# Observers for populations dynamics

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# Outline

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# Model

$$\begin{cases} \partial_t p(a,x,t) + \partial_a p(a,x,t) \\ \quad = -\mu(a)p(a,x,t) + k\Delta p(a,x,t), & a \in (0,a^*),\ x \in \Omega,\ t > 0, \\ p(a,x,t) = 0, & a \in (0,a^*),\ x \in \partial\Omega,\ t > 0, \\ p(a,x,0) = p_0(a,x), & a \in (0,a^*),\ x \in \Omega, \\ p(0,x,t) = \displaystyle\int_0^{a^*} \beta(a)p(a,t,x)\,\mathrm{d}a, & x \in \Omega,\ t > 0. \end{cases} \tag{1}$$

- $p(a,x,t)$ denotes the distribution density of the population of age $a$ at spatial position $x$ at time $t$;
- $p_0$ denotes the initial distribution and $a^*$ is the maximal life expectancy;
- $\beta(a)$ and $\mu(a)$ are positive functions denoting respectively the birth and death rates (which are supposed to be independent of $x$);
- $\Omega$ is a smooth bounded domain, $k$ is a positive constant diffusion coefficient and $\Delta$ the laplacian with respect to $x$.

# Model

Typical assumptions on $\beta$ and $\mu$ are the following

- $\beta \in L^2(0,a^*) \cap L^\infty(0,a^*)$, $\beta \geqslant 0$ a.e. in $(0,a^*)$ ;
- $\mu \in L^1_{\mathrm{loc}}(0,a^*)$, $\mu \geqslant 0$ a.e. in $(0,a^*)$ and

$$\lim_{a\to a^*} \int_0^a \mu(s)\,\mathrm{d}s = +\infty.$$

We also introduce the function

$$\pi(a) := \exp\left(-\int_0^a \mu(s)\,\mathrm{d}s\right)$$

which represents the probability to survive at age $a > 0$.
Note that $\lim_{a\to a^*} \pi(a) = 0$.

# Goal

Assume that $p_0$ is unknown.

The aim of this talk is to design an observer for (1). More precisely, we want to recover the distribution density of the population $p(a,x,t)$ for arbitrary $a$ and $x$ and for $t$ large enough, from the knowledge of $p(a,x,t)$ for arbitrary $t$ but only for an age interval $(a_1,a_2)$ (where $0 \leqslant a_1 < a_2 \leqslant a^*$) and a subdomain $\mathcal{O} \subset \Omega$. In other words, the available output is $y(t) = p|_{(a_1,a_2)\times\mathcal{O}}$ where $t \in (0,T)$ ($T$ has to be chosen large enough).

# Some references on population dynamics

- Semigroup properties: Song et al., Chan, Guo, Li et al., Langlais, Walker
- Controllability problems: Ainseba, Anita, Iannelli, Langlais, Traoré, Kavian
- Inverse problems: Traoré, Rundell, Filin, Perasso, Picart
- Numerical aspects: Lopez, Trigiante, Douglas, Milner, Huyer, Guo, Gerardo-Giorda

# First order system

We introduce the Hilbert space $X := L^2((0,a^*)\times\Omega)$, and define the following unbounded operator $A$ of $X$:

$$\mathcal{D}(A) = \{\varphi \in X \mid \varphi(a,\cdot) \in H^2(\Omega)\cap H_0^1(\Omega) \text{ for almost all } a \in (0,a^*);$$

$$\varphi(0,x) = \int_0^{a^*} \beta(a)\varphi(a,x)\,\mathrm{d}a \text{ for almost all } x \in \Omega;$$

$$-\partial_a\varphi - \mu\varphi + k\Delta\varphi \in X\}$$

$$A\varphi = -\partial_a\varphi - \mu\varphi + k\Delta\varphi, \qquad \forall\varphi \in \mathcal{D}(A).$$

Then problem (1) reads in the compact form

$$\begin{cases} \dot{p}(t) := Ap(t), & t>0 \\ p(0) = p_0. \end{cases} \tag{2}$$

# Well-posedness result [Guo, Chan, 1994]

**Theorem**

*The operator $A$ is the infinitesimal generator of a $C_0$ semigroup $e^{tA}$ on $X$.*
*Consequently, if $p_0 \in X$, there exists a unique solution $p \in C([0,\infty), X)$ of (2), and if $p_0 \in \mathcal{D}(A)$, there exists a unique solution $p \in C([0,\infty), \mathcal{D}(A)) \cap C^1([0,\infty), X)$ of (2).*

# Outline

1. The model
2. Spectral properties
   - The diffusion free population dynamics
   - The population dynamics with diffusion
3. Detectability
4. Application : observer design for populations dynamics
5. Numerical approximation

# McKendrick–Von Foerster model

The diffusion free case is described by the so-called McKendrick–Von Foerster model (1959):

$$\begin{cases} \partial_t p(a,t) + \partial_a p(a,t) = -\mu(a) p(a,t), & a \in (0,a^*),\ t>0, \\ p(a,0) = p_0(a), & a \in (0,a^*), \\ p(0,t) = \displaystyle\int_0^{a^*} \beta(a) p(a,t)\,\mathrm{d}a, & t>0. \end{cases}$$

