TALK 1: DEFINITION OF THE DUAL GROUP AND THE SATAKE ISOMORPHISM

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1. Introduction and Motivation

Let G be a split reductive group over O, the ring of integers of a non-archimedean local field F, and let $T \subset G$ be a split maximal torus. The spherical Hecke algebra $\mathcal{H}_G$ associated with G is by definition the set of compactly supported $\mathbb{Z}$-valued functions $G(F) \to \mathbb{Z}$, which are invariant under left and right multiplications by elements of $G(O)$. This set is equiped with a natural convolution product, making it a ring (which is actually commutative). Denote by $X_*(T) := \mathrm{Hom}(\mathbb{G}_{\mathrm{m},O}, T)$ the group of cocharacters, and recall that the group ring $\mathbb{Z}[X_*(T)]$ is equiped with a canonical action of the Weyl group W of G. The goal of these notes will be to explain the construction of the following isomorphism, due to Satake ([Sat63, Chapter 2]).

Theorem 1.1. *Let q be the cardinality of the residue field of O. There exists a $\mathbb{Z}[q^{\pm 1/2}]$-algebra isomorphim*

$$\mathcal{H}_G \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}] \xrightarrow{\sim} \mathbb{Z}[X_*(T)]^W \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}].$$

This is the isomorphism that the Geometric Satake equivalence (or a version of it) will categorify. Namely, taking characters induces an isomorphism between the Grothendieck ring $K_0(\mathrm{Rep}(\hat{G}_{\mathbb{C}}))$ and the ring $\mathbb{Z}[X_*(T)]^W$ (where $\mathrm{Rep}(\hat{G}_{\mathbb{C}})$ denotes the monoidal abelian category consisting of finite dimensional algebraic representations of the Langlands dual group $\hat{G}_{\mathbb{C}}$ over $\mathbb{C}$), and one would want to find a monoidal abelian category $\mathcal{A}$ with an isomorphism of rings $K_0(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}] \simeq \mathcal{H}_G \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$ and a monoidal equivalence of categories $\mathcal{A} \xrightarrow{\sim} \mathrm{Rep}(\hat{G}_{\mathbb{C}})$ that recovers the Satake isomorphism when passing to Grothendieck rings and tensoring with $\mathbb{Z}[q^{\pm 1/2}]$ over $\mathbb{Z}$.

The question of finding the right category $\mathcal{A}$ that does this will not be discussed here, but one can find the necessary explanations in [Zhu16, §5.6].

Another motivation for the study of $\mathcal{H}_G$ is its relation with smooth unramified representation of $G(F)$ over $\mathbb{C}$. Recall that a smooth representation V of $G(F)$ over $\mathbb{C}$ is a $\mathbb{C}$-vector space together with a group morphism $\pi : G(F) \to \mathrm{GL}(V)$ such that each vector v has an open stabilizer in $G(F)$; we call V unramified if the subspace $V^{G(O)}$ of vectors fixed by $G(O)$ is non-zero. It is well known that the association $V \mapsto V^{G(O)}$ induces a functor between smooth representation of $G(F)$ and left $\mathcal{H}_G \otimes_{\mathbb{Z}} \mathbb{C}$-modules (for any $v \in V^{G(O)}$, the action of $f \in \mathcal{H}_G \otimes_{\mathbb{Z}} \mathbb{C}$ is given by $f \cdot v = \sum_{g \in G(F)/G(O)} f(g)\pi(g)v$). Moreover, this functor induces a bijection

$$\{\text{Irreducible smooth unramified } G(F)\text{-rep.}\}/\sim \longrightarrow \{\text{Irreducible } \mathcal{H}_G \otimes_{\mathbb{Z}} \mathbb{C}\text{-modules}\}/\sim .$$

The Satake isomorphism allows to give a simple description of the right-hand side above as the set of $\mathbb{C}$-algebra morphisms $\mathbb{C}[X_*(T)]^W \to \mathbb{C}$ (the ring $\mathbb{C}[X_*(T)]^W$ is commutative, so an irreducible representation has dimension one). These considerations are used for instance in [C+79, §4], and we will not come back to these in the sequel.

In the next few pages, we will first focus on the definition of the Langlands dual group, and then we will pass to the construction of the classical Satake isomorphism.

2. Definition of the dual group

The goal of this section is to explain the definition of the Langlands dual group of a split reductive group scheme. Here we follow mainly [Jan03, II.1] and [Gil13]. For the next few general definitions, we work over a unitary commutative ring R. For all $x \in \mathrm{Spec}(R)$, we denote by $\kappa(x)$ the residue field of $\mathrm{Spec}(R)$ at x, and by $\overline{\kappa(x)}$ an algebraic closure. If G is an R-group scheme, the notation G_x (resp. $G_{\overline{x}}$) will denote the group scheme $G \times_R \kappa(x)$ (resp. $G \times_R \overline{\kappa(x)}$), called the fiber of G at x (resp. the geometric fiber of G at x).

Definition 2.1. A group scheme G over R is called reductive if it satisfies the following conditions:

(1) G is affine and smooth over R;
(2) For every $x \in \mathrm{Spec}(R)$, the geometric fiber $G_{\overline{x}}$ is a (connected[1]) reductive $\overline{\kappa(x)}$-algebraic group.

In the sequel, we will denote by $G_{R'}$ the R'-group scheme corresponding to $G \times_R R'$ for any $R \to R'$.

