CATEGORICAL TRACES OF CONVOLUTION CATEGORIES

KOSTAS I. PSAROMILIGKOS

In this text I explain what I understand from [Ben+22, §3]. In particular, I expand on the strategy described in [Dos24, §2.5], following the notation and conventions in [Ric24].

These notes are incomplete and full of errors - please feel free to send me comments and questions.

Much of the beautiful theory developed in this section depends on a lot of previous work, especially by the authors. An incomplete list that I found particularly helpful is:

(1) For the monoidal structure on the category of ind-coherent sheaves, see David Ben-Zvi, David Nadler, and Anatoly Preygel. "Integral transforms for coherent sheaves". In: *Journal of the European Mathematical Society* 19.12 (Nov. 2017), pp. 3763–3812. ISSN: 1435-9863. DOI: 10.4171/JEMS/753

(2) For the notion of singular support for ind-coherent sheaves, see D. Arinkin and D. Gaitsgory. "Singular support of coherent sheaves and the geometric Langlands conjecture". In: *Selecta Mathematica* 21.1 (Nov. 2014), pp. 1–199. ISSN: 1420-9020. DOI: 10.1007/s00029-014-0167-5

(3) For the relevant derived algebraic geometry and infinity category, see D. Gaitsgory and N. Rozenblyum. *A study in derived algebraic geometry*. Ed. by American Mathematical Society. Mathematical Surveys and Monographs, 2017

(4) For Theorem 7 which is the main technical tool behind the use of categorical traces for the construction of the Springer sheaf, see D. Gaitsgory et al. "A toy model for the Drinfeld–Lafforgue shtuka construction". In: *Indagationes Mathematicae* 33.1 (Jan. 2022), pp. 39–189. ISSN: 0019-3577. DOI: 10.1016/j.indag.2021.11.002

(5) For the computation of the categorical trace and the theory of descent with supports, see David Ben-Zvi, David Nadler, and Anatoly Preygel. "A spectral incarnation of affine character sheaves". In: *Compositio Mathematica* 153.9 (June 2017), pp. 1908–1944. ISSN: 1570-5846. DOI: 10.1112/S0010437X17007278

1. Generalities on categorical traces

The purpose of Section 3 is to develop a theory of higher categorical traces on convolution categories coming from derived algebraic geometry so that it can be applied to the mixed affine Hecke category based on the results of Section 2.

1. **Recall.** We start by recalling some constructions from the previous talks, in particular, see [Ric24].

Let G be a reductive group, $\mathfrak{g}$ its Lie algebra, $\mathcal{N} \subseteq \mathfrak{g}$ its nilpotent cone, and $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ the Springer resolution.

We define $\mathcal{Z} := \tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ to be the Steinberg stack, where the fiber product is derived.

Also,

$$\mathbf{H} := \mathrm{Coh}(\mathcal{Z}/G),\ \mathbf{H}^m := \mathrm{Coh}(\mathcal{Z}/(G \times \mathbb{G}_m)),$$

are the affine and mixed affine Hecke categories.

We denote by $\mathcal{H}_{\mathrm{aff}}$ the affine Hecke algebra of the affine Weyl group of G.

In Simon's talk, we learned that

$$\mathcal{H}_{\mathrm{aff}} \cong \mathrm{HH}^0(\mathrm{Ind}(\mathbf{H}^m))$$

and that the other Hochschild homology groups vanish, so we can identify $\mathcal{H}_{\mathrm{aff}}$ with the dg-algebra concentrated in degree 0, and turn this into an isomorphism

$$\mathcal{H}_{\mathrm{aff}} \cong \mathrm{HH}(\mathrm{Ind}(\mathbf{H}^m))$$

of dg-algebras.

The affine Hecke and mixed affine Hecke categories are special examples of convolution categories coming from derived algebraic geometry, ie. categories of some kind of sheaves on $X \times_Y X$, which are usually monoidal using some kind of convolution. In this section (and a multitude of other papers) the authors develop a formalism based on categorical traces that applies nicely to convolution categories to give another computation of the Hochschild homology, and in particular construct the universal Springer sheaf and study its properties. All these categories are naturally ∞-categories.