# First order system

The population operator $A_0$ corresponding to the above system is defined as follows

$$\mathcal{D}(A_0) = \left\{ \varphi \in H^1(0,a^*) \mid \varphi(0) = \int_0^{a^*} \beta(a)\varphi(a)\,\mathrm{d}a; \right.$$

$$\left. -\frac{\mathrm{d}\varphi}{\mathrm{d}a} - \mu\varphi \in L^2(0,a^*) \right\}.$$

$$A_0\varphi = -\frac{\mathrm{d}\varphi}{\mathrm{d}a} - \mu\varphi, \qquad \forall \varphi \in \mathcal{D}(A_0).$$

Then the McKendrick–Von Foerster model reads in the compact form

$$\begin{cases} \dot{p}(t) := A_0 p(t), & t > 0 \\ p(0) = p_0. \end{cases}$$

# Spectral properties [Song et al., 1982]

## Theorem

*The operator $A_0$ has compact resolvent and its spectrum is constituted of a countable (infinite) set of isolated eigenvalues with finite algebraic multiplicity. The eigenvalues $(\mu_n)_{n\geqslant 1}$ of $A_0$ (counted without multiplicity) are the solutions of the characteristic equation*

$$F(\mu) := \int_0^{a^*} \beta(a) e^{-\mu a} \pi(a) \, \mathrm{d}a = 1.$$

*The eigenvalues $(\mu_n)_{n\geqslant 1}$ are of geometric multiplicity one, the eigenspace associated to $\mu_n$ being the one-dimensional subspace of $L^2(0, a^*)$ generated by the function*

$$\psi_n(a) = e^{-\mu_n a} \pi(a) = e^{-\mu_n a - \int_0^a \mu(s) \, \mathrm{d}s}.$$

# Spectral properties [Song et al., 1982]

**Theorem**

*Finally, every vertical strip of the complex plane* $\alpha_1 \leqslant \mathrm{Re}\,(z) \leqslant \alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$, *contains a finite number of eigenvalues of* $A_0$.

# Spectral properties [Song et al., 1982]

**Theorem**

*The operator $A_0$ has a unique real eigenvalue $\mu_1$. Moreover, we have the following properties*

1. $\mu_1$ *is of algebraic multiplicity one;*
2. $\mu_1 > 0$ *(reps.* $\mu_1 < 0$*) if and only if* $F(0) > 1$ *(resp.* $F(0) < 1$*);*
3. $\mu_1$ *is a real dominant eigenvalue:*

$$\mu_1 > \mathrm{Re}\,(\mu_n), \forall n \geqslant 2.$$

# The population dynamics with diffusion

Recall:

$$\mathcal{D}(A) = \{\varphi \in X \mid \varphi(a,\cdot) \in H^2(\Omega) \cap H_0^1(\Omega) \text{ for almost all } a \in (0,a^*);$$

$$\varphi(0,x) = \int_0^{a^*} \beta(a)\varphi(a,x)\,\mathrm{d}a \text{ for almost all } x \in \Omega; -\partial_a\varphi - \mu\varphi + k\Delta\varphi \in X\}$$

$$A\varphi = -\partial_a\varphi - \mu\varphi + k\Delta\varphi, \qquad \forall\varphi \in \mathcal{D}(A).$$

Let $0 < \lambda_1^D < \lambda_2^D \leqslant \lambda_3^D \leqslant \cdots$ be the increasing sequence of eigenvalues of $-k\Delta$ with Dirichlet boundary conditions and let $(\varphi_n)_{n\geqslant 1}$ be a corresponding orthonormal basis of $L^2(\Omega)$.

Let $(\mu_n)_{n\geqslant 1}$ and $(\psi_n)_{n\geqslant 1}$ be respectively the sequence of eigenvalues and eigenfunctions of the free diffusion operator $A_0$.

# Spectral properties [Chan and Guo, 1989]

**Theorem**

1. *The operator $A$ has compact resolvent and its eigenvalues are*

$$\sigma(A) = \left\{ \mu_i - \lambda_j^D \,|i, j \in \mathbb{N}^* \right\}$$

2. *$A$ has a real dominant eigenvalue:*

$$\lambda_1 = \mu_1 - \lambda_1^D > \mathrm{Re}\,(\lambda), \forall \lambda \in \sigma(A),\ \lambda \neq \lambda_1.$$

*$\lambda_1$ is a simple eigenvalue, the corresponding eigenspace being generated by*

$$\Phi_1(a, x) := \psi_1(a)\varphi_1(x) = e^{-\mu_1 a}\pi(a)\varphi_1(x).$$

3. *The eigenspace associated to an eigenvalue $\lambda$ of $A$ is given by*

$$\text{Span} \left\{ \psi_i(a)\varphi_j(x) = e^{-\mu_i a}\pi(a)\varphi_j(x) \mid \mu_i - \lambda_j^D = \lambda \right\}.$$

# Spectrums

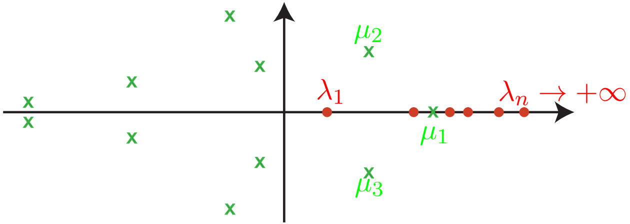



Figure: The spectrums of the free diffusion operator $A_0$ (green) and of $-k\Delta$ (red).