Example 2.2. Let M be an R-module, and define the R-group scheme GL_M via its R-points by the formula $A \mapsto \mathrm{Aut}_A(M \otimes_R A)$, (where $\mathrm{Aut}_A(N)$ denotes the group of A-linear automorphism of an A-module N). When $M \simeq R^n$ for some $n \geqslant 1$, GL_M is represented by the affine scheme $\mathrm{Spec}(R[T_{i,j}]_{1 \leqslant i,j \leqslant n}[\det(T_{i,j})^{-1}])$. We put $\mathrm{GL}_{n,R} := \mathrm{GL}_{R^n}$ and $\mathrm{GL}_n := \mathrm{GL}_{n,\mathbb{Z}}$. Then $\mathrm{GL}_{n,R}$ is reductive, just as the torus $\mathbb{G}_{m,R}^n := \prod_{k=1}^n \mathrm{GL}_{1,R}$.

Definition 2.3. Let T be an R-group scheme. We say that T is a torus if every $x \in \mathrm{Spec}(R)$ has an open neighborhood U such that there exists an fpqc morphism $U' \to U$ for which $T_{U'}$ is isomorphic to $(\mathbb{G}_m^n)_{U'}$ over U' for some n. If T is isomorphic to $(\mathbb{G}_m^n)_R$, then we say that T is split.

Definition 2.4. Let G be an R-group scheme, T a torus over R and $T \to G$ a closed immersion. We say that T is a maximal torus if $T_{\overline{x}}$ is a maximal torus of $G_{\overline{x}}$ (in the usual sens), for every $x \in \mathrm{Spec}(R)$.

In order to simplify the statements, we assume from now on that R is a connected ring. We recall a few results concerning the structure of split reductive groups over R. Let $G_{\mathbb{Z}}$ be a reductive group over $\mathbb{Z}$ equiped with split maximal torus $T_{\mathbb{Z}} \simeq \mathbb{G}_m^n$. We call $G_{\mathbb{Z}}$ a *split reductive group* over $\mathbb{Z}$.

Remark 2.5. One has to be carefull with the definition of a split reductive group over an arbitrary ring (or scheme), cf. [Gil13, Definition 13.3.6]. However by Remark 13.3.7 of *loc.cit.*, a group scheme G over a unique factorization domain R is a split reductive group for the general definition iff it has a split maximal torus (this is because the Picard group of a UFD is trivial).

We consider the R-reductive group $G := G_{\mathbb{Z}} \times \mathrm{Spec}(R)$, which admits $T := T_{\mathbb{Z}} \times \mathrm{Spec}(R)$ as a split maximal torus. We will now explain how to attach a root datum to G. We denote by $\mathrm{Lie}(G)$ the Lie algebra of G, which is a free R-module of finite rank. The action of G on itself by conjugation allows us to define the adjoint representation of G, i.e. a morphism of group schemes

$$\mathrm{Ad} : G \to \mathrm{GL}_{\mathrm{Lie}(G)}.$$

The restriction of this action yields a linear action of T on $\mathrm{Lie}(G)$, and thus a decomposition

$$\mathrm{Lie}(G) = \mathrm{Lie}(T) \oplus \bigoplus_{\alpha \in X^*(T),\ \alpha \neq 0} \mathrm{Lie}(G)_\alpha,$$

where $\mathrm{Lie}(G)_\alpha := \{x \in \mathrm{Lie}(G) \mid \mathrm{Ad}(t)(x \otimes 1) = x \otimes \alpha(t)\ \forall t \in T(A),\ \forall\ R\text{-algebra } A\}$, and $\mathrm{Lie}(G)_0 = \mathrm{Lie}(T)$. We denote by $\Phi \subset X^*(T) := \mathrm{Hom}(T, \mathbb{G}_{\mathrm{m},R})$ the roots relative to T, i.e. the set of characters $\alpha \in X^*(T)$ such that $\mathrm{Lie}(G)_\alpha \neq 0$. For each root $\alpha \in X^*(T)$, there exists a *root morphism* $x_\alpha : \mathbb{G}_{a,R} \to G$ satisfying

$$t x_\alpha(a) t^{-1} = x_\alpha(\alpha(t)a)$$

for every R-algebra A and all $t \in T(A)$, $a \in A$, and such that the tangent map dx_α induces an isomorphism of R-modules $\mathrm{Lie}(\mathbb{G}_{\mathrm{a},R}) \simeq \mathrm{Lie}(G)_\alpha$. Such a root morphism is unique up to multiplication by an invertible element $R^\times$, acting on $\mathbb{G}_{a,R}$.

[1]We follow the convention that a reductive group over an algebraically closed field is connected.

There is also a morphism $\varphi_\alpha : \mathrm{SL}_{2,R} \to G$ sending the maximal diagonal torus of $\mathrm{SL}_{2,R}$ to T and such that, for all $a \in A$,

$$\varphi_\alpha \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = x_\alpha(a) \qquad \text{and} \qquad \varphi_\alpha \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = x_{-\alpha}(a).$$

Thus the formula

$$\alpha^\vee(a) := \varphi_\alpha \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

defines an element $\alpha^\vee \in \mathrm{Hom}(\mathbb{G}_{\mathrm{m},R}, T) =: X_*(T)$, called the coroot corresponding to α. One can easily see that this definition does not depend on the choice of our root morphisms. We write $\Phi^\vee := \{\alpha^\vee,\ \alpha \in \Phi\} \subset X_*(T)$, and the application $\alpha \mapsto \alpha^\vee$ is a bijection from Φ to $\Phi^\vee$.