Let Vect_k to be the ∞-category of complexes of vector spaces over k. For us, a "monoidal ∞-category" means "an algebra object in $\mathrm{Pr}^L_{\mathrm{St},k}$", where $\mathrm{Pr}^L_{\mathrm{St},k}$ stands for "the ∞-category of Vect_k-modules in the ∞-category of stable presentable ∞-categories" and left adjoint functors, as in Simon's talk - so we also require our infinity categories to be stable presentable and k-linear, as we will work in the setting of derived algebraic geometry. $\mathrm{Pr}^L_{\mathrm{St},k}$ is symmetric monoidal by defining the tensor product to be the product over Vect_k. As an example, the category Vect_k is itself an obvious object of $\mathrm{Pr}^L_{\mathrm{St},k}$, and also the unit for the symmetric monoidal structure.

We also denoted by $\mathrm{Cat}^{\mathrm{perf}}_k$ the ∞-category of small, stable, idempotent complete ∞-categories, with morphisms given by exact functors. It is also symmetric monoidal and we have a symmetric monoidal ind-completion functor

$$\mathrm{Ind} : \mathrm{Cat}^{\mathrm{perf}}_k \to \mathrm{Pr}^L_{\mathrm{St},k}.$$

For details, see [Ric24, §1.2] and the references therein.

Example 1. *Let X be a QCA derived stack over k, which for the authors means quasi-compact stacks of finite presentation with affine finitely-presented diagonal. Then,*

1. $\mathrm{Perf}(X)$*, the category of perfect complexes on X, is in* $\mathrm{Cat}^{\mathrm{perf}}_k$.
2. $\mathrm{QC}(X) \cong \mathrm{Ind}(\mathrm{Perf}(X))$.
3. $\mathrm{QC}^!(X) \cong \mathrm{Ind}(\mathrm{Coh}(X))$*, see [GR17].*
4. *By the above, we have*

$$\mathrm{QC}(X)^c = \mathrm{Perf}(X),\ \mathrm{QC}^!(X)^c = \mathrm{Coh}(X).$$

Definition 2. *We call a monoidal ∞-category $\mathcal{C}$* compactly generated*, when it is of the form $\mathcal{C} = \mathrm{Ind}(\mathcal{A})$ for a category $\mathcal{A} \in \mathrm{Cat}^{\mathrm{perf}}_k$.*

2. **Definitions of categorical trace.** For an extensive discussion of the notions in this subsection, and generally for applications of this formalism, see the online notes by Harrison Chen on his webpage, or [Dos24, §2].

If $\mathcal{C}$ is compactly generated, $\mathcal{C}$ is dualizable, and we have defined its *Hochschild homology* $\mathrm{HH}(\mathcal{C}, F) \in \mathrm{Vect}_k$ see [Ric24, Definition 1.3].

Definition 3. *We also call this the* vertical trace $\mathrm{tr}(\mathcal{C}, f) := \mathrm{HH}(\mathcal{C}, f)$.

Morphisms of pairs $(F, \psi) : (\mathcal{C}, f) \to (\mathcal{D}, g)$ induce maps on traces $\mathrm{tr}(F, \psi) : \mathrm{HH}(\mathcal{C}, f) \to \mathrm{HH}(\mathcal{D}, g)$ via functoriality of traces, see [Ric24, §1.4].

Consider the Morita category of $\mathcal{A}$-module categories, with morphisms given by $(\mathcal{B}, \mathcal{A})$-bimodule categories, 2-morphisms are functors between them etc. If we apply our construction to this category, we get the following.

Definition 4. *Alternatively, we have the* horizontal *or 2-categorical trace*

$$\mathrm{Tr}(\mathcal{C}, F) := \mathcal{C} \otimes_{\mathcal{C} \otimes \mathcal{C}^{rv}} \mathcal{C}^F,$$

where $\mathcal{C}^{rv}$ is defined by reversing just the monoidal product, and $\mathcal{C}^F$ is $\mathcal{C}$ where the left action has been twisted by F.

For more details on the above, consult [Ben+22, Definition 3.1].

By this definition, we also have functoriality of traces for Tr.

If $\mathcal{C}$ is a compactly generated monoidal ∞-category and F a monoidal endofunctor, there is a natural morphism of pairs $\eta : (\mathrm{Vect}_k, id) \to (\mathcal{C}, F)$. We have $\mathrm{Tr}(\mathrm{Vect}_k, id) \cong \mathrm{Vect}_k$, so we can consider k as a complex concentrated in degree 0 and define the following object.