# Compactness [Chan and Guo, 1989]

## Proposition

*The semigroup $e^{tA}$ generated on $X$ by $A$ is compact for $t \geqslant a^*$.*

According to Zabczyk, this implies in particular that

$$\omega_a(A) = \omega_0(A)$$

where $\omega_a(A) := \lim_{t \to +\infty} t^{-1} \ln \|e^{tA}\|$ denotes the type of $e^{tA}$ and $\omega_0(A) := \sup \{\mathrm{Re}\,\lambda \mid \lambda \in \sigma(A)\}$ the spectral bound of $A$. It is worth noticing that the above condition ensures that the exponential stability of $e^{tA}$ is equivalent to the condition $\omega_0(A) < 0$.

# Outline

# Abstract framework

Let $A : \mathcal{D}(A) \to X$ be a linear operator with compact resolvent on a Hilbert space $X$ generating a $C_0$-semigroup in $X$, and let $C \in \mathcal{L}(X, Y)$, where $Y$ is another Hilbert space. We assume

**(A1.)** $A$ admits $M$ eigenvalues (counted without multiplicities) with real part greater than $0$. More precisely we can reorder the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ of $A$ so that the sequence $(\operatorname{Re}\lambda_n)_{n\in\mathbb{N}^*}$ is nonincreasing, and we suppose that $M \in \mathbb{N}^*$ is such that

$$\cdots \leqslant \operatorname{Re}\lambda_{M+1} < 0 \leqslant \operatorname{Re}\lambda_M \leqslant \cdots \leqslant \operatorname{Re}\lambda_2 \leqslant \operatorname{Re}\lambda_1.$$

**(A2.)** We have the equality

$$\omega_a(A) = \omega_0(A).$$

# Detectability

## Definition

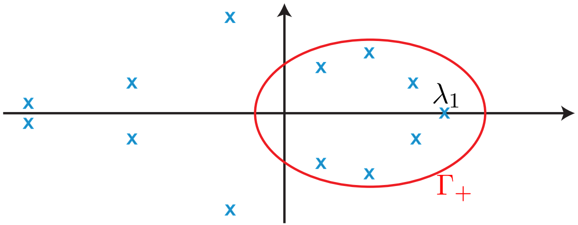
The pair $(A, C)$ is detectable if there exists $H \in \mathcal{L}(Y, X)$ such that $(A + HC)$ generates an exponentially stable semigroup. Such an operator $H$ is called a stabilizing output injection operator for $(A, C)$.

In this section, our goal is to show that the detectability of the infinite dimensional system $(A, C)$ can be deduced from the detectability of the finite dimensional system corresponding to the unstable part of $A$ (namely the projection on the finite dimensional space associated with the unstable eigenvalues $\lambda_1, \cdots, \lambda_M$) provided a Hautus type assumption is satisfied.

# Projector

We set $\Sigma_+ := \{\lambda_1, \ldots, \lambda_M\}$ and let $\Gamma_+$ be a positively oriented curve enclosing $\Sigma_+$ but no other point of the spectrum $\sigma(A)$ of $A$. Let $P_+ : X \to X$ be the projection operator defined by

$$P_+ := -\frac{1}{2\pi i} \int_{\Gamma_+} (\xi - A)^{-1} \, d\xi.$$



# Splitting

We set $X_+ := P_+ X$ and $X_- := (I - P_+)X$, and then $P_+$ provides the following decomposition of $X$

$$X = X_+ \oplus X_-.$$

Following [Triggiani, Raymond, Thevenet, Badra & Takahashi], we can decompose our system into two subsystems:

- a finite dimension one to be stabilized,
- a stable infinite dimensional one.

More precisely, $X_+$ and $X_-$ are invariant subspaces under $A$ (since $A$ and $P_+$ commute) and the spectra of the restricted operators $A\big|_{X_+}$ and $A\big|_{X_-}$ are respectively $\Sigma_+$ and $\sigma(A) \setminus \Sigma_+$. We also set

$$A_+ := A|_{\mathcal{D}(A)\cap X_+} : \mathcal{D}(A) \cap X_+ \to X_+,$$

$$A_- := A|_{\mathcal{D}(A)\cap X_-} : \mathcal{D}(A) \cap X_- \to X_-.$$

# Remark

The space $X_+$ is the finite dimensional space spanned by the generalized eigenfunctions of $A$ associated to the unstable eigenvalues:

$$X_+ = \bigoplus_{k=1}^{M} \text{Ker}\,(A - \lambda_k)^{m_k^{\mathrm{P}}}$$

where $m_k^{\mathrm{P}}$ is the multiplicity of the pole $\lambda_k$ in the resolvent. If $m_k^{\mathrm{A}} := \dim \text{Ker}\,(A - \lambda_k)^{m_k^{\mathrm{P}}}$ denotes the algebraic multiplicity of $\lambda_k$, $k = 1, \ldots, M$, then the dimension of $X_+$ is the sum of the algebraic multiplicities:

$$\dim X_+ = \sum_{k=1}^{M} m_k^{\mathrm{A}}.$$

If $A_+ := A|_{\mathcal{D}(A)\cap X_+}$ is diagonalizable, then $m^{\mathrm{G}}(\lambda_k) = m^{\mathrm{A}}(\lambda_k)$ for all positive eigenvalues $\lambda_k$ of $A$. This implies in particular that the unstable space is $X_+ = \bigoplus_{k=1}^{M} \text{Ker}\,(A - \lambda_k)$, since $m^{\mathrm{A}}(\lambda_k) = 1$, for all $k = 1, \ldots, M$.