The sets $X_*(T)$ and $X^*(T)$ carry a structure of abelian group, for which we will denote the group law additively. As $T = \mathbb{G}^n_{\mathrm{m},R}$ and R is connected, it follows that $X_*(T)$ and $X^*(T)$ are isomorphic to $\mathbb{Z}^n$ as $\mathbb{Z}$-modules ([Gil13, Lemma 3.4.3]). For all $(\lambda, \alpha) \in X^*(T) \times X_*(T)$, there exists a unique integer $\langle \lambda, \alpha \rangle$ corresponding to the morphism $\lambda \circ \alpha \in \mathrm{End}(\mathbb{G}_{m,R})$ under the isomorphism $\mathrm{End}(\mathbb{G}_{m,R}) \simeq \mathbb{Z}$. Thus, the application $\langle \cdot, \cdot \rangle$ is bilinear on $X_*(T) \times X^*(T)$ and induces an isomorphism $X_*(T) \simeq \mathrm{Hom}_{\mathbb{Z}}(X^*(T), \mathbb{Z})$.

One can then show that the quadruple $\mathcal{R}(G,T) := (\Phi, \Phi^\vee, X^*(T), X_*(T))$ defines a *reduced root datum* (cf. [SGA, Exposé XXI] for a complete exposition on root data). Moreover, all of the preceding construction applies by extension of scalars: we have $\mathrm{Lie}(G) = \mathrm{Lie}(G_{\mathbb{Z}}) \otimes_{\mathbb{Z}} R$, $X_*(T) \simeq X_*(T_{\mathbb{Z}})$, $X^*(T) \simeq X^*(T_{\mathbb{Z}})$ and $(\mathcal{R}, \mathcal{R}^\vee, X^*(T), X_*(T))$ is the root datum associated with $G_{\mathbb{Z}}$. In particular, the root datum of G does not depend on the connected ring R.

Example 2.6. We now recall what the root datum of $G := \mathrm{GL}_n$ is. We let T be the split maximal torus consisting of diagonal matrices. One can then easily check the following classical results:

$$X^*(T) \simeq \mathbb{Z}^n \qquad X_*(T) \simeq \mathbb{Z}^n$$
$$\Phi \simeq \{e_i - e_j,\ 1 \leqslant i,j \leqslant n\} \qquad \Phi^\vee \simeq \{e_i - e_j,\ 1 \leqslant i,j \leqslant n\},$$

where the isomorphism $X^*(T) \simeq \mathbb{Z}^n$ (resp. $X_*(T) \simeq \mathbb{Z}^n$) sends an element of the canonical basis e_i onto the character

$$\mathrm{diag}(z_1, \cdots, z_i, \cdots, z_n) \mapsto z_i$$

(resp. onto the cocharacter $z \mapsto \mathrm{diag}(1, \cdots, \underbrace{z}_{\text{place } i}, \cdots, 1)$).

We can now state the following unicity theorem, followed by Chevalley's existence theorem (cf. [Gil13, Theorems 19.4.1,19.4.2]).

Theorem 2.7. *Let (G,T) and (G',T') be two split reductive groups over R equipped with split maximal tori. Then (G,T) and (G',T') are isomorphic iff their root data $\mathcal{R}(G,T)$ and $\mathcal{R}(G',T')$ are isomorphic.*

Theorem 2.8. *Let $\mathcal{R} = (\Phi, \Phi^\vee, X^*, X_*)$ be a reduced root datum. Then there exists a split $\mathbb{Z}$-reductive group $G_{\mathbb{Z}}$ equipped with a split maximal torus $T_{\mathbb{Z}}$ such that $\mathcal{R}(G_{\mathbb{Z}}, T_{\mathbb{Z}}) \simeq \mathcal{R}$.*

The split reductive group $G_{\mathbb{Z}}$ is called the *Chevalley's group* associated with $\mathcal{R}$. It is defined only up to isomorphim.

We can finally state the definition of the Langlands dual group. Recall that if $\mathcal{R} = (\Phi, \Phi^\vee, X^*, X_*)$ is a reduced root datum, then so is $\mathcal{R}^\vee = (\Phi^\vee, \Phi, X_*, X^*)$. We call $\mathcal{R}^\vee$ the *dual root datum* of $\mathcal{R}$.

Definition 2.9. Let (G,T) be a split reductive group over R with split maximal torus, and let $\mathcal{R}(G,T)^\vee$ be the dual root datum of $\mathcal{R}(G,T)$. Let $\hat{G}_{\mathbb{Z}}$ be the Chevalley's group associated with $\mathcal{R}(G,T)^\vee$. For any ring A, the split reductive group $\hat{G}_{\mathbb{Z}} \times A$ is called the Langlands dual group of G over A (it is defined up to isomorphism).

Example 2.10. We give a few examples of Langlands dual groups:

G	$\mathrm{GL}_{n,R}$	$\mathrm{SL}_{n,R}$	$\mathrm{SO}_{2n+1,R}$	$\mathrm{SO}_{2n,R}$
$\hat{G}_{\mathbb{Z}}$	GL_n	PGL_n	Sp_{2n}	SO_{2n}

We end this section with a few more definitions that will be useful in the sequel. Let G be a split reductive group over R with split maximal torus T, and pick a Borel subgroup B containing T. We define the positive roots Φ_+ as the set of roots $\alpha \in \Phi$ which appear in the T-weights of $\mathrm{Lie}(B)$, and set $\Phi_+^\vee := \{\alpha^\vee,\ \alpha \in \Phi_+\}$. The set Φ_+ defines a positive root system $\Phi_+ \subset \Phi$. Finally, the set of dominant cocharacters is

$$X_*(T)^+ := \{\lambda \in X_*(T) \mid \forall \alpha \in \Phi_+,\ \langle \lambda, \alpha \rangle \geqslant 0\}$$

and, for any pair of cocharacters $\lambda, \mu \in X_*(T)$, we will write $\mu \leqslant \lambda$ if $\lambda - \mu \in \mathbb{Z}_{\geqslant 0} \cdot \Phi_+^\vee$.