Definition 5. *The* character $[\mathcal{C}, F]$ *of $(\mathcal{C}, F)$ is defined to be the image*

$$[\mathcal{C}, F] := \mathrm{Tr}(\eta)(k) \in \mathrm{Tr}(\mathcal{C}, F).$$

The theory we will study works nicely in the following setting

Definition 6. *A compactly generated monoidal ∞-category $\mathcal{C}$ is called* rigid *if the monoidal unit is compact, the multiplication map sends compact objects to compact objects, and every compact object of $\mathcal{C}$ is dualizable.*

In particular, the vertical trace will be an algebra object of Vect_k, not just an object.

See also [Ric24, Remark 1.5].

The first part of the following is [Gai+22, Theorem 3.8.5] stated in our setting similarly to [Ben+22, Theorem 3.4], [Dos24, Theorem 2.5].

Theorem 7. *Let $\mathcal{C}$ be a compactly generated rigid monoidal ∞-category and F a monoidal endofunctor. There is an equivalence*

$$\mathrm{HH}(\mathcal{C}, F) \cong End_{\mathrm{Tr}(\mathcal{C},F)}([\mathcal{C}, F]),$$

of dg-algebras, inducing an equivalence of functors

$$\mathrm{HH}(-) \cong \mathrm{Hom}_{\mathrm{Tr}(\mathcal{C},F)}([\mathcal{C}, F], -) : (\mathcal{C}, F)-\mathbf{mod} \to \mathrm{HH}(\mathcal{C}, F)-\mathrm{mod}$$

In particular, assuming that $[\mathcal{C}, F]$ is a compact object, then the left adjoint to the functor $\mathrm{Hom}_{\mathrm{Tr}(\mathcal{C},F)}([\mathcal{C}, F], -)$ defines a fully faithful embedding which preserves compact objects, whose essential image is the category generated by $[\mathcal{C}, F]$.

$$\begin{tikzcd} \mathrm{HH}(\mathcal{C},F)\text{-mod} \arrow[rr, shift left, "{[\mathcal{C},F]\otimes_{End_{\mathrm{Tr}(\mathcal{C},F)}}-}"] \arrow[dr, "\cong"'] & & \mathrm{Tr}(\mathcal{C},F) \arrow[ll, shift left, "{\mathrm{Hom}_{\mathrm{Tr}(\mathcal{C},F)}([\mathcal{C},F],-)}"] \\ & \langle[\mathcal{C},F]\rangle \arrow[ur, hook] & \end{tikzcd}$$

For the second part, notice that since $[\mathcal{C}, F]$ is compact and the category is rigid, the tensor sends compact objects to compact objects. I think the reason the embedding is fully faithful is that the category generated by $[\mathcal{C}, F]$ embeds in $\mathrm{Tr}(\mathcal{C}, F)$ (but this may just be circular).

2. Traces in geometric settings

1. **Monoidal structures of convolution categories.** Let $p_1 : X \to S,\ p_2 : Y \to S$ be proper maps of perfect stacks over k.

There is an equivalence

$$\mathrm{QC}(X) \otimes_{\mathrm{QC}(S)} \mathrm{QC}(Y) \xrightarrow{\cong} \mathrm{QC}(X \times_S Y),$$

and since $\mathrm{QC}(X)$ is a dualizable $\mathrm{QC}(S)$-module, the integral transform

$$\Phi : \mathrm{QC}(X \times_S Y) \xrightarrow{\cong} \mathrm{Fun}^L_{\mathrm{QC}(S)}(\mathrm{QC}(X), \mathrm{QC}(Y)),$$

given by $\Phi(\mathcal{K}) = \mathcal{F} \mapsto p_{2*}(p_1^*\mathcal{F} \otimes \mathcal{K})$, is an equivalence.

Remark 8. *Alternatively there is the tensor product*

$$\mathrm{QC}^!(X) \otimes_{\mathrm{QC}^!(S)} \mathrm{QC}^!(Y) \to \mathrm{QC}^!(X \times_S Y)$$

which we will heavily use, but it is not an equivalence anymore. Indeed, if X, Y, S are assumed to be smooth, the left hand side is equal to $\mathrm{QC}(X \times_S Y)$, but the fiber product does not have to remain smooth (it does not for the affine Hecke category), therefore $\mathrm{QC}(X \times_S Y) \neq \mathrm{QC}^!(X \times_S Y)$.

