THE MANIFOLD-VALUED DIRICHLET PROBLEM FOR SYMMETRIC DIFFUSIONS

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Abstract. Harmonic maps between two Riemannian manifolds M and N are often constructed as energy minimizing maps. This construction is extended for the Dirichlet problem to the case where the Riemannian energy functional on M is replaced by a more general Dirichlet form. We obtain weakly harmonic maps and prove that these maps send the diffusion to N-valued martingales. The basic tools are the reflected Dirichlet space and the stochastic calculus for Dirichlet processes.

Key-words. Harmonic maps, Energy minimizing maps, Stochastic calculus on manifolds, Dirichlet forms, Dirichlet processes.

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1 Introduction

An harmonic map h between two Riemannian manifolds M and N is a smooth map satisfying some partial differential equation linked with the Laplacian Δ_M on M and the Riemannian metric on N. More precisely, for $y \in N$, we can use the exponential map $\exp_y$ and its inverse $\exp_y^{-1}$ which is defined on a neighbourhood of y. If $h : M \to N$ is C^2, then

$$h_x : z \mapsto \exp_{h(x)}^{-1} h(z)$$

is defined and C^2 on a neighbourhood of x, and takes its values in the tangent space $T_{h(x)}(N)$; thus the Laplacian Δ_M acts on it and we can define

$$\Delta_M^N h(x) = \Delta_M h_x(x) \in T_{h(x)}(N).$$

This map is called the tension field of h, and h is said to be harmonic if $\Delta_M^N h = 0$. The problem of constructing harmonic maps has been widely studied in the literature, see for instance [4, 9] or the lectures in [7]. In particular, if $\overline{M} = M \cup \partial M$ is a manifold with boundary ∂M, the Dirichlet problem consists in finding a map h which is harmonic on M, and which converges to a prescribed function g on ∂M. If

$$dh(x) : T_x(M) \to T_{h(x)}(N)$$

is the derivative of h and if μ is the Riemannian measure on M, one can consider the energy functional

$$\mathcal{E}_N(h) = \frac{1}{2} \int |dh(x)|^2 \mu(dx).$$

If $(h_\varepsilon; \varepsilon \in \mathbb{R})$ is a smooth family of maps, then

$$\frac{d}{d\varepsilon} \mathcal{E}_N(h_\varepsilon) = - \int \Big(\frac{d}{d\varepsilon} h_\varepsilon(x), \Delta_M^N h_\varepsilon(x) \Big) \mu(dx),$$

so the harmonic maps appear to be the critical points of $\mathcal{E}_N$. Thus a classical method for solving the Dirichlet problem is to look for a map which is energy minimizing in the class of maps converging to g; this map is a critical point for the energy functional, but it is not necessarily smooth (it is in the Sobolev space $W^{1,2}$); it is said to be weakly harmonic; then one has to study its smoothness in order to prove that it is harmonic. This program has been completed in [8] when the boundary condition g has a small image (see the survey [4] for other results).

Our aim is firstly to extend the construction of energy minimizing maps when the Laplacian Δ_M is replaced by a more general symmetric operator, and secondly to study the probabilistic properties of these maps; the problem of the smoothness is studied in [17]. More precisely, the manifold M is replaced by a locally compact space endowed with a local Dirichlet form $\mathcal{E}$. The main tool is the reflected Dirichlet space; this space was introduced in [20], and was further studied in [3]. We extend it to maps $h : M \to N$, and define the energy $\mathcal{E}_N(h)$. The ellipticity of the Laplacian Δ_M is replaced here by an absolute continuity condition on the semigroup associated to $\mathcal{E}$, and it appears that this condition is sufficient for the existence of energy minimizing maps. In particular, we obtain weakly harmonic maps. Then we can begin the stochastic analysis of the problem; we consider the diffusion associated to $\mathcal{E}$; harmonic maps send the diffusion to martingales on N, and we prove that weakly harmonic maps have the same property except on a set of zero capacity; they are said to be quasi harmonic. The main probabilistic tool is the stochastic calculus for Dirichlet processes which has been worked out in [12, 11]. By using the results of [17], one can then deduce under some conditions that these quasi harmonic maps are smooth, so they are harmonic in the strong sense.

Notice that another stochastic construction of harmonic maps can be deduced from the existence of a martingale on N with prescribed limit, see [10, 16, 1]; this method does not require the symmetry of the diffusion or any smoothness on the boundary condition g; however, one has to assume a strong convexity condition on the image of g (the map g has to take its values in a set with convex geometry). The advantage of the variational approach is to get rid of this convexity condition; the convexity is then only used for the smoothness.

We first introduce the framework, and state the main existence theorem for the Dirichlet problem in Section 2, in the case where N is complete. This theorem is proved in two steps; we solve the variational problem in Section 3, and the probabilistic analysis involving Dirichlet processes is worked out in Section 4. Finally, in Section 5, we explain how the convexity can be used in order to consider non complete manifolds N.

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2 The main result

Let N be a C^∞ separable Riemannian manifold which is viewed as a Riemannian submanifold of the Euclidean space $\mathbb{R}^n$; more precisely, if N is complete, we suppose that N is closed in $\mathbb{R}^n$, and in the general case we suppose that its closure $\overline{N}$ is a subset of a smooth manifold $\tilde{N}$ with the same dimension. Nevertheless, we will verify that our results do not depend on the choice of this isometric embedding of N into $\mathbb{R}^n$.

If F is a C^2 function on N, then its Hessian is at y a symmetric bilinear form on $T_y(N) \times T_y(N)$, and F is said to be convex if its Hessian is non negative. We can also consider the notion of N-valued continuous martingales (we rely on [5] for the stochastic calculus on manifolds and do not list the original papers in which the results appeared); actually, since we do not use non continuous martingales, we will omit the word "continuous".

Definition 1 *Let (Y_t) be a N-valued adapted process.*

1. *We say that $(Y_t; 0 \leq t \leq \infty)$ is a martingale if it is a continuous semimartingale such that the process*

$$\Lambda_t^F = F(Y_t) - \frac{1}{2}\int_0^t \operatorname{Hess} F(Y_s)(dY_s, dY_s) \tag{1}$$

   is a local martingale for any C^∞ real-valued function F.

2. *If σ is an optional time, we say that (Y_t) is a martingale on $[0, \sigma]$ if the process stopped at σ is a martingale.*

3. *If τ is a predictable time, we say that (Y_t) is a martingale on $[0, \tau)$ if it is a martingale on $[0, \sigma]$ for any optional time $\sigma \leq \tau$ such that $\sigma < \tau$ on $\{\tau > 0\}$.*

The N-valued martingales can also be characterized locally as the continuous adapted processes which are transformed into submartingales by real-valued convex C^2 functions. This characterization has useful consequences; if τ is a predictable time, a martingale on $[0,\tau)$ is a martingale on $[0,\tau]$ if and only if it is left continuous at time τ; if τ is the increasing limit of optional times σ_k, one can also check that (Y_t) is a martingale on $[0,\tau)$ if and only if it is a martingale on each $[0,\sigma_k]$.