# Result of detectability

**Theorem**

*Let $Q_+ : Y \to Y_+ := CX_+$ be the orthogonal projection operator from $Y$ to $Y_+$, $i_{X_+} : X_+ \to X$ be the injection operator from $X_+$ into $X$ and let*

$$C_+ = Q_+ C i_{X_+} \in \mathcal{L}(X_+, Y_+).$$

*Assume that the finite dimensional projected system $(A_+, C_+)$ is detectable through a stabilizing output injection operator $H_+ \in \mathcal{L}(Y_+, X_+)$. Then, the infinite dimensional system $(A, C)$ is detectable through the stabilizing output injection operator*

$$H = i_{X_+} H_+ Q_+ \in \mathcal{L}(Y, X). \tag{3}$$

# Proof: splitting

For $H \in \mathcal{L}(Y, X)$, consider the system

$$\dot{z}(t) = Az(t) + HCz(t). \tag{4}$$

If we write $z = z_+ + z_-$ where $z_+ := P_+ z$ and $z_- := (I - P_+)z$, by applying $P_+$ and $(I - P_+)$ to (4), there is a corresponding splitting of (4) into two equations satisfied by $z_+$ and $z_-$ respectively

$$\dot{z}_+(t) = A_+ z_+(t) + P_+ HCz(t) \quad \text{and} \; \dot{z}_-(t) = A_- z_-(t) + (I - P_+)HCz(t).$$

Using the definition $H = i_{X_+} H_+ Q_+$ and the facts that $P_+ i_{X_+} = \mathrm{Id}_{X_+}$ and $(I - P_+) i_{X_+} = 0$, we obtain

$$\dot{z}_+(t) = A_+ z_+(t) + H_+ Q_+ Cz(t) \quad \text{and} \quad \dot{z}_-(t) = A_- z_-(t).$$

# Proof: use of assumptions

It follows from assumption **(A2.)** that $z_-$ is exponentially stable:

$$\|z_-(t)\| \leqslant Ke^{-\omega_- t}\|z(0)\| \tag{5}$$

where $\omega_- = -\mathrm{Re}\,\lambda_{N+1} > 0$. On the other hand, by using the definition $C_+ = Q_+ C i_{X_+}$ and since $i_{X_+} z_+ = z_+$, we have

$$\begin{aligned} \dot{z}_+(t) &= A_+ z_+(t) + H_+ Q_+ C(z_+(t) + z_-(t)) \\ &= A_+ z_+(t) + H_+ Q_+ C i_{X_+} z_+(t) + H_+ Q_+ C z_-(t) \\ &= (A_+ + H_+ C_+) z_+(t) + H_+ Q_+ C z_-(t). \end{aligned}$$

# Proof: Duhamel's formula

Using Duhamel's formula, we get

$$z_+(t) = \mathbb{T}_t^+ z_+(0) + \int_0^t \mathbb{T}_{t-s}^+ H_+ Q_+ C z_-(s) ds,$$

where $\mathbb{T}_t^+$ is the semigroup generated by $(A_+ + H_+ C_+)$, which is exponentially stable by the detectability assumption, i.e. there exists $\omega_+ > 0$ such that

$$\left\| \mathbb{T}_t^+ x \right\| \leqslant K e^{-\omega_+ t} \left\| x \right\| \qquad \forall x \in X_+, \, \forall t > 0.$$

Combined with exponential stability of $z_-$, this yields

$$\left\| z_+(t) \right\| \leqslant K \left\{ e^{-\omega_+ t} \left\| z_+(0) \right\| + \left\| H_+ \right\| \left\| C \right\| \int_0^t e^{-\omega_+ (t-s)} e^{-\omega_- s} \left\| z_-(0) \right\| ds \right\},$$

and consequently

$$\left\| z_+(t) \right\| \leqslant K \left( e^{-\omega_+ t} + \left\| H_+ \right\| \left\| C \right\| \frac{e^{-\omega_+ t} - e^{-\omega_- t}}{\omega_- - \omega_+} \right) \left\| z_0 \right\|.$$

# End of the proof

It is then sufficient to choose $\omega_+$ small enough such that $0 < \omega_+ < \omega_-$ to have the exponential decay of $t \mapsto z_+(t)$:

$$\|z_+(t)\| \leqslant K e^{-\omega_+ t} \|z_0\|, \qquad t > 0. \tag{6}$$

Relations (5) and (6) yield immediately the exponential decay of $z = z_+ + z_-$. ■

# Hautus test

The following result provide a Hautus type sufficient condition for the detectability of the projected system $(A_+, C_+)$.

## Proposition

*If the Hautus test*

$$\left(\varphi \in \mathcal{D}(A) \mid A\varphi = \lambda\varphi \text{ for } \lambda \in \Sigma_+ \text{ and } C\varphi = 0\right) \quad \Longrightarrow \quad \varphi = 0$$

*is satisfied, then $(A_+, C_+)$ is detectable.*

Proof : Since $C_+ z_+ = C z_+$ for any $z_+ \in X_+$ by definition of $Y_+$ and $Q_+$, if the Hautus criterion is satisfied, then it is clear that the following Hautus test is also satisfied:

$$\left(\ \varphi \in \mathcal{D}(A) \cap X_+ \mid A_+\varphi = \lambda\varphi \text{ and } C_+\varphi = 0\ \right) \quad \Longrightarrow \quad \varphi = 0.$$

As the above system is finite dimensional, $(A_+, C_+)$ is detectable. ∎

# Remarks

Combining the Theorem and Proposition shows that $(A, C)$ is detectable via the stabilizing output injection operator $H$ defined as previously provided that the Hautus test is satisfied.