To find the essential image of this tensor product, we will need the notion of singular support, which we will recall in the next subsection.

By [Ben+22, Theorem 1.1.3], and since for a smooth stack $\mathrm{QC}(X) = \mathrm{QC}^!(X)$, the compact objects are the same and in particular $\mathrm{Coh}(X) = \mathrm{Perf}(X)$, we have

Proposition 9. *The integral transform Φ restricts on the compact objects to an equivalence*

$$\Phi : \mathrm{Coh}(X \times_Y X) \xrightarrow{\cong} \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Perf}(Y)}(\mathrm{Perf}(X), \mathrm{Perf}(X)).$$

Grothendieck duality, ie. the functor $\mathbb{D}(\mathcal{F}) = \mathcal{F}^\vee \otimes \omega_X$, intertwines this equivalence, and we get the !-transform given by $\Phi(\mathcal{K}) = \mathcal{F} \mapsto p_{2*}(p_1^!\mathcal{F} \otimes \mathcal{K})$, since by properness of p_2 we have $p_{2*} = p_{2!}$.

On the right hand side, we have a monoidal structure by composition, which can be transformed into convolution on the right to give $\mathrm{Coh}(Z)$ a monoidal structure via convolution.

After ind-completion, we can turn $\mathrm{QC}^!(Z)$ into a monoidal ∞-category via the !-convolution. For $\mathrm{QC}(Z)$ we cannot do the same trick, but there is an equivalence by Ben-Zvi, Francis, Nadler which is the same as in the proposition by replacing all categories with QC, which immediately gives $\mathrm{QC}(Z)$ a monoidal structure, this time corresponding to $*$-convolution.

For $\mathrm{QC}^!$, this works as follows. Let X_i be smooth QCA stacks over k, proper over Y, and $Z_{ij} = X_i \times_Y X_j$. We have a correspondence

$$Z_{12} \times Z_{23} \xleftarrow{(p_{12},p_{23})} X_1 \times_Y X_2 \times_Y X_3 \xrightarrow{p_{13}} Z_{13},$$

wich allows us to define the convolution

$$\mathcal{F}_1 * \mathcal{F}_2 := p_{13*}(p_{12}^!\mathcal{F}_1 \otimes p_{23}^!\mathcal{F}_2) : \mathrm{QC}^!(Z_{12}) \otimes \mathrm{QC}^!(Z_{23}) \to \mathrm{QC}^!(Z_{13}).$$

Notice the maps p_{ij} are proper as base changes of the proper maps $X_k \to Y$. Smoothness of the stacks implies pullback along the diagonal preserves coherent objects, and properness of the stacks over Y implies the pushforward also preserves coherence. The convolution functor $*$ descents to a relative tensor product

$$* : \mathrm{QC}^!(Z_{12}) \otimes_{\mathrm{QC}^!(Z_{22})} \mathrm{QC}^!(Z_{23}) \to \mathrm{QC}^!(Z_{13}).$$

Proposition 10. *The monoidal unit of* $\mathrm{QC}^!(Z)$ *is* $\omega_\Delta := \Delta_*\omega_X$*, where* $\Delta : X \to X \times_Y X$ *is the relative diagonal.*

2. **Review on singular support.** Let X be a quasi-smooth stack.

Definition 11. *The odd cotangent bundle or scheme of singularities* $\mathbb{T}_X^{*[-1]}$ *is defined to be*

$$\mathbb{T}_X^{*[-1]} := \mathrm{Spec}_X \mathrm{Sym}_X H^0(\mathcal{T}_X[1]),$$

where $\mathcal{T}_{\mathcal{X}}$ *is the tangent complex of* X*.*

Example 12. *For a smooth stack* X*,* $\mathbb{T}_X^{*[-1]} = X$*. In general, the fibers of* $\mathbb{T}_X^{*[-1]} \to X$ *lie over the singular locus of* X*.*

For an ind-coherent sheaf $\mathcal{F} \in \mathrm{QC}^!(X)$, Arinkin and Gaitsgory constructed its *singular support* which is a closed conical subset of the odd cotangent bundle $SS(\mathcal{F}) \subseteq \mathbb{T}_X^{*[-1]}$, see [AG14, Definition 4.4]. The reason we review this notion is that we often want to restrict to the full subcategory $\mathrm{QC}^!_\Lambda(X) \subseteq \mathrm{QC}^!(X)$ of ind-coherent sheaves with singular support in Λ, in particular for this application to be able to do some kind of descent, and to study tensor products of ind-coherent sheaves.

