TALK 0: INTRODUCTION

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1. History

1.1. **Lusztig's work.** The Geometric Satake Equivalence can be seen as a "geometric" (or "categorical") version of the Satake isomorphism (see Talk 1 for details). However, the first clear indication that this isomorphism could have such a geometric version is to be found in work of Lusztig [Lu]. The main results of this paper are concerned with the Kazhdan–Lusztig basis of the Hecke algebra associated with the affine Weyl group of a reductive group G, but this algebra is known to be a "categorical trace" of the category of Iwahori-equivariant perverse sheaves on the affine flag variety $\mathcal{F}\ell$ of G; these results can therefore also be interpreted in terms of perverse sheaves (with coefficients in a field of characteristic 0) on $\mathcal{F}\ell$ or Gr. With this translation, the main results of [Lu] take the following form:

- [Lu, Theorem 6.1] the dimension of the fiber of the intersection cohomology complex associated with the L^+G-orbit on Gr associated with a dominant cocharacter λ along the orbit associated with a cocharacter μ is the multiplicity of μ in the simple representation of the Langlands dual group of highest weight λ;
- [Lu, Corollary 8.7] the convolution of two simple L^+G-equivariant perverse sheaves on Gr is perverse; moreover, the multiplicities of simple perverse sheaves in such a convolution product agree with the multiplicities of simple representations in the corresponding tensor product of simple representations for the dual group;
- [Lu, Last line on p. 228] for any dominant coweight λ, the dimension of the total cohomology of the intersection cohomology complex associated with the orbit of a dominant cocharacter λ is the dimension of the simple representation of the dual group with highest weight λ.

This paper also explains how to construct the affine Grassmannian of a reductive group G in terms of some lattices in $\mathfrak{g}(\!(z)\!)$ where $\mathfrak{g}$ is the Lie algebra of G; see [Lu, §11]. An immediate interpretation of these results is that they identify the basis corresponding to classes of simple modules under the Satake isomorphism as the trace of Frobenius on intersection cohomology complexes on Gr.

Lusztig remarked that the results of [Lu] suggest an equivalence between the category of spherical perverse sheaves on the affine Grassmannian (for coefficients in a field $\Bbbk$ of characteristic 0) and the category of representations of the dual group (over $\Bbbk$), and that the total cohomology functor should correspond to the functor sending a representation to the underlying vector space. However, as he did not see how to interpret the commutativity of the tensor product of representations on the other side, he did not pursue his work on this subject.

1.2. **Drinfeld's contribution.** Some years later, and in relation with his work with Beĭlinson [BD], Drinfeld also remarked this possible application of Lusztig's work, and understood how to define a commutativity constraint on the category of spherical perverse sheaves on the affine Grassmannian, based on a new description of the convolution product in terms of the *fusion product.* This construction uses a new family of schemes, now called *Beĭlinson–Drinfeld Grassmannians*, and defined as some "relative" versions of the affine Grassmannian over copies of a curve, which will be introduced in Talk 7. Drinfeld explained this construction to a number of people, including V. Ginzburg. It did not appear in written form before the announcement [MV00] (see §1.4 below).

1.3. **Ginzburg's work.** In [Gi], V. Ginzburg claimed to give the first proof of the Geometric Satake Equivalence (for coefficients in a field of characteristic 0). This proof relies on Lusztig's results explained in §1.1, but does *not* use Drinfeld's idea for the construction of the commutativity constraint. It proposes a different construction of this isomorphism, based on the localization theorem in equivariant cohomology. Unfortunately, this construction has a gap: it defines the isomorphism as the unique morphism satisfying an appropriate property (see the proof of [Gi, Proposition 2.3.1]), but there is no proof that such a morphism exists. In fact, in later work Zhu [Zh17a] observed that Ginzburg's condition is *not* the property that the commutativity constraint should satisfy; an appropriate sign must be added.

The strategy of proof relies of the Tannakian formalism: one constructs some structures on the category of spherical perverse sheaves on the affine Grassmannian, and then uses these structures to apply general results that guarantee that this category must be equivalent to the category of representations of a group scheme. One then proves that this group scheme is in fact the Langlands dual group.

The preprint [Gi] introduced many ideas and geometric constructions that were later useful in various applications of the Geometric Satake Equivalence, but it does not contain a complete proof of this equivalence.

1.4. **Mirković–Vilonen's work.** The paper [MV07] by Mirković–Vilonen contains the first complete proof of the Geometric Satake Equivalence, and in fact proves such an equivalence for (essentially) any ring of coefficients.[1] The proof again uses the ideas of the Tannakian formalism, but requires new ingredients to work over rings since there is no Tannakian formalism for such coefficients. The construction of the commutativity constraint follows Drinfeld's suggestion. The main new ingredient compared to previous works is the *semi-infinite orbits*, whose construction will be explained in Talk 8. These orbits appeared earlier in some form in the p-adic group literature and in [Lu] (in the guise of the periodic module); here they are given a geometric structure, and their relation with spherical orbits is studied.

The semi-infinite orbits are used in particular in an essential way:

- in the construction of the maximal torus of the dual group;
- in the application of Braden's "hyperbolic localization theorem;"
- in the construction of the Tannakian group scheme over rings.

They also provide a "canonical" basis of weight spaces of certain representations of the dual group, in terms of some varieties now called *Mirković–Vilonen cycles*, which gave rise to an extensive study by various authors.

Note that this proof was first announced in [MV00]. However, the proof outlined there had a gap, related to the proof that the group scheme constructed by the authors has reduced fibers. This gap was filled in [MV07] using a general result on reductive group schemes proved in the meantime by Prasad–Yu. Note also that the proof in [MV07] is written in a topological setting. The authors claim that a similar proof can be given for étale perverse sheaves, but some important steps need to be modified. Some indications on how to perform the required modifications can be found in [Zh17b].

1.5. **Fargues–Scholze's work.** The proof that we will study in this workshop is that given recently by Fargues–Scholze in [FS21, Chap. VI]. This proof is written in the language of spatial diamonds, but we will explain how it can be "translated" in the world of étale sheaves on schemes. The proof follows closely the strategy of [MV07]. The main new ingredient is a way of defining a "Satake category" of sheaves on the Beĭlinson–Drinfeld Grassmannians (via ULA relative perverse sheaves), which makes the proof of the compatibility between the various structures of the Satake category more transparent.

1.6. **Related work.** Several alternative proofs and variants of the Geometric Satake Equivalence have appeared in the literature. For some references on these works, see [BR18] and [Zh17b].

[1] An erratum correcting some minor inaccuracies later appeared in [MV18].

2. Leitfaden

The workshop will start with an introductory talk on the Satake isomorphism (Talk 1). Then we will cover the sheaf-theoretic background needed to make sense of Fargues–Scholze's constructions in Talks 2–5. In Talks 6–8 we will cover the geometric background that is involved in the proof, and in particular the construction of the various versions of the affine Grassmannian that will be used. In Talks $8\frac{1}{2}$–10 we will explain how the Satake category and its structures can be defined using this background. Finally, in Talks 11–12 we will explain the proof of the Geometric Satake Equivalence.

References

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