Let us now consider the space M. We adopt the framework of [6] with moreover a carré du champ operator ([2]). Let M be a separable locally compact space which is endowed with a Radon measure μ with support M. Let $\mathcal{E}$ be a regular strongly local Dirichlet form on $L^2(\mu)$ with domain $\mathbb{D}$, and which is transient; in the Beurling-Deny formula, the strong locality means that there is no jump and no killing inside M; we suppose that $\mathcal{E}$ admits a carré du champ Γ defined on $\mathbb{D}\times\mathbb{D}$, so that

$$\mathcal{E}(\phi,\psi)=\frac{1}{2}\int\Gamma(\phi,\psi)d\mu. \tag{2}$$

We will often use the short notation $\Gamma(f)$ or $\mathcal{E}(f)$ for $\Gamma(f,f)$ and $\mathcal{E}(f,f)$. The space $\mathbb{D}$ endowed with the form

$$\mathcal{E}_1(\phi,\psi)=\mathcal{E}(\phi,\psi)+\int\phi\,\psi\,d\mu \tag{3}$$

is a Hilbert space. We will need the reflected Dirichlet space $\mathbb{D}^r$ (see [20, 3]); it consists of the functions, the truncations of which are locally in $\mathbb{D}$ and have bounded energy. Then Γ and $\mathcal{E}$ can be extended to $\mathbb{D}^r$. Notice that the functions of $\mathbb{D}^r$ have a quasi continuous modification.

The diffusion associated to $\mathcal{E}$ can be realized on the space Ω of M-valued continuous paths with finite or infinite lifetime and which diverge to infinity; this means that if $\omega\in\Omega$ and if $\zeta(\omega)\in(0,\infty]$ is its lifetime, then $\omega(t)$ is M-valued and continuous on $[0,\zeta(\omega))$, $\omega(t)$ quits all the compact subsets of M as $t\uparrow\zeta(\omega)$, and $\omega(t)=\partial\notin M$ (a cemetery point) for $t\geq\zeta(\omega)$. Let X_t be the canonical process with filtration $\mathcal{F}_t$; the time ζ is predictable. Let $(\mathbb{P}^x; x\in M)$ be a family of probabilities on Ω such that $X_0=x$ almost surely under $\mathbb{P}^x$. We suppose that $(\Omega,\mathcal{F}_t,X_t;\mathbb{P}^x,x\in M)$ is the symmetric diffusion associated to $(\mathbb{D},\mathcal{E})$; we also consider the σ-finite measure

$$\mathbb{P}^\mu=\int\mathbb{P}^x\mu(dx).$$

If f is quasi continuous, then $(f(X_t);t\geq 0)$ is $\mathbb{P}^\mu$ almost surely continuous.

Definition 2 *Let h be a map from M into N. We say that h is quasi harmonic if $h(X_t)$ is a $\mathbb{P}^\mu$ martingale on $[0, \zeta)$.*

We now introduce the Dirichlet problem which is the main purpose of this work. If (θ_t) is the shift operator on Ω, then a random variable U is said to be terminal if $U \circ \theta_t = U$ on $\{t < \zeta\}$.

Definition 3 *Let U be a N-valued terminal variable. We say that $h : M \to N$ is a quasi solution of the Dirichlet problem with terminal condition U if $h(X_t)$ is a $\mathbb{P}^\mu$ martingale converging almost surely to U as $t \uparrow \zeta$. This means that the process*

$$Y_t = h(X_t) \text{ on } \{t < \zeta\}, \quad U \text{ on } \{t \geq \zeta\} \tag{4}$$

is a $\mathbb{P}^\mu$ martingale on $[0, \infty]$.

Our main assumption on $\mathcal{E}$ will be an absolute continuity condition on the associated semigroup; we say that the process X_t satisfies the absolute continuity assumption at $x \in M$ if

$$\mu(B) = 0 \Longrightarrow \mathbb{P}^x[X_t \in B] = 0 \tag{5}$$

for $t > 0$. We will suppose that this condition holds for almost any x (this is needed for Lemma 1 below).

Let Q be the set of points x such that the process Y_t of (4) is a $\mathbb{P}^x$ martingale. Definition 3 says that X_t lives $\mathbb{P}^\mu$ almost surely in Q (otherwise, from the optional section theorem, there would exist an optional time σ such that X_σ is in Q^c with positive $\mathbb{P}^\mu$ measure, and therefore $Y_{\sigma+t}$ would not be a martingale), so Q^c has zero capacity. This is a justification for the prefix "quasi" of our terminology. If moreover (5) holds for any x, then Q^c is polar, and the process $(Y_t; t > 0)$ is for any x a $\mathbb{P}^x$ martingale. We can deduce that x is in Q if and only if h is finely continuous at x (this means that $h(X_t)$ converges $\mathbb{P}^x$ almost surely to $h(x)$ as $t \downarrow 0$). If $Q = M$, we can say that h is a solution of the Dirichlet problem. In this work, we only construct quasi solutions; a condition for the fine continuity is given in [17].

Remark. Suppose that $\mathcal{E}$ is irreducible; we have assumed that it is transient; if this is not the case, then U is a constant under $\mathbb{P}^\mu$, and this constant is a trivial quasi solution of the Dirichlet problem.

Example 1. Let $\tilde{M}$ be a C^∞ manifold and let μ be a measure with positive C^∞ density with respect to the Lebesgue measure on each local chart. If $(\Xi_i; 1 \leq i \leq q)$ are C^∞ vector fields, we can consider the operator

$$\Gamma(\phi, \psi) = \sum_{i=1}^{q} (\Xi_i \phi)(\Xi_i \psi) \tag{6}$$

and the form $\tilde{\mathcal{E}}$ associated by (2). A core for this form is the space of C^∞ functions with compact support. We can close this form and obtain a Dirichlet space $\tilde{\mathbb{D}}$. The associated diffusion $\tilde{X}_t$ under $\mathbb{P}^x$ can be realized on the Wiener space as the solution up to its explosion time $\tilde{\zeta}$ of the Stratonovich equation

$$d\tilde{X}_t^x = \Xi_0(\tilde{X}_t^x)dt + \sum_{i=1}^{q} \Xi_i(\tilde{X}_t^x) \circ dW_t^i, \qquad \tilde{X}_0^x = x,$$

with

$$\Xi_0 = \frac{1}{2} \sum_{i=1}^{q} (\mathrm{div}_\mu \Xi_i) \Xi_i.$$

If now M is an open subset of $\tilde{M}$, one can let $(\mathbb{D}, \mathcal{E})$ be the part of $(\tilde{\mathbb{D}}, \tilde{\mathcal{E}})$ on M; this is the subset of functions, a quasicontinuous modification of which is zero on $\tilde{M} \setminus M$ (Section 4.4 of [6]). If $\tilde{\mathcal{E}}$ is irreducible and $\tilde{M} \setminus M$ has positive capacity, then $\mathcal{E}$ is transient. The process X_t is $\tilde{X}_t$ absorbed (or killed) at the first exit time ζ of M; if we consider $g : \partial M \to N$ and $y \in N$, then the variable

$$U = g(\tilde{X}_\zeta) = g(X_{\zeta-}) \text{ on } \{\zeta < \tilde{\zeta}\}, \qquad y \text{ on } \{\zeta = \tilde{\zeta}\}$$

is terminal. Thus we obtain the classical Dirichlet problem with boundary condition g on ∂M, and y at infinity. If $\zeta < \tilde{\zeta}$, there is no condition at infinity; this is for instance the case when M is relatively compact in $\tilde{M}$, or when $\tilde{\mathcal{E}}$ is recurrent. If the Lie algebra generated by the vector fields $(\Xi_i; 1 \leq i \leq q)$ is at x the whole tangent space $T_x(M)$ (this is the Hörmander condition), then the absolute continuity condition (5) holds at x. In good cases, the reflected Dirichlet space $\mathbb{D}^r$ can be identified to a regular Dirichlet form on $\overline{M}$, and the associated process is the classical reflected diffusion (see the end of [3]).