Let Y be another quasi-smooth stack and $f : X \to Y$ a representable morphism. There is a natural correspondence

$$\mathbb{T}_X^{*[-1]} \xleftarrow{df} \mathbb{T}_Y^{*[-1]} \times_Y X \xrightarrow{p} \mathbb{T}_Y^{*[-1]},$$

where p is just the projection and df is a "derived differential"? We can then pushforward and pullback conical subsets via $f_*\Lambda_X = \overline{p(df^{-1}(\Lambda_X))}$, $f^!\Lambda_Y = \overline{p(df(\Lambda_Y))}$

We sum up the properties we need in the following proposition. For more details, see [AG14, §1.3.9].

Proposition 13. *For an ind-coherent sheaf* $\mathcal{F} \in \mathrm{QC}^!(X)$*, the following are true.*

(1) The intersection of $SS(\mathcal{F})$ *with the zero section is the classical support of* $\mathcal{F}$*.*
(2) $SS(f_*\mathcal{F}) = f_*SS(\mathcal{F})$ *[AG14, Proposition 7.1.3]*
(3) $SS(f^!\mathcal{F}) = f^!SS(\mathcal{F})$ *[AG14, Proposition 7.1.3]*
(4) $SS(\mathcal{F})$ *is the zero section if and only if* $\mathcal{F} \in \mathrm{QC}(X)$ *[AG14, Theorem 4.2.6], and if* $\mathcal{F}$ *is coherent* $SS(\mathcal{F}) = \{0\}_X \Leftrightarrow \mathcal{F}$ *to* $\mathcal{F}$ *is perfect.*
(5) The singular support is preserved by Serre duality [AG14, Proposition 4.7.2].

Remark 14. *If there exists a reasonable theory of this in characteristic p, then one could reasonably expect all these properties to remain true, except (2), which would probably be a very strict containment.*

Singular support can be checked locally: See [AG14, Theorem 1.3.8].

Example 15. *(1)* $\mathrm{QC}^!_{\{0\}_X}(X) = \mathrm{QC}(X)$.

(2) $\mathrm{QC}^!_{\mathbb{T}^{*[-1]}_X}(X) \cong \mathrm{QC}^!(X)$

(3) More generally, if $i : Z \hookrightarrow X$ is a closed subscheme, and Λ_0, Λ_1 are the pullbacks to Z through the inclusion, $\mathrm{QC}^!_{\Lambda_1}(X) = \mathrm{QC}_Z(X), \mathrm{QC}^!_{\Lambda_2}(X) = \mathrm{QC}^!_Z(X)$*, ie. sheaves with classical support on Z.*

Example 16. *Notice that from the above we also get that for a smooth stack X, since $\mathbb{T}^{*[-1]}_X = X$,*

$$\mathrm{QC}(X) = \mathrm{QC}^!_{\{0\}_X}(X) = \mathrm{QC}^!_{\mathbb{T}^{*[-1]}_X}(X) \cong \mathrm{QC}^!(X).$$

We may thus view the singular support as measuring the failure of an object of $\mathrm{QC}^!(X)$ to be in $\mathrm{QC}(X)$.

We consider the (classical I think) category of pairs Pair, with objects (X, Λ) being X a quasi-smooth stack and Λ a closed conical subset of the odd cotangent bundle, and morphisms being maps of stacks such that $f_*\Lambda_X \subseteq \Lambda_Y$.

To a pair (X, Λ), we attach the corresponding category $\mathrm{QC}^!_\Lambda(X)$. The inclusion functor $i : \mathrm{QC}^!_\Lambda(X) \hookrightarrow \mathrm{QC}^!(X)$ has a right adjoint $R\Gamma_\Lambda$, which we intuitively think of as "taking local cohomology along Λ".