Classically, the stabilizing output injection operator $H^+$ of the finite dimensional projected system $(A^+, C^+)$ can be determined by solving a (finite dimensional) Riccati equation.

It is worth noticing that the matrices $A^+$ and $C^+$ are in practice of small size (their dimensions are respectively $\dim X_+ \times \dim X_+$ and $\dim Y_+ \times \dim X_+$), making the computation of the Riccati matrix affordable.

# Outline

# Goal

Going back to the population system (1)

$$\begin{cases} \partial_t p(a,x,t) + \partial_a p(a,x,t) = -\mu(a)p(a,x,t) + k\Delta p(a,x,t), & a \in (0,a^*), x \in \Omega, t > \\ p(a,x,t) = 0, & a \in (0,a^*), x \in \partial\Omega, t \\ p(a,x,0) = p_0(a,x), & a \in (0,a^*),\ x \in \Omega, \\ p(0,x,t) = \displaystyle\int_0^{a^*} \beta(a)p(a,t,x)\,\mathrm{d}a, & x \in \Omega,\ t > 0, \end{cases}$$

our aim is to design an observer for (1).

More precisely, we want to recover the distribution density of the population $p(a,x,t)$ for arbitrary $a$ and $x$ and for $t$ large enough, from the knowledge of $p(a,x,t)$ for arbitrary $t$ but only for an age interval $(a_1,a_2)$ (where $0 \leqslant a_1 < a_2 \leqslant a^*$) and a subdomain $\mathcal{O} \subset \Omega$.

In other words, the available output is $y(t) = p|_{(a_1,a_2)\times\mathcal{O}}$ where $t \in (0,T)$ ($T$ has to be chosen large enough).

More precisely, with the notation of Section 1, let $A$ be defined in Section 1 and let $p_0 \in X$ be given. We consider the abstract system (2), i.e.

$$\begin{cases} \dot{p}(t) = Ap(t), & t \in (0,T) \\ p(0) = p_0, & \end{cases}$$

with the output

$$y(t) = Cp(t), \qquad t \in (0,T),$$

where the observation operator $C \in \mathcal{L}(X,Y)$, $Y := L^2((a_1,a_2) \times \mathcal{O})$ is defined by

$$C\varphi := \varphi|_{(a_1,a_2)\times\mathcal{O}} \text{ for all } \varphi \in X.$$

# Observer

We introduce the observer

$$\begin{cases} \dot{\hat{p}}(t) = A\hat{p}(t) + HC\hat{p}(t) - Hy(t), & t \in (0,T) \\ \hat{p}(0) = 0, \end{cases} \tag{7}$$

where $H \in \mathcal{L}(Y,X)$ is a linear operator to be defined.

Then the error $e := \hat{p} - p$ satisfies

$$\begin{cases} \dot{e}(t) = (A + HC)e(t), & t \in (0,T) \\ e(0) = -p_0. \end{cases} \tag{8}$$

Assumptions **(A1.)** and **(A2.)** given in Section 3 are satisfied by the above system and the problem of determining the stabilizing output injection operator $H$ for $(A, C)$ fits into the framework described in Section 3. Therefore we can apply this approach.

Let $N$ denote the number of (distinct) eigenvalues of $A$ with positive real part and $\Sigma_+$ the set of these unstable eigenvalues. Let $\Gamma_+$ be a positively oriented curve enclosing $\Sigma_+$ but no other point of the spectrum of $A$, and let $P_+ : X \to X$ be the projection operator defined as previously.

# Hautus test

It remains to verify that the Hautus test is satisfied for our system $(A, C)$:

**Lemma**

*If $\Phi \in \mathcal{D}(A)$ satisfies $A\Phi = \lambda\Phi$ for $\lambda \in \Sigma_+$ and $C\Phi = 0$, then $\Phi$ vanishes identically.*

# Proof of the Hautus test

Let $\lambda$ be an unstable eigenvalue of $A$ and $\Phi$ an associated eigenfunction : $A\Phi = \lambda\Phi$, $\Phi \in \mathcal{D}(A)$, $\Phi \neq 0$ and $\operatorname{Re}\lambda \geqslant 0$. Then, we have

$$\Phi(a,x) = \sum_{i,j|\lambda=\mu_i-\lambda_j^D} \alpha_j \Phi_{ij}(a,x) = \sum_{i,j|\lambda=\mu_i-\lambda_j^D} \alpha_j e^{-\mu_i a}\pi(a)\varphi_j(x).$$

Assume now that $C\Phi = \Phi|_{(a_1,a_2)\times\mathcal{O}} = 0$, i.e. that

$$\sum_{i,j|\lambda=\mu_i-\lambda_j^D} \alpha_j e^{-\mu_i a}\pi(a)\varphi_j|_{\mathcal{O}} = 0, \qquad a \in (a_1,a_2).$$

As the eigenvalues $(\mu_i)_{i\in\mathbb{N}}$ are distinct, this equation reduces to

$$\sum_{j|\lambda+\lambda_j^D\in\sigma(A_0)} \alpha_j e^{-\mu_{i(j)} a}\varphi_j|_{\mathcal{O}}(x) = 0, \qquad a \in (a_1,a_2),$$

where $i(j) \in \mathbb{N}^*$ is the unique index s.t. $\mu_i = \lambda + \lambda_j^D$. Since the eigenfunctions of $-k\Delta$ with Dirichlet boundary cond. are analytic, $\Phi = 0$.