Example 2.11. We take back the example 2.6. Let $B \subset G$ be the Borel subgroup of upper triangular matrices. We have

$$\Phi_+ = \{e_i - e_j,\ 1 \leqslant i < j \leqslant n\} \qquad X_*(T)^+ \simeq \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geqslant \cdots \geqslant \lambda_n\},$$

For two cocharacters $\mu = (\mu_1, \cdots, \mu_n), \lambda = (\lambda_1, \cdots, \lambda_n)$, we have $\mu \leqslant \lambda$ iff

$$\mu_1 + \cdots + \mu_i \leqslant \lambda_1 + \cdots + \lambda_i \ \forall i < n \text{ and } \mu_1 + \cdots + \mu_n = \lambda_1 + \cdots + \lambda_n.$$

3. The spherical Hecke algebra

From now on, F will be a non-archimedean local field with ring of integers O and residue field $\mathbf{k}$ of cardinality q. Recall that this means that F is the completion of a global field for a non-trivial and non-archimedean valuation, endowed with an absolute value $|\cdot|$, and that $O = \{x \in F \mid |x| \leqslant 1\}$ is a DVR, with maximal ideal $\mathfrak{m}$. We pick a generator ϖ of $\mathfrak{m}$ (called a uniformizer of F). Every non-zero element $x \in F$ can be written uniquely as $x = \varpi^n u$ for some $n \in \mathbb{Z}, u \in O^\times = \{x \in F \mid |x| = 1\}$, and this defines a valuation $v : F \to \mathbb{Z} \cup \infty$ with $v(x) = n,\ v(0) = \infty$. We then have $|a| = q^{-v(a)}$ for every $a \in F$. If F is of characteristic zero, then it is a finite extension of $\mathbb{Q}_p$, and O its ring of integers (typically $F = \mathbb{Q}_p$, $O = \mathbb{Z}_p$, so $q = p$). Otherwise, we must have $F = \mathbb{F}_q((\varpi))$ and $O = \mathbb{F}_q[[\varpi]]$.

We let G be a split reductive group over O, with a (split) maximal torus T and Borel subgroup B containing T. Any cocharacter $\lambda \in X_*(T) := \mathrm{Hom}(\mathbb{G}_{\mathrm{m},O}, T)$ induces a morphism $F^\times \to T(F)$, and we will denote by $\lambda(\varpi) \in T(F)$ the image of ϖ under this morphism. When $G = \mathrm{GL}_{n,O}$, we see that $\lambda(\varpi) = \mathrm{diag}(\varpi^{\lambda_1}, \cdots, \varpi^{\lambda_n})$ for an element $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$. We denote by $\hat{G}$ the dual group of G over $\mathbb{C}$, with a dual maximal torus $\hat{T} \subset \hat{G}$.

In these notes, we will mainly be interested in the group $G(F)$, which we will endow with a topology making it a locally compact (even locally profinite) topological group. A basis of compact open neighborhoods of the identity are the subgroups $\{K_n\}_{n \geqslant 0}$ defined by

$$K_n = \ker(G(O) \to G(O/\varpi^n O)),$$

where the considered map is induced by the canonical projection (see [C+79, §1.1]). The locally compact group $G(F)$ is endowed with a unique Haar measure μ giving $G(O)$ volume 1 (since G is reductive, one can also show that $G(F)$ is unimodular, i.e. that μ is both left and right invariant). The spherical Hecke algebra is by definition the ring

$$\mathcal{H}_G := \mathcal{C}_{\mathrm{c}}(G(O) \backslash G(F) / G(O))$$

of locally constant compactly supported functions $f : G(F) \to \mathbb{Z}$ which are left and right invariant under multiplication by elements of $G(O)$. Multiplication is given by convolution:

$$f \star g(z) := \int_{G(F)} f(x) g(x^{-1} z) \mathrm{d}x.$$

Notice that, since f is compactly supported and $G(O)$-invariant, one can write

$$f = \sum_{i=1}^{n} f(x_i) \mathbf{1}_{x_i G(O)}$$

for some elements $x_i \in G(F)$, so the integral of f is just the finite sum $\sum_{i=1}^n f(x_i)$. In particular we have

$$f \star g(z) = \sum_{x \in G(F)/G(O)} f(x)g(x^{-1}z)$$

for all $f, g \in \mathcal{H}_G$. The following result will be crucial.

Proposition 3.1. *We have the Cartan decomposition*

$$G(F) = \bigsqcup_{\lambda \in X_*(T)^+} G(O)\lambda(\varpi)G(O).$$

In the case of GL_n, the Cartan decomposition is a consequence of the Gaussian elimination process, which implies (using the fact that O is a DVR with only prime ideal (ϖ)) that any element $g \in \mathrm{GL}_n(F)$ can be transformed into an element $\lambda(\varpi)$ by using elementary operations encoded by elements of $\mathrm{GL}_n(O)$ (cf. [Ser07, Theorem 6.2.1]). The unicity of such a $\lambda(\varpi)$ follows from the fact that, for every $k \leqslant n$, the gcd of minors of order k of the matrix g coincides with the gcd of minors of order k of the matrix $\lambda(\varpi)$, which is equal to $\varpi^{\lambda_1+\cdots+\lambda_k}$.

