors. Moreover, faithfulness of these fibre functors already determines the signs.

We conclude that, for any decomposition $I = I_1 \sqcup \ldots \sqcup I_k$, we get the *fusion* product

$$\ast : \mathrm{Sat}^{I_1}_{G,\Lambda} \times \ldots \times \mathrm{Sat}^{I_k}_{G,\Lambda} \to \mathrm{Sat}^{I}_{G,\Lambda}.$$

In particular, for any index set I, the composite

$$\mathrm{Sat}^{I}_{G,\Lambda} \times \ldots \times \mathrm{Sat}^{I}_{G,\Lambda} \xrightarrow{\ast} \mathrm{Sat}^{I\sqcup\ldots\sqcup I}_{G,\Lambda} \xrightarrow{\Psi_G} \mathrm{Sat}^{I}_{G,\Lambda}$$

makes $\mathrm{Sat}_{G,\Lambda}$ into a symmetric monoidal category, by the previous remark. Moreover, one readily checks that this agrees with the convolution product.

Now, by commutativity of (3.1), we see that the induced diagram

$$\begin{array}{ccc} \mathrm{Sat}^{I_1}_{G,\Lambda} \times \ldots \times \mathrm{Sat}^{I_k}_{G,\Lambda} & \xrightarrow{\ast} & \mathrm{Sat}^{I}_{G,\Lambda} \\ \downarrow{\scriptstyle (F^{I_j})_j} & & \downarrow{\scriptstyle F^{I}} \\ \mathrm{LocSys}(X^{I_1},\Lambda) \times \ldots \times \mathrm{LocSys}(X^{I_k},\Lambda) & \xrightarrow{\boxtimes} & \mathrm{LocSys}(X^{I},\Lambda) \end{array}$$

also naturally commutes. In particular, the fibre functor F^I is naturally symmetric monoidal for the fusion product, and hence also for the convolution product by Lemma 3.1.

Finally, let us go back to the setting of Lemma 1.9 and before, which attached to certain parabolics $P \subseteq G$ with Levi M the constant term functor

$$\mathrm{CT}_P[\deg_P] : \mathrm{Sat}^{I}_{G,\Lambda} \to \mathrm{Sat}^{I}_{M,\Lambda}.$$

It turns out these also have a symmetric monoidal structure.

Proposition 3.6. *For any decomposition $I = I_1 \sqcup \ldots \sqcup I_k$, the diagram*

$$\begin{array}{ccc} \mathrm{Sat}^{I_1}_{G,\Lambda} \times \ldots \times \mathrm{Sat}^{I_k}_{G,\Lambda} & \xrightarrow{\ast} & \mathrm{Sat}^{I}_{G,\Lambda} \\ \downarrow{\scriptstyle \left(\mathrm{CT}^{I_j}_P[\deg_P]\right)_j} & & \downarrow{\scriptstyle \mathrm{CT}^{I}_P[\deg_P]} \\ \mathrm{Sat}^{I_1}_{M,\Lambda} \times \ldots \times \mathrm{Sat}^{I_k}_{M,\Lambda} & \xrightarrow{\ast} & \mathrm{Sat}^{I}_{M,\Lambda} \end{array}$$

naturally commutes.

Proof. Over the open subset $X^{I;I_1\ldots,I_k}$, this follows from the Künneth formula. We conclude by Lemma 3.3 □

Again using Lemma 3.1, we conclude that the constant term functors are also symmetric monoidal for the convolution product.

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