## Main result

This immediately leads to the following result.

**Theorem**

*Let $p_0 \in X$ and $p$ the solution of (1). Assume that $y(t) = p|_{(a_1,a_2)\times\mathcal{O}}$ $(t > 0)$ is known. Let $\hat{p}$ the observer defined by (7), where $H \in \mathcal{L}(Y, X)$ is the (finite dimensional) stabilizing output injection operator defined by (3). Then, there exists $C > 0$ and $\omega > 0$ such that*

$$\|\hat{p}(t) - p(t)\| \leqslant Ce^{-\omega t}\|p_0\|, \qquad t > 0.$$

# Outline

From now on, we make the following assumptions for the sake of simplicity:

- the birth rate $\beta$ belongs to $C^1([0, a^*])$ and
$$\beta(0) = 0.$$
- $A_+ := A|_{\mathcal{D}(A)\cap X_+}$ is diagonalizable, i.e. $m^{\mathrm{G}}(\lambda_k) = m^{\mathrm{A}}(\lambda_k)$ for all positive eigenvalues $\lambda_k$ of $A$. This implies in particular that the unstable space is $X_+ = \bigoplus_{k=1}^{M} \mathrm{Ker}\,(A - \lambda_k)$, since $m^{\mathrm{A}}(\lambda_k) = 1$, for all $k = 1, \ldots, M$.

# Rescaling the problem

In order to overcome the difficulties due to unboundedness of the coefficient $\mu$, we introduce the auxiliary variable

$$u(a,x,t) = \frac{p(a,x,t)}{\pi(a)} = \exp\left(\int_0^a \mu(s)\,\mathrm{d}s\right) p(a,x,t).$$

which satisfies

$$\begin{cases} \partial_t u(a,x,t) + \partial_a u(a,x,t) - \Delta u(a,x,t) = 0, & a \in (0,a^*),\ x \in \Omega,\ t > 0, \\ u(a,x,t) = 0, & a \in (0,a^*),\ x \in \partial\Omega,\ t > 0, \\ u(a,x,0) = u_0(a,x), & a \in (0,a^*),\ x \in \Omega, \\ u(0,x,t) = \displaystyle\int_0^{a^*} m(a)u(a,x,t)\,\mathrm{d}a, & x \in \Omega,\ t > 0 \end{cases} \tag{9}$$

where we have set $u_0(a,x) = p_0(a,x)/\pi(a)$ and $m(a) = \beta(a)\pi(a)$ stands for the maternity function.

# Variational formulation

The variational formulation of the above problem leads to the following problem : Find $u(t) \in L^2(0, a^*; H_0^1(\Omega))$ such that for all $v \in H_0^1(\Omega)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u(a,x,t) v(x)\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}a} \int_\Omega u(a,x,t) v(x)\,\mathrm{d}x + \int_\Omega \nabla u(a,x,t) \cdot \nabla v(x)\,\mathrm{d}x = 0$$

with the initial conditions

$$\begin{cases} u(a,x,0) = u_0(a,x), & a \in (0,a^*),\ \ x \in \Omega, \\ u(0,x,t) = \displaystyle\int_0^{a^*} m(a) u(a,x,t)\,\mathrm{d}a, & x \in \Omega,\ \ t > 0. \end{cases}$$

# Finite element discretization in space

We consider a $\mathbb{P}^1$ finite element approximation of $u(a,\cdot,t)$, that is we seek for the solution $u_h(a,\cdot,t) \in V_h$ such that for all $v_h \in V_h$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_h(a,x,t)v_h(x)\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}a}\int_\Omega u_h(a,x,t)v_h(x)\,\mathrm{d}x + \int_\Omega \nabla u_h(a,x,t)\cdot\nabla v_h(x)\,\mathrm{d}x = 0$$

where
$V_h = \{v_h \in C^0(\Omega);\ v_{h|T_\ell} \in \mathbb{P}^1(T_\ell),\ \forall \ell = 1,\dots,L, v_{h|\partial\Omega} = 0\}$ is the finite element space associated to the triangulation $\Omega = (T_\ell)_{\ell=1}^L$.
We denote by $(w_i)_{i=1}^{N_h}$ the finite element basis of the finite dimensional approximation space $V_h$.