We put $c_\lambda := \mathbf{1}_{G(O)\lambda(\varpi)G(O)}$ for $\lambda \in X_*(T)^+$, this is the characteristic function of the subset $G(O)\lambda(\varpi)G(O)$ of $G(F)$. As a consequence of the Cartan decomposition, we get the following

Corollary 3.2. *The family of functions $\{c_\lambda\}_{\lambda \in X_*(T)^+}$ generates $\mathcal{H}_G$ as a $\mathbb{Z}$-module.*

Moreover, if we write an element $f \in \mathcal{H}_G$ as $f = \sum_\lambda a_\lambda c_\lambda$, we see that

$$a_\lambda = f(\lambda(\varpi)). \tag{3.1}$$

We can right away give the following beautiful and easy result (which will not be used in the sequel), known as "Gelfand's Lemma".

Proposition 3.3. *The ring $\mathcal{H}$ is commutative.*

Proof. The proof relies on the existence of an automorphism $\iota : G(F) \to G(F)$ such that $\iota \circ \iota = \mathrm{id}$ and $\iota(x) \in G(O)x^{-1}G(O)$ for all $x \in G(F)$. When $G = \mathrm{GL}_n$, one can take ι to be the transpose of the inverse.

For any $f \in \mathcal{C}_c(G(F))$, we set $\overline{f}(x) = f(\iota(x))$ and $\widetilde{f}(x) = f(x^{-1})$. If $f \in \mathcal{H}_G$, then we see that $\overline{f} = \widetilde{f}$. Since the measure $\mu \circ \iota$ is left invariant, this implies the existence of a scalar $c > 0$ such that

$$\int_{G(F)} \overline{f}(x)\mathrm{d}x = c\int_{G(F)} f(x)\mathrm{d}x.$$

Since $\iota^2 = \mathrm{id}$, we get that $c = 1$. We then see that $\overline{f \star g} = \overline{f} \star \overline{g}$ for every $f, g \in \mathcal{C}_c(G(F))$.

On the other hand the unimodularity of $G(F)$ also implies that $\int_{G(F)} \widetilde{f}(x)\mathrm{d}x = \int_{G(F)} f(x)\mathrm{d}x$, and an easy computation shows that $\widetilde{f \star g} = \widetilde{g} \star \widetilde{f}$. For $f, g \in \mathcal{H}_G$, the previous results put together finally yield

$$\overline{f \star g} = \overline{f} \star \overline{g} = \widetilde{f} \star \widetilde{g} = \widetilde{g \star f} = \overline{g \star f},$$

and thus $f \star g = g \star f$. □

The case when $G = T$ will be of particular interest for us. First, note that we have an exact sequence

$$0 \longrightarrow T(O) \longrightarrow T(F) \longrightarrow X_*(T) \longrightarrow 0.$$

The arrow $T(O) \to T(F)$ is just the inclusion, and $T(F) \to X_*(T)$ is the map sending t to the cocharacter $\gamma(t)$ satisfying

$$\langle \gamma(t), \chi \rangle = v(\chi(t))$$

for all $\chi \in X^*(T)$. This sequence is split, since the map $\lambda \mapsto \lambda(\varpi)$ provides a splitting for γ.

Since T is commutative, we have an obvious isomorphism

$$T(O)\backslash T(F)/T(O) \simeq T(F)/T(O).$$

Thus we get

$$\begin{aligned} c_\lambda \star c_\mu(z) &= \int_{T(F)} c_\lambda(x) c_\mu(x^{-1}z) \mathrm{d}x \\ &= \int_{T(F)} \mathbf{1}_{\{x \in \lambda(\varpi)T(O),\ x^{-1}z \in \mu(\varpi)T(O)\}}(x) \mathrm{d}x \\ &= c_{\lambda+\mu}(z), \end{aligned}$$

(we use the fact that $(\mu + \lambda)(\varpi) = \mu(\varpi)\lambda(\varpi)$) which means that we have an isomorphism of commutative rings

$$\begin{array}{ccc} \mathcal{H}_T & \to & \mathbb{Z}[X_*(T)] \\ c_\lambda & \mapsto & e^\lambda. \end{array}$$

4. The Satake Transform

In this section, we study the Satake transform, which induces an isomorphism over $\mathbb{Z}[q^{\pm 1/2}]$ between the spherical Hecke algebra and the Grothendieck ring of $\hat{G}$.

We let N be the unipotent radical of $B = T \cdot N$, and $\mathrm{d}n$ be the Haar measure on $N(F)$ giving $N(O)$ volume 1. The modulus character of B is the function $\delta : B(F) \to \mathbb{R}_{>0}$ defined by

$$\mathrm{d}(bnb^{-1}) = \delta(b)\mathrm{d}n,$$

for any $b \in B(F)$. Using this definition, one can check that $\delta(b) = |\det(\mathrm{Ad}(b)|_{\mathrm{Lie}(N_F)})|$. Moreover, since δ is obviously trivial on $N(F)$, we see that it defines a character $\delta : T(F) \to \mathbb{R}_{>0}$. We can be more explicit:

$$\begin{aligned} \delta(t) &= |\det(\mathrm{Ad}(t)|_{\mathrm{Lie}(N_F)})| \\ &= |2\rho(t)| \\ &= q^{-2\rho(t)} \end{aligned}$$

for all $t \in T$.