Example 2. If $\tilde{M}$ is a Riemannian manifold, if μ is the Riemannian volume measure, we can consider the operator

$$\Gamma(\phi, \psi) = (d\phi(x), d\psi(x)).$$

The associated diffusion $\tilde{X}_t$ is the Brownian motion on $\tilde{M}$. This example can enter the framework of Example 1 by considering $\tilde{M}$ as a submanifold of $\mathbb{R}^q$, and by letting $(\Xi_i(x))$ be the orthogonal projection of the canonical basis of $\mathbb{R}^q$ on the tangent space $T_x(M)$; then $\Xi_0 = 0$. This process is elliptic, so it satisfies (5) everywhere. If now μ has a smooth density u with respect to the Riemannian measure and Γ is unchanged, then $\tilde{X}_t$ is the Brownian motion with drift $\Xi_0 = \nabla u/(2u)$ on $\{u > 0\}$; this is a Nelson diffusion; it can be extended to $\tilde{M}$ by letting $\{u = 0\}$ be a trap for the process, and (5) holds on $\{u > 0\}$. If $\tilde{M}$ is a Euclidean space, this construction has been extended to some non smooth u such that $u > 0$ almost everywhere for the Lebesgue measure (see [13, 21]). If $\sqrt{u}$ is in the Sobolev space $W^{1,2}$ one obtains a recurrent Dirichlet form on $\tilde{M}$; the diffusion is absolutely continuous with respect to the Wiener measure, so (5) holds almost everywhere.

Example 3. If $\tilde{M}$ is a C^∞ finite dimensional Lie group with Lie algebra $\mathcal{M}$, if μ is a right invariant measure, and if $(\Lambda_i; 1 \leq i \leq q)$ are elements of $\mathcal{M}$, we can consider the left invariant vector fields Ξ_i associated to Λ_i and the operator (6); then $\Xi_0 = 0$ and the Hörmander condition is satisfied when the Lie algebra generated by $(\Lambda_i; 1 \leq i \leq q)$ is $\mathcal{M}$; in this case, the diffusion $\tilde{X}_t$ is a left invariant hypoelliptic Brownian motion (or continuous Lévy process) on $\tilde{M}$; it has infinite lifetime.

Let $\mathbb{D}^r(\mathbb{R}^n)$ be the set of $\mathbb{R}^n$-valued functions $f = (f_1, \ldots, f_n)$ such that each f_j is in the reflected space $\mathbb{D}^r$; for these functions we can put

$$\Gamma(\phi, \psi) = \sum_{j=1}^{n} \Gamma(\phi_j, \psi_j), \qquad \mathcal{E}(\phi, \psi) = \sum_{j=1}^{n} \mathcal{E}(\phi_j, \psi_j).$$

We let $\mathbb{D}^r(N)$ be the subset of functions which are N-valued, and introduce the energy functional $\mathcal{E}_N(f) = \mathcal{E}(f)$.

Proposition 1 *The space $\mathbb{D}^r(N)$ and the energy functional $\mathcal{E}_N$ do not depend on the embedding of N as a submanifold of $\mathbb{R}^n$.*

Proof. First consider the case when N is compact. Let J be an isometry between two copies N_1 and N_2 of N in $\mathbb{R}^n$. Let P be the orthogonal projection onto N_1; then J can be extended to a smooth Lipschitz function of $\mathbb{R}^n$ into itself such that $J = J \circ P$ on a neighbourhood V of N_1. Consider a map f of

$\mathbb{D}^r(N_1)$; the space $\mathbb{D}^r$ is stable by the action of Lipschitz transformations, so $J \circ f$ is in $\mathbb{D}^r(N_2)$. Moreover, J is a contraction on V, so

$$\Gamma(J \circ f) \leq \Gamma(f),$$

and therefore

$$\mathcal{E}_{N_2}(J \circ f) \leq \mathcal{E}_{N_1}(f).$$

The reverse inequality is obtained by a symmetric argument, so the result is proved. In the general non compact case, one can again consider a smooth extension of the function J (this follows from our assumptions on the embedding of N, see the beginning of the section), but this extension is only locally Lipschitz. Consider the sequence of functions

$$F_k(x) = ((|x| \wedge (2k - |x|)) \vee 0)\, x/|x|$$

on $\mathbb{R}^n$; then F_k is a contraction, has compact support, and is the identity on the ball of radius k. The function $F_k \circ J$ is Lipschitz, so $F_k \circ J \circ f$ is in $\mathbb{D}^r(\mathbb{R}^n)$ and

$$\mathcal{E}(F_k \circ J \circ f) \leq \mathcal{E}_{N_1}(f).$$

If $i \geq k$, one has

$$\begin{aligned}
\mathcal{E}(F_i \circ J \circ f - F_k \circ J \circ f) &= \frac{1}{2}\int_{\{|J\circ f| \geq k\}} \Gamma(F_i \circ J \circ f - F_k \circ J \circ f) d\mu \\
&\leq \int_{\{|J\circ f| \geq k\}} \Gamma(F_i \circ J \circ f) d\mu \\
&\qquad + \int_{\{|J\circ f| \geq k\}} \Gamma(F_k \circ J \circ f) d\mu \\
&\leq 2\int_{\{|J\circ f| \geq k\}} \Gamma(f) d\mu
\end{aligned}$$

because $F_k \circ J$ and $F_i \circ J$ are contractions on a neighbourhood of N_1. This expression converges to 0, so $(F_k \circ J \circ f)$ is a Cauchy sequence for the seminorm $\mathcal{E}$. Since it converges to $J \circ f$, we deduce that $J \circ f$ is in $\mathbb{D}^r(N_2)$ (this is a consequence of Theorem 3.3 of [3]), and

$$\mathcal{E}_{N_2}(J \circ f) = \lim \mathcal{E}(F_k \circ J \circ f) \leq \mathcal{E}_{N_1}(f).$$

Thus we can again conclude. □

If U is the terminal condition, we define $\mathbb{D}^r_U(N)$ as the subset of quasi continuous functions f of $\mathbb{D}^r(N)$ such that $f(X_t)$ converges $\mathbb{P}^\mu$ almost surely to U as $t \uparrow \zeta$. We are now ready to state the main result of this work when N is complete. The non complete case will be considered in Theorem 2.