Plugging the formula $u_h(a,x,t) = \sum_{i=1}^{N_h} u_i(a,t) w_i(x)$ in the above relation leads to the following problem for $\mathbf{U}(a,t) := \begin{bmatrix} u_1(a,t) \\ \vdots \\ u_{N_h}(a,t) \end{bmatrix}$:

$$\begin{cases} \mathbb{M}\dfrac{\mathrm{d}\mathbf{U}}{\mathrm{d}t}(a,t) + \mathbb{M}\dfrac{\mathrm{d}\mathbf{U}}{\mathrm{d}a}(a,t) + \mathbb{K}\mathbf{U}(a,t) = 0, \\ \mathbf{U}(a,0) = \mathbf{U}_{0h}(a) \\ \mathbf{U}(0,t) = \displaystyle\int_0^{a^*} m(a)\mathbf{U}(a,t)\,\mathrm{d}a, \end{cases} \tag{10}$$

where

- $\mathbb{M}$ and $\mathbb{K}$ denote respectively the mass and the stiffness matrices ($\mathbb{M}_{ij} = \int_\Omega w_i w_j$, $\mathbb{K}_{ij} = \int_\Omega \nabla w_i \cdot \nabla w_j$),
- $\mathbf{U}_{0h}(a) = \begin{bmatrix} u_0(a,x_1) \\ \vdots \\ u_0(a,x_{N_h}) \end{bmatrix}$, where $(x_i)_{i=1}^{N_h}$ denote the nodes of the mesh.

Note that $\mathbb{M}$ and $\mathbb{K}$ are symmetric matrices and that $\mathbb{M}$ is invertible.

# Finite difference discretization in age

Let us denote by $u_i^k(t)$ an approximation of $u_i(k\Delta a, t)$ and by $\mathbf{U}^k(t) := \begin{bmatrix} u_1^k(t) \\ \vdots \\ u^k_{N_h(t)} \end{bmatrix}$ an approximation of $\mathbf{U}(k\Delta a, t)$.
The approximation of $u(k\Delta a, x, t)$ is then given by

$$u(k\Delta a, x, t) \simeq \sum_{i=1}^{N_h} u_i^k(t) w_i(x).$$

# Equation for $U^k(t)$, $k = 1, \cdots, K$

Using a Crank-Nicholson scheme to approximate the solution of problem (10), we can move from age $(k-1)\Delta a$ to age $k\Delta a$ following the scheme

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}\left(\frac{\mathbf{U}^k(t)+\mathbf{U}^{k-1}(t)}{2}\right)+\frac{1}{\Delta a}\mathbb{M}\left(\mathbf{U}^k(t)-\mathbf{U}^{k-1}(t)\right)+\mathbb{K}\left(\frac{\mathbf{U}^k(t)+\mathbf{U}^{k-1}(t)}{2}\right)=0$$

with the initial condition

$$\mathbf{U}^k(0)=\mathbf{U}_{0h}(k\Delta a), \qquad \forall k=1,\dots,K.$$

This implies that, for any $k \in \{1,\dots,K\}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}\left(\mathbf{U}^k(t)+\mathbf{U}^{k-1}(t)\right)=-\left(\frac{2}{\Delta a}\mathbb{M}+\mathbb{K}\right)\mathbf{U}^k(t)+\left(\frac{2}{\Delta a}\mathbb{M}-\mathbb{K}\right)\mathbf{U}^{k-1}(t).$$

# Equation for $U^0(t)$

In order to obtain an equation for $\mathbf{U}^0(t)$, we consider the continuous equation satisfied at age $a = 0$:

$$u(0,x,t) = \int_0^{a^*} m(a)u(a,x,t)\,\mathrm{d}a, \qquad x \in \Omega,\ t > 0.$$

Taking the time derivative of the above equation, we obtain after integration by parts that

$$\partial_t u(0,x,t) = \int_0^{a^*} \left[m(a)\Delta u(a,x,t) - m'(a)u(a,x,t)\right]\,\mathrm{d}a, \qquad x \in \Omega,\ t > 0.$$

The above relation suggests the following equation for $\mathbf{U}^0(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}\mathbf{U}^0(t) = -\sum_{k=0}^{K} \omega_k \,\left[m(k\Delta a)\mathbb{K} + m'(k\Delta a)\mathbb{M}\right]\mathbf{U}^k(t).$$

where we have used the approximation

$$\int_0^{a^*} f(a)\,\mathrm{d}a \simeq \sum_{k=0}^{K} \omega_k\, f(k\Delta a).$$

Consequently, setting

$$\mathbb{B}^k := -\omega_k \left[ m(k\Delta a)\mathbb{K} + m'(k\Delta a)\mathbb{M} \right]$$

$$\mathbb{K}^- := -\left( \frac{2}{\Delta a}\mathbb{M} + \mathbb{K} \right) \qquad \mathbb{K}^+ := \frac{2}{\Delta a}\mathbb{M} - \mathbb{K},$$

we have

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}\mathbf{U}^0(t) = \displaystyle\sum_{k=0}^{K} \mathbb{B}^k \mathbf{U}^k(t), \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbb{M}\left(\mathbf{U}^k(t) + \mathbf{U}^{k-1}(t)\right) = \mathbb{K}^-\mathbf{U}^k(t) + \mathbb{K}^+\mathbf{U}^{k-1}(t), \qquad \forall k = 1, \ldots, K \\ \mathbf{U}^k(0) = \mathbf{U}_{0h}(k\Delta a), \qquad \forall k = 0, \ldots, K. \end{cases}$$

This can be written as

$$\mathbf{M}\dot{\mathbf{Z}}(t) = \begin{bmatrix} \mathbb{B}^0 & \mathbb{B}^1 & \dots & \dots & \dots & \mathbb{B}^K \\ \mathbb{K}^+ & \mathbb{K}^- & 0 & \dots & \dots & 0 \\ 0 & \mathbb{K}^+ & \mathbb{K}^- & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \mathbb{K}^+ & \mathbb{K}^- \end{bmatrix} \mathbf{Z}(t),$$

where $\mathbf{M} = \begin{bmatrix} \mathbb{M} & 0 & \dots & 0 \\ \mathbb{M} & \mathbb{M} & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \mathbb{M} & \mathbb{M} \end{bmatrix}$ and $\mathbf{Z}(t) = \begin{bmatrix} \mathbf{U}^0(t) \\ \mathbf{U}^1(t) \\ \vdots \\ \mathbf{U}^K(t) \end{bmatrix}$ is a vector of size $(K+1)N_h$.