We can now define the Satake transform

$$\begin{array}{ccc} \mathcal{H}_G & \to & \mathcal{H}_T \otimes_{\mathbb{Z}} \mathbb{Z}[q^{1/2}, q^{-1/2}] \\ f & \mapsto & S(f), \end{array} \tag{4.1}$$

where $S(f)(t) := t \mapsto \delta(t)^{1/2} \int_{N(F)} f(tn)\mathrm{d}n$ for all $t \in T(F)$ (it is easy to check that $S(f)(tk) = S(f)(t)$ for all $k \in T(O)$). One can check from the construction that this map is a morphism of rings. The presence of the character $\delta(t)^{1/2}$ appearing in this definition might seem artificial at first. In fact, it allows $S(f)$ to be invariant under the Weyl group W, which is definitely not obvious from this construction... For the next theorem, recall that by construction of the Langlands dual group we have an identification $X_*(T) = X^*(\hat{T})$, and thus an isomorphism of rings $\mathcal{H}_T \xrightarrow{\sim} X^*(\hat{T})$ thanks to (4.1).

Theorem 4.1. *The Satake transform induces an isomorphism of $\mathbb{Z}[q^{\pm 1/2}]$-algebras*

$$\mathcal{H}_G \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}] \xrightarrow{\sim} \mathbb{Z}[X^*(\hat{T})]^W \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}] \simeq K_0(\mathrm{Rep}(\hat{G})) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$$

Recall that the second isomorphism in the theorem is due to the well known following fact (cf. [Mil17, Theorem 22.38]).

Proposition 4.2. *Let (H, K) be a split reductive group over a field with split maximal torus, and denote by W the associated Weyl group. For any $V \in \mathrm{Rep}(H)$, define the character map*

$$\mathrm{ch}(V) := \sum_{\lambda \in X^*(K)} \dim(V_\lambda) e^\lambda \in \mathbb{Z}[X^*(K)],$$

where $V_\lambda := \{v \in V \mid t \cdot v = \lambda(t)v\}$. Then the character induces an isomorphism of rings

$$K_0(\mathrm{Rep}(H)) \simeq \mathbb{Z}[X^*(K)]^W.$$

The proof that $S(f)$ is indeed invariant under W involves orbital integrals. In fact, any coset $T(F)/T(O)$ contains a regular element ν (i.e. such that $\langle\gamma(\nu),\chi\rangle \neq 0$ for every $\chi \in X^*(T)$), and one can rewrite the Satake transform as

$$S(f)(\nu) = |D(\nu)|^{1/2}\mathcal{O}_\nu(f),$$

where $\mathcal{O}_\nu(f)$ is the orbital integral of f at ν, and $D(\cdot)$ is a function on $G(F)$ which is invariant under conjugation by W. Since $\nu \mapsto \mathcal{O}_\nu(f)$ is also invariant by conjugation under W, this shows the invariance of $S(f)$ under W (cf. [C+79, Theorem 4.1]).

We will now explain why S gives a bijection. For any $\lambda \in X_*(T)^+$, we will denote by $V(\lambda) \in \operatorname{Rep}(\hat{G})$ the simple highest weight representation of highest weight λ. The idea is to show that, when expressed in the basis $\{c_\lambda\}_{\lambda\in X_*(T)^+}$ and $\{\operatorname{ch}(V(\lambda))\}_{\lambda\in X_*(T)^+}$, ordered with $\leqslant$, the Satake morphism takes the form of an upper triangular matrix, with invertible elements on the diagonal. This is the motivation for the following statement (also recall (3.1)).

Proposition 4.3. *For every $\lambda, \mu \in X_*(T)^+$, we have*

$$S(c_\lambda)(\mu(\varpi)) \neq 0 \Rightarrow \mu \leqslant \lambda \tag{4.2}$$

$$S(c_\lambda)(\lambda(\varpi)) \in \mathbb{Z}[q^{1/2}, q^{-1/2}]^\times. \tag{4.3}$$

Sketch of the proof. We first sketch the proof of (4.2) in the case where $F = \mathbb{F}_q((\varpi))$ (cf. [Zhu16, Lemma 5.3.7 and §5] for the justifications). By definition we have

$$S(c_\lambda)(\mu(\omega)) = q^{-\langle\rho,\mu\rangle}\int_{N(F)} c_\lambda(\mu(\varpi)n)\mathrm{d}n.$$

Thus, if the left-hand side of this equation is non-zero we must have

$$\mu(\varpi)N(F) \cap G(O)\lambda(\varpi)G(O) \neq \varnothing,$$

which is equivalent to

$$N(F)\mu(\varpi) \cap G(O)\lambda(\varpi)G(O) \neq \varnothing. \tag{4.4}$$

We claim that (4.4) implies that $\mu \leqslant \lambda$. This can be shown using algebro-geometric arguments related to the affine Grassmannian Gr_G, which is an ind-projective $\mathbb{F}_q$-ind-scheme whose $\mathbb{F}_q$-points coincide with $G(F)/G(O)$, and where the notions of closures and limits of a $\mathbb{G}_{\mathrm{m}}$-action make sense. In fact one shows that

$$S_\mu := N(F)\mu(\varpi)G(O)/G(O) = \{x \in G(F)/G(O) \mid \lim_{s\to 0}2\rho^\vee(s)x = \mu(\varpi) \bmod G(O)\},$$