Theorem 1 *Let $\mathcal{E}$ be a regular strongly local transient Dirichlet form on $L^2(M, \mu)$ with a carré du champ; we suppose that $\mathcal{E}$ satisfies the absolute continuity assumption (5) for almost any x. Let N be a complete separable Riemannian manifold. Let U be a N-valued terminal variable, and suppose that $\mathbb{D}_U^r(N)$ is not empty. Then the Dirichlet problem with terminal condition U has a quasi solution in $\mathbb{D}_U^r(N)$.*

We have assumed that $\mathbb{D}_U^r(N)$ is not empty, so let us comment this assumption. A necessary condition is that $\mathbb{D}_U^r(\mathbb{R}^n)$ should be non empty; this is the case when U is in the universal Dirichlet space ([20, 3]), so that the harmonic function $g_0(x) = \mathbb{E}^x[U]$ is well defined and in $\mathbb{D}_U^r(\mathbb{R}^n)$. In the framework of the above examples with $U = g(X_{\zeta-})$, this condition holds when the boundary condition g is in the domain of the trace of $\tilde{\mathcal{E}}$ on $\tilde{M} \setminus M$ (Theorem 4.6.5 of [6]). If moreover U takes its values in a closed subset N_0 of N and if N_0' is the closed convex hull of N_0, then the function g_0 is in $\mathbb{D}_U^r(N_0')$. Thus, if there exists a Lipschitz projection of N_0' onto N_0, then $\mathbb{D}_U^r(N_0)$ is not empty, and therefore $\mathbb{D}_U^r(N)$ is not empty. This can be applied for instance when N_0 is a closed subset of the sphere $N = S^{n-1}$ with $N_0 \neq N$. However, $\mathbb{D}_U^r(N)$ may be empty when this Lipschitz projection does not exist. Consider for instance the case where X_t is the Brownian motion on the unit ball M of $\mathbb{R}^2$, let N be the unit circle, and let $U = g(X_{\zeta-})$ with g the identity map; if $\mathbb{D}_U^r(N)$ is not empty, then the analytical theory of [18, 19] shows that g can be extended to a weakly harmonic map which is continuous on $\overline{M}$ (because M has dimension 2); but g has no continuous extension, so $\mathbb{D}_U^r(N)$ is empty in this case. Notice however that $h(x) = x/|x|$ is a quasi solution of the Dirichlet problem ; moreover, since $N = S^1$, there are many solutions (because one can easily construct martingales on the circle with prescribed final value, see [15]), but these solutions have infinite energy.

Theorem 1 will be proved in two steps. We first prove the existence of an energy minimizing map (Proposition 3); then we prove that it is a quasi solution (Proposition 4).

3 Potential analysis

We first adapt to our framework the classical notion of weakly harmonic maps which are critical points for the energy functional.

Definition 4 *Consider a map h of $\mathbb{D}^r(N)$.*

1. *Let $(h_\varepsilon; \varepsilon \in \mathbb{R})$ be a family of maps of $\mathbb{D}^r(N)$ such that $h_0 = h$. We say that (h_ε) is a good perturbation of h if $h_\varepsilon(x) = h(x)$ when $(x, h(x))$ is outside a compact set, and $(h_\varepsilon - h)/\varepsilon$ converges in $(\mathbb{D}^r(\mathbb{R}^n), \mathcal{E}) \cap L^\infty$ as $\varepsilon \to 0$.*
2. *We say that h is weakly harmonic if the derivative of $\mathcal{E}_N(h_\varepsilon)$ at $\varepsilon = 0$ is 0 for any good perturbation (h_ε).*

This definition has been stated in order to be intrinsic.

Proposition 2 *The notion of good perturbation, and therefore of weakly harmonic map, does not depend on the embedding of N in $\mathbb{R}^n$.*

Proof. Let J be an isometry between two copies N_1 and N_2 of N, which is extended to a smooth function of $\mathbb{R}^n$ into itself as in the proof of Proposition 1. Let h be in $\mathbb{D}^r(N_1)$ and (h_ε) be a good perturbation; thus

$$h_\varepsilon - h = \varepsilon\, f + o(\varepsilon) \tag{7}$$

in $\mathbb{D}^r(\mathbb{R}^n) \cap L^\infty$. Notice that f is bounded, has compact support, so it is in the Dirichlet space $\mathbb{D}(\mathbb{R}^n)$; notice also that $f(x) = 0$ when $h(x)$ is outside a compact set. If dJ is the derivative of J, we can deduce that the function $((dJ) \circ h, f)$ is in $\mathbb{D}^r(\mathbb{R}^n) \cap L^\infty$, and the proposition will be proved if we check that

$$J \circ h_\varepsilon - J \circ h = \varepsilon((dJ) \circ h, f) + o(\varepsilon)$$

in $\mathbb{D}^r(\mathbb{R}^n) \cap L^\infty$. From (7), it is sufficient to check that

$$J \circ h_\varepsilon - J \circ h - ((dJ) \circ h, h_\varepsilon - h) = o(\varepsilon). \tag{8}$$

The left hand side can be written in the form $F(h, h_\varepsilon - h)$, where $F(y, z)$ is a smooth function with compact support such that $F(y, z) = O(|z|^2)$ as $z \to 0$. The estimation (8) in L^∞ is easy. For the estimation in Dirichlet seminorm, we use the estimates

$$\partial_y F(y, z) = O(|z|^2), \qquad \partial_z F(y, z) = O(|z|)$$

and deduce

$$\Gamma(F(h, h_\varepsilon - h)) \le C|h_\varepsilon - h|^4\Gamma(h) + C|h_\varepsilon - h|^2\Gamma(h_\varepsilon - h).$$

By integrating with respect to μ, we obtain

$$\begin{aligned}\mathcal{E}(F(h, h_\varepsilon - h)) &\le C\|h_\varepsilon - h\|_\infty^4\mathcal{E}(h) + C\|h_\varepsilon - h\|_\infty^2\mathcal{E}(h_\varepsilon - h)\\ &= O(\varepsilon^4) = o(\varepsilon^2)\end{aligned}$$

so (8) is proved. □

The aim of this section is to prove the following result.

Proposition 3 *Under the conditions of Theorem 1, there exists a map h which minimizes $f \mapsto \mathcal{E}_N(f)$ in $\mathbb{D}_U^r(N)$. This map is weakly harmonic.*

We have to find an energy minimizing map; then the fact that it is weakly harmonic is evident because the good perturbations do not modify the terminal condition and are therefore in $\mathbb{D}_U^r(N)$. A basic step for the proof of this proposition is the following extension of the classical Rellich compactness theorem; it is due to [22] and is reproduced here with the kind permission of its author.

Lemma 1 *Assume that (5) holds almost everywhere and let (f_k) be a sequence of functions which is bounded in $(\mathbb{D}, \mathcal{E}_1)$ defined in (3). Then there exists a subsequence which converges almost everywhere.*

Proof. Consider the sequence $(\arctan f_k)$; it is also bounded in $(\mathbb{D}, \mathcal{E}_1)$, and if it has a converging subsequence, then the subsequence of (f_k) will also converge from the boundedness in $L^2(\mu)$. Thus we can suppose that (f_k) is uniformly bounded. Let ν be a probability measure on M with a positive bounded density with respect to μ; it is sufficient to find a subsequence converging in $L^1(\nu)$. If P_t is the semigroup associated to $\mathcal{E}$, the property

$$\begin{aligned}\frac{d}{dt}\|P_t f - f\|_{L^2(\mu)}^2 &= 2\mathcal{E}(P_t f, f) - 2\mathcal{E}(P_t f, P_t f)\\ &\le 2\mathcal{E}(P_{t/2} f, P_{t/2} f) \le 2\mathcal{E}(f, f)\end{aligned}$$

implies

$$\|P_t f_k - f_k\|_{L^2(\mu)}^2 \le 2\mathcal{E}(f_k)t \le C\,t,$$

so the approximation of f_k by $P_t f_k$ shows that it is sufficient to prove that for any $t > 0$, the sequence $(P_t f_k)$ has a converging subsequence. If $p(t, x, z)$ is the transition density of P_t which exists for almost any x, then

$$P_t f_k(x) = \int f_k(z) p(t, x, z) \mu(dz).$$

Bounded subsets of $L^\infty(\mu)$ are sequentially relatively compact for the topology $\sigma(L^\infty, L^1)$, so (f_k) has a subsequence $(f_{k(j)})$ converging for this topology, and therefore $P_t f_{k(j)}(x)$ converges for almost any x; since it is uniformly bounded, we deduce that $P_t f_{k(j)}$ converges in $L^1(\nu)$. □