For the proof, see [BNP17a, Lemma 2.3.1]. A map of pairs f is called strict along Λ'_X if

$$f^!\mathbb{T}^{*[-1]}_Y \cap \Lambda_X \cap \Lambda'_X = f^!\Lambda_Y \cap \Lambda'_X$$

Proposition 17. *Let f be a quasi-smooth map of pairs. There is a natural morphism*

$$f_* \circ R\Gamma_{\Lambda_X} \to R\Gamma_{\Lambda_Y} \circ f_*$$

of functors $\mathrm{QC}^!(X) \to \mathrm{QC}^!_{\Lambda_Y}(Y)$, which restricts to an equivalence in $\mathrm{QC}^!_{\Lambda'_X}$ if f is strict along Λ'_X.

Definition 18. *Suppose that f is a quasi-smooth map of pairs. We can define adjoint functors with support conditions $f_* : \mathrm{QC}^!_{\Lambda_X}(X) \to \mathrm{QC}^!_{\Lambda_Y}(Y)$ and $f^!$, via the formulas (in the right hand side we consider the actual pushorward and pullback - we will heavily abuse notation in this)*

$$f_* := R\Gamma_{\Lambda_Y} \circ f_* \circ i_{\Lambda_X}, \;\; f^! := R\Gamma_{\Lambda_Y} \circ f^! \circ i_{\Lambda_Y}$$

The next definitions are important building blocks for the theory of descent with support in [BNP17a].

Definition 19. *[BNP17a, Definition 2.3.5] A* strict Cartesian diagram of pairs *is a Cartesian diagram of quasi-smooth stacks*

$$\begin{tikzcd}
(Z = X \times_S X', \Lambda_Z) \arrow[r, "p_2"] \arrow[d, "p_1"] & (X', \Lambda'_X) \arrow[d, "q"] \\
(X, \Lambda_X) \arrow[r, "p"] & (Y, \Lambda_Y)
\end{tikzcd}$$

which also satisfies the strictness condition

$$\Lambda_Z \supseteq p_1^! \Lambda_X \cap p_2^! \Lambda'_X.$$

Remark 20. *If p_1, p_2 are also quasi-smooth, which will be true in all cases we study, the strictness condition is equivalent to p_1 being strict along $p_2^! \Lambda'_X$ and p_2 being strict along $p_1^! \Lambda_X$.*

In such diagrams we restore natural properties like base change on the underlying categories with restricted support, see [BNP17a, §2.3]. This is important for the definition of augmented simplicial diagrams, which basically means all possible Cartesian diagrams are strict, and we usually want stacks to be smooth and maps proper - these allow us to compute the categorical trace by identifying the terms in a relative bar resolution.

Let Δ denote the simplex category of totally ordered non-empty finite sets, and Δ^+ the augmented category of possibly empty finite sets.

Definition 21. *A simplicial diagram of pairs is a functor $\Delta^{op} \to \mathrm{Pair}$ where* Pair *denotes the obvious category of pairs with morphisms. An augmented simplicial diagram is a functor $\Delta^{+,op} \to \mathrm{Pair}$. We denote a simplicial diagram of pairs as $(X^\bullet, \Lambda^\bullet)$ as is usual in these settings.*

We will call an (augmented) simplicial diagram nice *if*

1. *All X_n are quasi-smooth*
2. *Any map $[m] \to [n]$ induces a strict Cartesian diagram of pairs.*
3. *The categories $\mathrm{QC}^!_{\Lambda_n}(X_n)$ are compactly generated.*
4. *If it is an augmented diagram, we ask that the last map is conservative.*

For nice augmented simplicial diagrams, we can use [BNP17a, Corollary 2.4.2] to provide an isomorphism between the resolution $|\mathrm{QC}^!_{\Lambda^\bullet}(Z^\bullet)|$ and the augmentation $\mathrm{QC}^!_{\Lambda_{-1}}(Z_{-1})$. This process is called *descent with support*, and the support conditions are necessary for base change arguments to be true - in Lurie language, these are called "adjointability conditions".

We will apply it first to study the convolution of $\mathrm{QC}^!(Z)$. There is a similar convolution of closed conical subsets where $\Lambda_{12} * \Lambda_{23}$ is defined via the same diagram.