where the limit appearing above is the translation on $\mathbb{F}_q$-points of the fact that the ind-scheme associated with S_μ is the attractor space of Gr_G for the $\mathbb{G}_{\mathrm{m}}$-action induced by $2\rho^\vee$ (see §4 of Talk 8). So now set [2] $\mathrm{Gr}_{G,\lambda} := G(O)\lambda(\varpi)G(O)/G(O)$ and assume that there exists $x \in S_\mu \cap \mathrm{Gr}_{G,\lambda}$. Since $2\rho^\vee(s) \in G(\mathbb{F}_q) \subset G(O)$ for all $s \in \mathbb{F}_q^\times$, we see that $2\rho^\vee(s)x \in \mathrm{Gr}_{G,\lambda}$. This implies that

$$\mu(\varpi) = \lim_{s\to 0}2\rho^\vee(s)x \in \overline{\mathrm{Gr}_{G,\lambda}}.$$

But one can show that $\overline{\mathrm{Gr}_{G,\lambda}} = \bigsqcup_{\mu\leqslant\lambda}\mathrm{Gr}_{G,\mu}$, and so $\mu \leqslant \lambda$.

Let us prove[3] (4.3) in the case where $G = \mathrm{GL}_r$, for some $r \geqslant 1$. For every $n \in N(F)$, we have that $c_\lambda(\lambda(\varpi)n) \neq 0$ iff $n \in \lambda(\varpi)^{-1}G(O)\lambda(\varpi)G(O)$, so $n \mapsto c_\lambda(\lambda(\varpi)n)$ is the characteristic function of the set $X_1 := N(F) \cap \lambda(\varpi)^{-1}G(O)\lambda(\varpi)G(O)$. Using the fact that X_1 is a disjoint union of left $N(O)$-cosets, we have

$$\begin{aligned}\operatorname{vol}(X_1) &= \#(N(F) \cap \lambda(\varpi)^{-1}G(O)\lambda(\varpi)G(O))/N(O)\\ &= \#(N(F) \cap \lambda(\varpi)^{-1}G(O)\lambda(\varpi))/N(O)\\ &= \operatorname{vol}(X_2),\end{aligned}$$

[2] It turns out that $G(O)\lambda(\varpi)G(O)/G(O)$, resp. $N(F)\mu(\varpi)/G(O)$, is the set of $\mathbb{F}_q$-points of a locally-closed sub-variety, resp. of an ind-sub-variety, of the affine Grassmannian.

[3] This part of the proof is inspired from [Rev16, §1.4].

where $X_2 := N(F) \cap \lambda(\varpi)^{-1}G(O)\lambda(\varpi)$ is stable by multiplication on the right with elements of $N(O)$ (indeed, since λ is dominant, $N(O)$ is stable under conjugation by $\lambda(\varpi)$, which implies that $\lambda(\varpi)^{-1}G(O)\lambda(\varpi)$ is stable by multiplication on the right with elements of $N(O)$), and the second equality follows after injecting $X_1/G(O)$ into $G(F)/G(O) = B(F)G(O)/G(O) \simeq B(F)/B(O)$ (here we use the Iwasawa decomposition $G(F) = B(F)G(O)$). We see that

$$X_2 = \{(n_{i,j}) \in N(F) \mid v(n_{i,j}) \geqslant \lambda_j - \lambda_i \ \forall 1 \leqslant i < j \leqslant r\}.$$

Thus we get

$$S(c_\lambda)(\lambda(\varpi)) = q^{-\langle \rho,\mu\rangle}\mathrm{vol}(X_2) = q^{-\langle \rho,\mu\rangle}q^{\sum_{1\leqslant i<j\leqslant r}(\lambda_i-\lambda_j)} = q^{\langle \rho,\mu\rangle} \in \mathbb{Z}[q^{\pm 1/2}]^\times.$$

□

Remark 4.4. Since $N(F)\mu(\varpi) \cap G(O)\lambda(\varpi)G(O)$ is stable under left multiplication by elements of $N(O)$, it must have a non-zero $\mathrm{d}n$-volume whenever it is non-empty. Therefore the implication (4.2) is actually an equivalence.

Knowing that the elements $\{\mathrm{ch}(V(\mu))\}_{\mu \in X_*(T)}$ generate the $\mathbb{Z}$-module $\mathbb{Z}[X_*(T)]^W$ and that each $\mathrm{ch}(V(\mu))$ has highest weight μ, one can thus express the Satake transform in this basis:

$$S(c_\lambda) = q^{\langle \rho,\mu\rangle}\mathrm{ch}(V(\lambda)) + \sum_{\mu<\lambda} a_{\lambda,\mu}\mathrm{ch}(V(\mu)),$$

for some elements $a_{\lambda,\mu} \in \mathbb{Z}[q^{\pm 1/2}]$.

Remark 4.5. A natural question is to ask what element of $\mathcal{H}_G \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$ is mapped to $\mathrm{ch}(V(\lambda))$ under S. The answer is naturally given by the Geometric Satake equivalence, cf. [Zhu16, Lemma 5.6.3].