Consider the subspace $\mathbb{D}_0^r$ of functions of $\mathbb{D}^r$ with terminal condition 0; this is the extended Dirichlet space; actually, $\mathbb{D}^r$ is the $\mathcal{E}$-orthogonal sum of $\mathbb{D}_0^r$ and of the space of harmonic functions $\mathbb{E}^x[U]$, for U in the universal Dirichlet space ([20, 3]). In particular, the transience implies that $(\mathbb{D}_0^r, \mathcal{E})$ is a Hilbert space, and that

$$\int |f(x)| \overline{\mu}(dx) \leq \mathcal{E}(f)^{1/2} \tag{9}$$

for any $f \in \mathbb{D}_0^r$ and for a measure $\overline{\mu}$ such that μ and $\overline{\mu}$ are mutually absolutely continuous (Theorem 1.5.3 of [6]).

Lemma 2 *Let (f_k) be a sequence which is bounded in $(\mathbb{D}_0^r, \mathcal{E})$. Then there exists a subsequence $(f_{k(j)})$ which converges μ-almost everywhere to a function f of $\mathbb{D}_0^r$ such that*

$$\mathcal{E}(f) \leq \liminf \mathcal{E}(f_k). \tag{10}$$

Proof. Let ϕ be a bounded function of $\mathbb{D}$ with compact support; then the sequence $(\phi \arctan f_k)$ is bounded in $(\mathbb{D}, \mathcal{E}_1)$, and has therefore a converging subsequence from Lemma 1; moreover, for any x of M, one can choose ϕ so that it is 1 in a neighbourhood of x, so there exists a subsequence $(f_{k(j)})$ such that $(\arctan f_{k(j)})$ converges almost everywhere; the sequence $(f_{k(j)})$ is bounded in $L^1(\overline{\mu})$ from (9), so it also converges almost everywhere to a function f. On the other hand, the space $(\mathbb{D}_0^r, \mathcal{E})$ is a Hilbert space, so one can choose the subsequence so that it converges weakly to some $\widetilde{f}$ in $\mathbb{D}_0^r$ satisfying (10). There exists a convex combination of $(f_{k(j)})$ which converges almost everywhere to f, and strongly in $\mathbb{D}_0^r$ to $\widetilde{f}$; from (9), it converges also in $L^1(\overline{\mu})$ to $\widetilde{f}$, so $f = \widetilde{f}$ almost everywhere and the lemma is proved. □

Proof of Proposition 3. Consider the harmonic function $g_0(x) = \mathbb{E}^x[U]$ which is in $\mathbb{D}^r_U(\mathbb{R}^n)$; then g_0 is $\mathcal{E}$-orthogonal to $\mathbb{D}^r_0(\mathbb{R}^n)$. Consider also a sequence h_k in $\mathbb{D}^r_U(N)$ such that

$$\lim \mathcal{E}(h_k) = \inf\Big\{\mathcal{E}(f);\ f \in \mathbb{D}^r_U(N)\Big\}.$$

The sequence $(h_k - g_0)$ is bounded in $(\mathbb{D}^r_0(\mathbb{R}^n), \mathcal{E})$, so Lemma 2 says that there exists a subsequence $(h_{k(j)} - g_0)$ converging almost everywhere to some function of $\mathbb{D}^r_0(\mathbb{R}^n)$, that we write in the form $h - g_0$, and which satisfies

$$\mathcal{E}(h - g_0) \leq \liminf \mathcal{E}(h_k - g_0). \tag{11}$$

Thus $(h_{k(j)})$ converges almost everywhere to h and h is in $\mathbb{D}^r_U(\mathbb{R}^n)$. The functions $h_{k(j)}$ take their values in N and N is closed, so there exists a quasi continuous modification of h taking its values in N. The function h is therefore in $\mathbb{D}^r_U(N)$. Moreover, the functions $h_k - g_0$ and $h - g_0$ are $\mathcal{E}$-orthogonal to g_0, so

$$\mathcal{E}(h_k) = \mathcal{E}(h_k - g_0) + \mathcal{E}(g_0), \qquad \mathcal{E}(h) = \mathcal{E}(h - g_0) + \mathcal{E}(g_0),$$

and we deduce from (11) that

$$\mathcal{E}(h) \leq \liminf \mathcal{E}(h_{k(j)}) = \inf\Big\{\mathcal{E}(f); f \in \mathbb{D}^r_U(N)\Big\}.$$

Thus h is energy minimizing. □

4 Stochastic analysis

The aim of this section is to prove the following result which will complete the proof of Theorem 1. It is not necessary to suppose that N is complete in this section.

Proposition 4 *Assume the conditions of Theorem 1, except that N is not necessarily complete. Let h be a quasi continuous map of $\mathbb{D}^r(N)$. Then h is quasi harmonic if and only if it is weakly harmonic.*

We will need the following reduction of the problem.

Lemma 3 *The proof of Proposition 4 can be reduced to the case where the diffusion is conservative ($\zeta = \infty$).*

Proof. We are going to slow down the diffusion so that it does not explode. Let G_k be relatively compact open subsets of M such that $\overline{G}_k \subset G_{k+1}$ and $G_k \uparrow M$, and let τ_k be the first exit time of G_k. Consider the function

$$\rho(x) = \sum_k \rho_k 1_{G_k \setminus G_{k-1}}(x)$$

where $\rho_k > 0$ is chosen sufficiently large so that

$$\mathbb{E}^\mu \Big[1_{G_k}(X_0) \exp - \int_0^{\tau_k} \rho(X_s) ds \Big] \leq 1/k.$$

By letting $k \uparrow \infty$, it appears that

$$A_t = \int_0^t \rho(X_s) ds$$

diverges $\mathbb{P}^\mu$ almost surely as $t \uparrow \zeta$. Consider now the change of time associated to the additive functional A_t (Section 6.2 of [6]); then the new diffusion has infinite lifetime. The measure μ is replaced by the measure μ_1 with density ρ (this is the Revuz measure of A_t), and the carré du champ is divided by ρ, so the values of $\mathcal{E}$ are not modified. The new Dirichlet space is $\mathbb{D}_0^r \cap L^2(\mu_1)$, and the reflected Dirichlet space $\mathbb{D}^r$ is not modified. The sets of weakly harmonic and quasi harmonic maps are also unchanged. □