Proposition 22. *Convolution defines an equivalence of categories*

$$\mathrm{QC}^!_{\Lambda_{12}}(Z_{12}) \otimes_{\mathrm{QC}^!(Z_{22})} \mathrm{QC}^!_{\Lambda_{23}}(Z_{23}) \xrightarrow{\cong} \mathrm{QC}^!_{\Lambda_{13}}(Z_{13})$$

Proof. For convenience, let $M = \mathrm{QC}^!_{\Lambda_{12}}(Z_{12}), N = \mathrm{QC}^!_{\Lambda_{23}}(Z_{23}), A = \mathrm{QC}^!(Z_{22}), B = \mathrm{QC}^!_{\Lambda_{13}}(Z_{13})$. We write

$$M \otimes_A N \cong M \otimes_A A \otimes_A N$$

and we resolve A as an $A \otimes_B A^{rv}$ module via the relative bar complex.

We define

$$q_n : Z_n := X_1 \times_Y X_2^{n+1} \times_Y X_3 \to Z_{12} \times Z_{22}^n \times Z_{23} =: W_n$$

by the projections, and $\Lambda_n := q_n^!(\Lambda_{12} \boxtimes (\mathbb{T}^{*[-1]}_{Z_{22}})^n \boxtimes \Lambda_{23})$.

Then, we have

$$M \otimes_A N = \mathrm{colim}(\mathrm{QC}^!_{\Lambda_n}(Z_n)).$$

The augmented simplicial diagram $(Z^\bullet, \Lambda^\bullet) \to (Z_{13}, \Lambda_{13})$ is nice, by definition in the other terms and for the augmentation by definition of Λ_{13}.

Therefore, by [BNP17a, Corollary 2.4.2], this colimit is $\mathrm{QC}^!_{\Lambda_{13}}(Z_{13})$. $\square$

3. Categorical trace of a convolution category

From now on let $f : X \to Y$ to be a proper morphism of smooth, QCA stacks over k, and $Z = X \times_Y X$. Let also $\phi_X : X \to X$, $\phi_Y : Y \to Y$ be representable proper self-maps commuting with f, inducing a proper self-map $\phi : Z \to Z$.

We have the *derived fixed points* $\mathcal{L}_\phi X := X \times_{X\times X} X$, where the derived fiber product is taken over the diagonal $\Delta : X \to X \times X$ and the graph map $\Gamma_\phi : X \to X \times X$. This construction is functorial. The special case $\phi = id$ is called the derived loop space, and it can also be written

$$\mathcal{L}(X) = \mathrm{Map}(S^1, X) \cong X \times_{X\times X} X$$

by thinking of S^1 as two points connected by two line segments.

We want to prove the following [Ben+22, Theorem 3.23].

Let

$$Z = X \times_Y X \xleftarrow{\delta} \mathcal{L}_\phi Y_X = Z \times_{X\times X} X \cong X \times_{Y\times X} X \xrightarrow{\pi} \mathcal{L}_\phi Y,$$

and we define $\Lambda_\phi := \delta_* \pi^! \mathbb{T}^{*[-1]}_Z$

Theorem 23. *We have*

$$\mathrm{Tr}(\mathrm{QC}^!(Z), \phi_*) \cong \mathrm{QC}^!_{\Lambda_\phi}(\mathcal{L}_\phi Y),$$

and

$$[\mathrm{QC}^!(Z), \phi_*] \cong \mathcal{L}_\phi f_* \omega_{\mathcal{L}_\phi X}.$$

For $\phi = id$, this is [BNP17a, Theorem 3.3.1]. It is extended for general ϕ in [Ben+22, Appendix A2]. We give a rough sketch of the ideas involved in this proof.

Proof. Let $\mathcal{A} := \mathrm{QC}^!(Z)$, $\mathcal{B} = \mathrm{Perf}(X)$ and $\mathcal{M}$ the $\mathcal{B} \otimes B$-module defined by the graph map Γ_ϕ. The diagonal map also makes $\mathcal{A}$ an algebra in $\mathcal{B}$-bimodules. This comes from the fact that both maps are proper by our assumptions so pushforward is monoidal.