5. The example of $\mathrm{PGL}_{2,O}$

We now treat in detail the example of $G = \mathrm{PGL}_{2,O}$, since the set $G(F)/G(O)$ can be represented by the vertices of a tree in this case, on which Schubert cells and varieties are visible[4]. We fix a maximal torus T and a Borel B of G consisting of the image of the diagonal matrices and upper triangular matrices respectively under the canonical projection $\mathrm{GL}_{2,O} \to \mathrm{PGL}_{2,O}$. We then have $\hat{G} = \mathrm{SL}_{2,\mathbb{C}}$ and $X_*(T) = X^*(\hat{T}) = \mathbb{Z}\cdot\chi$, where $\hat{T}$ is the diagonal torus and χ sends a diagonal element $\mathrm{diag}(\alpha,\alpha^{-1})$ to α. So $X_*(T)$ is isomorphic to $\mathbb{Z}$, and this isomorphism also yields an identification $X_*(T)^+ = \mathbb{Z}_{\geqslant 0}$.

Let X_q denote the set of vertices of the $(q+1)$-regular tree (cf. Figure 1). This tree is a realization of the Bruhat-Tits building for $\mathrm{PGL}_2(F)$, and the set X_q is in bijection with $G(F)/G(O)$, the $\mathbb{F}_q$-points of the affine Grassmannian (cf. [Gör10, §2.8]). In particular, if we pick a vertex $e \in X_q$ corresponding to the coset of the identity, we then have

$$\mathrm{Gr}_{G,\mu} \simeq \{x \in X_q \mid d(e,x) = \mu\}$$

for any $\mu \in \mathbb{Z}_{\geqslant 0}$, where $\mathrm{Gr}_{G,\mu} := G(O)\mu(\varpi)G(O)/G(O)$ is the Schubert cell associated with μ, and where $d(\cdot,\cdot)$ is the usual distance function on a tree. On the drawing below, the red dot is the vertex e, and the green circled vertices represent $\mathrm{Gr}_{G,2}$.

[4]I thank Timo Richarz for sharing some of his notes with me, which I follow in this section.

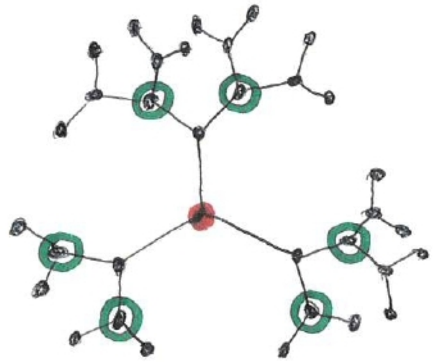

FIGURE 1. Portion of the Bruhat-Tits tree, $q = 2$.

For any $x \in X_q$ and $\mu \in \mathbb{Z}_{\geqslant 0}$, we let δ_x be the characteristic function of $\{x\}$ on X_q, and put

$$d_\mu := \sum_{x \in X_q,\ d(e,x)=\mu} \delta_x.$$

We then define $\mathcal{H}_q \subset \mathcal{C}_c(X_q, \mathbb{Z})$ as the $\mathbb{Z}$-span of $\{d_\mu,\ \mu \in \mathbb{Z}_{\geqslant 0}\}$, and put the ring structure on it induced by the formula

$$d_\mu \star \delta_y := \sum_{x \in X_q,\ d(y,x)=\mu} \delta_x.$$

One can check that we have the following equalities

$$\begin{cases} d_1 \star d_1 & = d_2 + (q+1)\delta_e \\ d_1 \star d_\mu & = d_{\mu+1} + q d_{\mu-1},\ \forall \mu \geqslant 2, \end{cases}$$

which easily imply that $\mathcal{H}_q$ is commutative, with unit d_0. The following proposition is also straightforward.

Proposition 5.1. *We have an isomorphism of rings*

$$\begin{array}{ccc} \mathcal{H}_q & \to & \mathcal{H}_G \\ d_\lambda & \mapsto & c_\lambda. \end{array}$$

We can thus consider the isomorphism of rings

$$S : \mathcal{H}_q \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}] \xrightarrow{\sim} \mathbb{Z}[X_*(T)]^W \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}],$$

which is the composition of the Satake isomorphism with the isomorphism of the preceding proposition. We can actually give an answer to the question raised by Remark 4.5 in this simple case of $\mathrm{PGL}_{2,O}$. For any $\mu \in \mathbb{Z}_{\geqslant 0}$, set

$$f_\mu := \sum_{0 \leqslant \nu \leqslant \mu,\ \nu = \mu \bmod 2} d_\nu,$$

and recall that $V(\mu) \in \mathrm{Rep}(\mathrm{SL}_{2,\mathbb{C}})$ denotes the simple highest weight representation of highest weight μ.

Proposition 5.2. *For any $\mu \in \mathbb{Z}_{\geqslant 0}$, we have $S(f_\mu) = q^{\mu/2}\mathrm{ch}(V(\mu))$.*

Proof. On the one hand, one can check that we have

$$\begin{cases} f_0 = d_0 \\ f_1 \star f_\mu = f_{\mu+1} + q f_{\mu-1}\ \forall \mu \geqslant 1. \end{cases} \tag{5.1}$$

On the other hand, the characters of Weyl modules are well known for $\mathrm{SL}_{2,\mathbb{C}}$:

$$\mathrm{ch}(V(\mu)) = \sum_{-\mu \leqslant \nu \leqslant \mu,\ \nu = \mu \bmod 2} e^\nu,$$

so that the sequence $\left(q^{\mu/2}\mathrm{ch}(V(\mu))\right)_{\mu\geqslant 0}$ satisfies the same relation as (5.1), with d_0 replaced by 1, and $\star$ replaced by the usual multiplication in $\mathbb{Z}[X_*(T)]\otimes_{\mathbb{Z}}\mathbb{Z}[q^{\pm 1/2}]$. This concludes the proof. □

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