The main probabilistic tool for the proof of Proposition 4 is the stochastic calculus for Dirichlet processes which has been worked out for conservative symmetric diffusions in [12, 11]; this calculus is based on a decomposition of Dirichlet processes used in the construction of Nelson diffusions ([13]), and enables to study a stochastic integral which was previously introduced in [14]. Functions of $\mathbb{D}$ or $\mathbb{D}^r$ have a quasi continuous modification, so we will always choose such a modification. Consider the measure $\mathbb{P}^\mu$ on the subset of Ω consisting of paths with infinite lifetime; it is reversible. If ψ is in $\mathbb{D}$, then the additive functional $\psi(X_t) - \psi(X_0)$ is the sum of a martingale additive functional and of an additive functional with zero energy; this means that $\psi(X_t)$ is a Dirichlet process (the sum of a martingale and of a process with zero quadratic variation), but it has also another interesting decomposition. More precisely, for any fixed $T > 0$, there exist a square integrable martingale

$(Z_t; t \leq T)$ for the filtration of X_t and a square integrable martingale $(Z^\star_t; t \leq T)$ for the filtration of X_{T-t} such that $Z_0 = Z^\star_0 = 0$ and

$$\psi(X_t) - \psi(X_0) = \frac{1}{2} Z_t - \frac{1}{2}(Z^\star_T - Z^\star_{T-t}). \tag{12}$$

Moreover

$$\langle Z, Z \rangle_t = \int_0^t \Gamma(\psi)(X_s) ds$$

and $\langle Z^\star, Z^\star \rangle_t$ satisfies a similar relation, so

$$\mathbb{E}^\mu \langle Z, Z \rangle_T = \mathbb{E}^\mu \langle Z^\star, Z^\star \rangle_T = 2T\, \mathcal{E}(\psi). \tag{13}$$

This decomposition made possible in [12, 11] the definition of a stochastic integral of bounded processes $\phi(X_t)$ with respect to the process $\psi(X_t)$; the symmetric (or Stratonovich) integral is defined from Ito integrals by

$$\int_0^t \phi(X_s) \circ d\psi(X_s) = \frac{1}{2}\Big(\int_0^t \phi(X_s) dZ_s - \int_{T-t}^T \phi(X_{T-s}) dZ^\star_s \Big). \tag{14}$$

The result does not depend on T, and from (13), it is in $L^2(\mathbb{P}^\mu)$. More precisely,

$$\begin{aligned} \mathbb{E}^\mu \Big[\sup_{t \leq T} \Big| \int_0^t \phi(X_s) \circ d\psi(X_s) \Big|^2 \Big] &\leq C\, \mathbb{E}^\mu \int_0^T \phi(X_s)^2 \Gamma(\psi)(X_s) ds \\ &\leq 2C\, T \|\phi\|_\infty^2 \mathcal{E}(\psi). \end{aligned}$$

If ϕ has compact support, this relation enables to extend the integral to functions ψ in $\mathbb{D}^r$; one indeed first defines the integral for functions ψ which are locally in $\mathbb{D}$ (by considering a function in $\mathbb{D}$ which coincides with ψ on the support of ϕ), and one notices that these functions are dense in $\mathbb{D}^r$ for $\mathcal{E}$.

Consider again a bounded function ϕ with compact support, and suppose moreover that it is in $\mathbb{D}$; we want to prove that the integral is in $L^1(\mathbb{P}^\mu)$ (recall that μ is not finite, so $L^1(\mathbb{P}^\mu)$ is not included in $L^2(\mathbb{P}^\mu)$). Since ϕ is in $\mathbb{D}$, it has a decomposition similar to (12), and the analogue of (13) enables to show that

$$\mathbb{E}^\mu \Big[\sup_{s \leq t} |\phi(X_s) - \phi(X_0)|^2 \Big] \leq C\, t\, \mathcal{E}(\phi),$$

so

$$\mathbb{E}^\mu \Big[\sup_{s \leq t} |\phi(X_s)|^2 \Big] \leq C(1+t) \mathcal{E}_1(\phi).$$

Thus, from the Burkholder-Davis-Gundy inequalities,

$$\begin{aligned}\mathbb{E}^\mu\Big|\int_0^t \phi(X_s)dZ_s\Big| &\leq C\mathbb{E}^\mu\Big[\sup_{s\leq t}|\phi(X_s)|^2\Big]^{1/2}\mathbb{E}^\mu\Big[\langle Z,Z\rangle_t\Big]^{1/2} \\ &\leq C'(t+\sqrt{t})\mathcal{E}_1(\phi)^{1/2}\mathcal{E}(\psi)^{1/2}.\end{aligned}$$

Moreover, since Z is a martingale, the $\mathbb{P}^\mu$ expectation of this integral is 0. The integral with respect to $Z^\star$ satisfies a similar estimate, so the integral of (14) is in $L^1(\mathbb{P}^\mu)$ and

$$\mathbb{E}^\mu\int_0^t \phi(X_s)\circ d\psi(X_s) = 0,$$

$$\mathbb{E}^\mu\Big|\int_0^t \phi(X_s)\circ d\psi(X_s)\Big| \leq C(t+\sqrt{t})\mathcal{E}_1(\phi)^{1/2}\mathcal{E}(\psi)^{1/2}.$$

If we again suppose that ϕ is bounded, in $\mathbb{D}$ and has compact support, then $\phi(X_t)$ is a Dirichlet process, and the processes $\phi(X_t)$ and $\psi(X_t)$ have a quadratic covariation given by

$$\langle\phi(X),\psi(X)\rangle_t = \int_0^t \Gamma(\phi,\psi)(X_s)ds.$$

We can define the forward (or Ito) integral by

$$\int_0^t \phi(X_s)d\psi(X_s) = \int_0^t \phi(X_s)\circ d\psi(X_s) - \frac{1}{2}\langle\phi(X),\psi(X)\rangle_t. \tag{15}$$

It is in $L^1(\mathbb{P}^\mu)$ and satisfies

$$\mathbb{E}^\mu\int_0^t \phi(X_s)d\psi(X_s) = -t\,\mathcal{E}(\phi,\psi), \tag{16}$$

$$\mathbb{E}^\mu\Big|\int_0^t \phi(X_s)d\psi(X_s)\Big| \leq C(t+\sqrt{t})\mathcal{E}_1(\phi)^{1/2}\mathcal{E}(\psi)^{1/2}. \tag{17}$$

The symmetric and forward integrals can also be defined as limits of Riemann sums (see Theorem 2.3.1 of [11] for the symmetric integral, the forward integral is obtained by discretizing the quadratic variation). In particular, if $\psi(X_t)$ is a semimartingale, then the forward integral coincides with the classical Ito integral.