We will compute $\mathrm{Tr}(\mathrm{QC}^!(Z), \phi_*)$ via the relative bar resolution $|\mathcal{A}^{\otimes_B(\bullet+2)}|$. We denote by $\mathcal{A}^{\phi_*}$ the same category where the left action is twisted by ϕ_*.

$$\mathcal{C}^\bullet = |\mathcal{A}^{\otimes_B(\bullet+2)}| \otimes_{\mathcal{A}\otimes\mathcal{A}^{rv}} \otimes \mathcal{A}^{\phi_*} = |\mathcal{A}^{\otimes_B(\bullet+1)} \otimes_{\mathcal{B}\otimes\mathcal{B}} \otimes \mathcal{M}|,$$

(I am not sure why the second equality is true)

Let

$$Z^\bullet := X^{\bullet+1} \times_Y \mathcal{L}_\phi Y,\ W^\bullet = (X \times_Y X)^\bullet \times \mathcal{L}_\phi X.$$

The complex $Z^\bullet \to \mathcal{L}_\phi Y$ is the Cech nerve [GR17, §2.2] of $Z_0 = X \times_Y \mathcal{L}_\phi Y \to \mathcal{L}_\phi Y$.

If $\phi = id$, this can be more intuitively written as

$$Z_n \cong \mathrm{Map}([n] \hookrightarrow S^1, X \to Y) \cong \mathrm{Map}([n], S^1) \times_{\mathrm{Map}([n],Y)} \mathrm{Map}(S^1, Y).$$

We have natural maps $q_n : Z_n \to W_n$, obtained by taking relative diagonals.

The fully faithful map of simplicial diagrams $\mathcal{C}^\bullet \to \mathrm{QC}^!(Z^\bullet)$,

has essential image $\mathrm{QC}^!_{\Lambda^\bullet}(Z^\bullet)$ where $\Lambda_n := q_n^!(\mathbb{T}^{*[-1]}_{W_n})$.

Since $(Z^\bullet, \Lambda^\bullet) \to (\mathcal{L}_\phi Y, \Lambda_\phi)$ is an augmented simplicial diagram satisfying all the assumptions, we just have to show it is nice.

The augmented simplicial diagram is the Cech nerve of $X \times_Y \mathcal{L}_\phi Y \to \mathcal{L}_\phi Y$, so the fact that the diagrams are cartesian is immediate, and because this map is a base change of the proper map $x \to Y$, we get the properness and quasi-smoothness of all face maps.

For strictness, see the concrete computations in [BNP17a, Proposition 3.3.8].

Therefore, by [BNP17a, Corollary 2.4.2] we have the identification of the categorical trace.

For the identification of the Springer sheaf: The monoidal unit of $\mathrm{QC}^!(Z)$ is $\omega_\Delta := \Delta_*\omega_X$, where $\Delta : X \to X \times_Y X = Z$ is the relative diagonal.

$$\begin{tikzcd}
\mathcal{L}_\phi X \arrow[r, "\tilde{\Delta}"] \arrow[d, "p"] & \mathcal{L}_\phi Y^X \arrow[r, "\pi"] \arrow[d, "\delta"] & \mathcal{L}_\phi Y \\
X \arrow[r, "\Delta"] & Z = X \times_Y X &
\end{tikzcd}$$

and we have $[\omega_\Delta] = \pi_*\delta^!\Delta_*\omega_X$, which by base change is $\mathcal{L}_\phi f_*\omega_{\mathcal{L}_\phi X} =: \mathcal{S}_\phi$. □

Then, the first sentence in the following theorem, which we will prove implies the rest.

Theorem 24. *[Ben+22, Theorem 3.25] The convolution category* $\mathrm{QC}^!(Z)$ *is a rigid monoidal* ∞*-category . In particular, Theorem 7 applies, and we have that the vertical trace of* $\mathrm{QC}^!(Z)$ *is identified as an algebra with the endomorphisms of the* Springer sheaf *(universal trace sheaf)* $\mathcal{S}_\phi := \mathcal{L}_\phi f_*\omega_{\mathcal{L}_\phi X} \in \mathrm{QC}^!(\mathcal{L}_\phi Y)$, *therefore*

$$\mathrm{HH}(\mathrm{QC}^!(Z), \phi_*) \cong End_{\mathrm{QC}^!(\mathcal{L}_\phi Y)}(\mathcal{S}_\phi)$$

Proof. To be written, but if everything else is set up correctly, it follows from the definition of the monoidal structure and the fact that ω_Δ is compact, upon checking that integral transforms coming from compact/coherent kernels preserve compactness - this comes I think from the fact that compact objects commute with colimits... □

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