Lemma 4 *Let L be the generator of the diffusion X_t in $L^2(\mu)$. Let ℓ be a bounded function which is in $L^2(\mu)$, and let*

$$\ell_1(x) = \lambda(\lambda - L)^{-1}\ell(x) = \lambda\,\mathbb{E}^x \int_0^\infty e^{-\lambda t}\ell(X_t)dt$$

for some $\lambda > 0$. For $\phi \in \mathbb{D}$ bounded with compact support, and ψ in $\mathbb{D}^r$, one has

$$\int_0^\infty e^{-\lambda t}\mathbb{E}^\mu\Big[\ell(X_0)\int_0^t \phi(X_s)d\psi(X_s)\Big]dt$$
$$= \int_0^\infty e^{-\lambda t}\mathbb{E}^\mu\Big[\int_0^t \ell_1(X_s)\phi(X_s)d\psi(X_s)\Big]dt. \tag{18}$$

Proof. First notice that ℓ_1 is bounded and in $\mathbb{D}$, so $\ell_1\phi$ is also bounded, in $\mathbb{D}$, and it has compact support. The stochastic integrals involved in the lemma are therefore well defined, and the two sides of (18) have a sense from (17). If ψ is in the domain of L, then $\psi(X_t)$ is a semimartingale and

$$\begin{aligned}\mathbb{E}^\mu\Big[\ell(X_0)\int_0^t \phi(X_s)d\psi(X_s)\Big] &= \mathbb{E}^\mu\Big[\ell(X_0)\int_0^t \phi(X_s)L\psi(X_s)ds\Big] \\ &= \mathbb{E}^\mu\Big[\ell(X_t)\int_0^t \phi(X_s)L\psi(X_s)ds\Big]\end{aligned}$$

from the reversibility of the process (X_t) under $\mathbb{P}^\mu$. Thus

$$\begin{aligned}&\int_0^\infty e^{-\lambda t}\mathbb{E}^\mu\Big[\ell(X_0)\int_0^t \phi(X_s)d\psi(X_s)\Big]dt \\ &\quad = \mathbb{E}^\mu\int_0^\infty \mathbb{E}\Big[\int_s^\infty e^{-\lambda t}\ell(X_t)dt \;\Big|\; X_s\Big]\phi(X_s)L\psi(X_s)ds \\ &\quad = \lambda^{-1}\mathbb{E}^\mu\int_0^\infty \ell_1(X_s)\phi(X_s)L\psi(X_s)e^{-\lambda s}ds\end{aligned}$$

which is the right-hand side of (18). If now ψ is in $\mathbb{D}_0^r$, we can approximate it in $(\mathbb{D}_0^r, \mathcal{E})$ by functions in the domain of L; from (17), the two sides of (18) are robust with respect to this approximation, so (18) again holds. Finally, if ψ is only in $\mathbb{D}^r$, we can replace it by a function of $\mathbb{D}_0^r$ taking the same values on the support of ϕ, and both sides of (18) are unchanged. □

We now introduce the stochastic integrals for manifold-valued processes. If h is in $\mathbb{D}^r(N)$ and if f is a bounded Borel function such that $f(x)$ is in the tangent space $T_{h(x)}(N)$ (which is embedded in $\mathbb{R}^n$), we can consider its adjoint $f^\star(x) \in T^\star_{h(x)}(N)$ with respect to the Riemannian metric, and we define the symmetric integral by

$$\int_0^t f^\star(X_s) \circ dh(X_s) = \sum_{j=1}^n \int_0^t f_j(X_s) \circ dh_j(X_s).$$

One can actually verify that this stochastic integral does not depend on the embedding of N in $\mathbb{R}^n$ (this is because it satisfies the rules of the classical differential calculus, see [14] for a particular case). If moreover f is in $\mathbb{D}(\mathbb{R}^n)$ and has compact support, then we can consider the quadratic covariation $\langle f(X), h(X)\rangle$ (the sum of the quadratic covariations of their components), and the forward integral can be defined as in (15). Properties (16) and (17) are of course also satisfied.

Lemma 5 *If $h \in \mathbb{D}^r(N)$ is quasi harmonic, then it is weakly harmonic.*

Proof. Let (h_ε) be a good perturbation of h, and let f be the limit of $(h_\varepsilon - h)/\varepsilon$; then f is bounded, has compact support, is in $\mathbb{D}(\mathbb{R}^n)$, and moreover $f(x)$ is in the tangent space $T_{h(x)}(N)$. The derivative of the energy $\mathcal{E}_N(h_\varepsilon)$ at $\varepsilon = 0$ is $2\mathcal{E}(f,h)$, so we have to prove that $\mathcal{E}(f,h) = 0$. We can consider the forward integral of $f^\star(X_t)$ with respect to $h(X_t)$, and (16) becomes

$$\mathbb{E}^\mu \int_0^t f^\star(X_s) dh(X_s) = -t\mathcal{E}(f,h).$$

It is therefore sufficient to prove that the $\mathbb{P}^x$ expectation of the forward integral is 0 for almost any x. But $h(X_t)$ is a semimartingale in $\mathbb{R}^n$, so the forward integral is an Ito integral computed in $\mathbb{R}^n$; by using the Riemannian structure of N, it is also the Ito integral of the form $f^\star(X_t)$ along the martingale $h(X_t)$, so it is a local martingale (see (7.35) in [5]); it is square integrable for almost any x (because it is in $L^2(\mathbb{P}^\mu)$), so it has zero expectation. □

Proof of Proposition 4. A part of the proof has been made in Lemma 5, so we now have to prove that if h is a weakly harmonic map, then $Y_t = h(X_t)$ is a $\mathbb{P}^\mu$ martingale. Thus we consider smooth real-valued functions F on N and prove that $F(Y_t)$ is a semimartingale (this implies that Y_t is a semimartingale), and that the process Λ^F_t of (1) is a local martingale. It is sufficient to consider functions F with compact support. We apply the above construction of stochastic integrals with respect to $h(X)$. We choose functions f of the form

$$f(x) = -\psi(x)(\nabla F \circ h)(x) \tag{19}$$

where ψ is a bounded function with compact support which is in $\mathbb{D}$, and ∇F is the gradient of F computed for the Riemannian metric (it is the adjoint of the derivative dF); then f is in $\mathbb{D}(\mathbb{R}^n)$ and $f(x)$ is in $T_{h(x)}(N)$. Notice that F can be extended to a smooth function with compact support on $\mathbb{R}^n$ which, in a neighbourhood of N, is the composition of F with the orthogonal projection on N; then, the gradient of F computed in $\mathbb{R}^n$ at $y \in N$ can be identified with $\nabla F(y)$; similarly, the Hessian matrix of F computed in $\mathbb{R}^n$ at $y \in N$ coincides on $T_y(N) \times T_y(N)$ with the Hessian bilinear form $\operatorname{Hess} F$ associated to the Riemannian metric. We consider the perturbation of h defined by

$$\frac{d}{d\varepsilon} h_\varepsilon(x) = -\psi(x)(\nabla F \circ h_\varepsilon)(x), \qquad h_0(x) = h(x). \tag{20}$$

This equation has a N-valued solution for $\varepsilon \in \mathbb{R}$ because ∇F is Lipschitz and the perturbation only acts on a compact part of N; the function $h_\varepsilon(x)$ is a Lipschitz function of $\psi(x)$ and $h(x)$, so h_ε is in $\mathbb{D}^r(N)$; notice also that $h_\varepsilon(x) = h(x)$ when x is outside the support of ψ, or $h(x)$ is outside the support of F. Moreover

$$\begin{aligned} h_\varepsilon(x) &= h(x) - \psi(x) \int_0^\varepsilon (\nabla F \circ h_\eta)(x) d\eta \\ &= h(x) + \varepsilon f(x) - \psi(x) \int_0^\varepsilon (\nabla F \circ h_\eta - \nabla F \circ h)(x) d\eta \\ &= h(x) + \varepsilon f(x) - \psi(x) \int_0^\varepsilon G_\eta(\psi(x), h(x)) d\eta \end{aligned}$$

where $G_\eta(u, y)$ is a C^1 function on $[-C, C] \times N$ (C is an upper bound of $|\psi|$) such that $G_\eta(0, y) = 0$. Since F is smooth with compact support, one can check that G_η and its first order derivatives converge uniformly to 0 as $\eta \to 0$; this implies

$$h_\varepsilon - h = \varepsilon f + o(\varepsilon)$$

in $(\mathbb{D}^r(\mathbb{R}^n), \mathcal{E}) \cap L^\infty$. Thus (h_ε) is a good perturbation of h. The map h is weakly harmonic, so $\mathcal{E}(f, h)$ is zero. Thus, for these functions f, the relation (16) becomes

$$\mathbb{E}^\mu \int_0^t f^\star(X_s) dh(X_s) = 0.$$

If ℓ is a real-valued bounded function of $L^2(\mu)$, and if ℓ_1 is defined as in Lemma 4 for some $\lambda > 0$, we can replace $f(x)$ by $\ell_1(x) f(x)$ which has the same form (replace ψ by $\psi\ell_1$ which is bounded and in $\mathbb{D}$); thus

$$\mathbb{E}^\mu \int_0^t \ell_1(X_s) f^\star(X_s) dh(X_s) = 0.$$

We deduce from Lemma 4 that

$$\mathbb{E}^\mu \Big[\ell(X_0) \int_0^t f^\star(X_s) dh(X_s)\Big] = 0$$

because it is continuous in t and its Laplace transform is 0. We can deduce from this relation and the Markov property that the process $\int_0^t f^\star(X_s)dh(X_s)$ is a $\mathbb{P}^\mu$ martingale. On the other hand, an application of Ito's formula for Dirichlet processes (see [12, 11]) shows that

$$(F \circ h)(X_t) - (F \circ h)(X_0) = \int_0^t (dF \circ h)(X_s) \circ dh(X_s).$$

Consider relatively compact open subsets G_j of M such that $G_j \uparrow M$; the regularity of $\mathcal{E}$ implies the existence of bounded non negative functions ψ_j in $\mathbb{D}$, with compact support, and such that $\psi_j = 1$ on $\overline{G}_j$; the function f_j associated to ψ_j by (19) is equal to $(-\nabla F \circ h)$ on $\overline{G}_j$, so

$$\begin{aligned} (F \circ h)(X_t) - (F \circ h)(X_0) &= -\int_0^t f_j^\star(X_s) \circ dh(X_s) \\ &= -\int_0^t f_j^\star(X_s) dh(X_s) + V_t^F \end{aligned} \quad (21)$$

up to the first exit time τ_j of G_j, with

$$V_t^F = -\frac{1}{2}\langle f_j(X), h(X)\rangle_t = \frac{1}{2}\langle \nabla F \circ h(X), h(X)\rangle_t.$$

In particular, the process $(F \circ h)(X_t)$ is a semimartingale on $[0, \tau_j]$ and its finite variation part if V_t^F. Thus $h(X_t)$ is a N-valued semimartingale on $[0, \tau_j]$, and we can apply the classical Ito formula

$$\nabla F \circ h(X_t) = \nabla F \circ h(X_0) + \int_0^t (\mathrm{Hess}\, F \circ h)(X_s) \circ dh(X_s)$$

to obtain

$$V_t^F = \frac{1}{2} \int_0^t (\mathrm{Hess}\, F \circ h)(X_s)(dh(X_s), dh(X_s)) \quad (22)$$

up to τ_j. This expression proves that the process Λ_t^F of (1) is a martingale on $[0, \tau_j]$; thus $h(X_t)$ is a martingale on the same time interval. By letting $j \to \infty$, one has $\tau_j \uparrow \zeta$, so $h(X_t)$ is actually a martingale on $[0, \zeta)$. $\square$

5 The non complete case

We want to extend Theorem 1 to the case where N is not complete. Recall that in this case we have supposed that $\overline{N}$ is embedded in a manifold $\tilde{N}$ with the same dimension. We are going to make a convexity assumption on N.

Theorem 2 *Assume the conditions of Theorem 1 except the completeness of N. Suppose that N is relatively compact in $\tilde{N}$. Suppose also that there exists a smooth function γ on $\tilde{N}$ such that γ is convex on N and $N = \{\gamma < 0\}$. Then the Dirichlet problem has again a quasi solution in $\mathbb{D}^r_U(N)$.*

Example. This theorem can be applied to the manifolds

$$N = S^{n-1}_\varepsilon = \{z \in \mathbb{R}^n;\ |z| = 1,\ z_n > \varepsilon\}, \qquad \varepsilon \geq 0$$

up to the open hemisphere S^{n-1}_0.

The problem is to prove the existence of an energy minimizing map in $\mathbb{D}^r_U(N)$; then this map will be weakly harmonic, and Proposition 4 will show that it is a quasi solution. The method of Proposition 3 can be used to prove the existence of an energy minimizing map in $\mathbb{D}^r_U(\overline{N})$. Thus Theorem 2 is a consequence of the following result.

Lemma 6 *If h is a quasi continuous energy minimizing map in $\mathbb{D}^r_U(\overline{N})$, then h has a quasi modification taking its values in N.*

Proof. We can let γ be constant outside a compact subset of $\tilde{N}$. Suppose that we have proved that $(\gamma \circ h)(X_t)$ is a $\mathbb{P}^\mu$ submartingale on $[0, \zeta)$; then it converges to $\gamma(U)$ as $t \uparrow \zeta$, so

$$(\gamma \circ h)(x) \leq \mathbb{E}^x[\gamma(U)] < 0$$

almost everywhere; the set $\{h \in \overline{N} \setminus N\} = \{\gamma \circ h = 0\}$ is invariant for the process X_t (a non positive submartingale which hits 0 must remain at 0) and is negligible; thus it has zero capacity and the lemma holds. Let us now prove the submartingale property for the process $(\gamma \circ h)(X_t)$. We consider the conservative case, and apply the method used in the proof of Proposition 4 by choosing $F = \gamma$ on $\tilde{N}$. The perturbations h_ε of (20) corresponding to

$\psi = \psi_j \geq 0$ are in $\mathbb{D}^r_U(\tilde{N})$. The function $\varepsilon \mapsto (\gamma \circ h_\varepsilon)(x)$ is non increasing, and is decreasing except when $\varepsilon \mapsto h_\varepsilon(x)$ is constant; thus h_ε remains in $\mathbb{D}^r_U(\overline{N})$ for $\varepsilon \geq 0$. Since h minimizes the energy in this space, the derivative of $\mathcal{E}_N(h_\varepsilon)$ at $\varepsilon = 0$ is non negative, so $\mathcal{E}(f, h)$ is non negative. We deduce from (16) that

$$\mathbb{E}^\mu \int_0^t f^\star(X_s) dh(X_s) \leq 0,$$

and by proceeding as in Proposition 4 that

$$\mathbb{E}^\mu \Big[\ell(X_0) \int_0^t f^\star(X_s) dh(X_s) \Big] \leq 0$$

for non negative ℓ. Thus $\int_0^t f^\star(X_s) dh(X_s)$ is a supermartingale. The process $(\gamma \circ h)(X_t)$ stopped at $\tau_j \uparrow \zeta$ is therefore a semimartingale which is from (21) the sum of a submartingale and of V_t^γ given by (22); the process V_t^γ is non decreasing because γ is convex on $\overline{N}$, so $(\gamma \circ h)(X_t)$ is a submartingale. □

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