# Discrete system

This is equivalent to

$$\begin{cases} \mathbf{E}\dot{\mathbf{Z}}(t) = \mathbf{A}\mathbf{Z}(t) \\ \mathbf{Z}(0) = \mathbf{Z_0}, \end{cases} \tag{11}$$

where

$$\mathbf{E} = \mathbf{M}^{\mathbf{T}}\mathbf{M}, \quad \mathbf{A} = \mathbf{M}^{\mathbf{T}} \begin{bmatrix} \mathbb{B}^0 & \mathbb{B}^1 & \dots & \dots & \dots & \mathbb{B}^K \\ \mathbb{K}^+ & \mathbb{K}^- & 0 & \dots & \dots & 0 \\ 0 & \mathbb{K}^+ & \mathbb{K}^- & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \mathbb{K}^+ & \mathbb{K}^- \end{bmatrix}, \quad \mathbf{Z_0} = \begin{bmatrix} U_{0h}(0) \\ U_{0h}(\Delta a) \\ \vdots \\ U_{0h}(K\Delta a) \end{bmatrix}.$$

Note that $\mathbf{E}$ and $\mathbf{A}$ are squared matrices of size $(K+1)N_h$ and that $\mathbf{E}$ is symmetric definite positive (since $\mathbb{M}$ is invertible).

# Detectability at the discrete level

Given a discretization $\mathbf{C} \in \mathbb{R}^{m\times(K+1)N_h}$ of $C$, our goal here is to show that the detectability of the system $(\mathbf{A}, \mathbf{C})$ can be deduced from the detectability of the smaller finite dimensional system corresponding to the unstable part of $\mathbf{A}$ (namely the projection on the finite dimensional space associated with the unstable eigenvalues $\lambda_1, \cdots, \lambda_N$).

To achieve this, we adapt the ideas developed in Section 3 for the continuous case, using arguments introduced in the PhD thesis of L. Thevenet.

# Eigenvalue

We recall that the complex $\lambda$ is an eigenvalue of the matrix pencil $(\mathbf{A}, \mathbf{E})$ if there exists $\mathbf{Z} \neq 0$ such that $\mathbf{AZ} = \lambda \mathbf{EZ}$. We assume that $(\mathbf{A}, \mathbf{E})$ admits $N$ eigenvalues (counted with multiplicities) with real part greater than $0$ and, having in mind the estimation problem in population dynamics, we assume that there is only one real eigenvalue, which is moreover dominant. More precisely we reorder the eigenvalues $(\lambda_n)_n$ of $\mathbf{A}$ so that the sequence $(\mathrm{Re}\,\lambda_n)_n$ is nonincreasing, and we suppose that $N \in \mathbb{N}^*$ is such that

$$\cdots \leqslant \mathrm{Re}\,\lambda_{N+1} < 0 \leqslant \mathrm{Re}\,\lambda_N \leqslant \cdots \leqslant \mathrm{Re}\,\lambda_2 < \lambda_1.$$

Moreover, we assume that $(\mathbf{A}, \mathbf{E})$ admits $N$ eigenvectors, denoted by $\widetilde{\mathbf{\Phi}}_1$,...,$\widetilde{\mathbf{\Phi}}_N$, corresponding to the $N$ unstable eigenvalues $\lambda_1$,..., $\lambda_N$. Similarly, we denote by $\widetilde{\mathbf{\Psi}}_1, \cdots, \widetilde{\mathbf{\Psi}}_N$ the $N$ eigenvectors of $(\mathbf{A}^{\mathbf{T}}, \mathbf{E})$ corresponding to the $N$ unstable eigenvalues $\lambda_1, \overline{\lambda_2}, \cdots, \overline{\lambda_N}$ of $\mathbf{A}^{\mathbf{T}}$.

Let

$$\mathbf{\Lambda} = \mathbf{diag}\left(\lambda_1, \cdots, \lambda_N\right),$$

be the diagonal matrix corresponding to the unstable eigenvalues of $(\mathbf{A}, \mathbf{E})$ and $\mathbf{\Lambda}^*$ its conjugate matrix. We denote by

$$\widetilde{\mathbf{\Phi}} = \left[\widetilde{\mathbf{\Phi}}_1 | \cdots | \widetilde{\mathbf{\Phi}}_N\right] \in \mathbb{C}^{(K+1)N_h \times N},$$

the matrix of $N$ complex eigenvectors of $(\mathbf{A}, \mathbf{E})$ and by

$$\widetilde{\mathbf{\Psi}} = \left[\widetilde{\mathbf{\Psi}}_1 | \cdots | \widetilde{\mathbf{\Psi}}_N\right] \in \mathbb{C}^{(K+1)N_h \times N},$$

the matrix of $N$ complex eigenvectors of $(\mathbf{A}^{\mathbf{T}}, \mathbf{E})$. In other words, we have

$$\mathbf{A}\widetilde{\mathbf{\Phi}} = \mathbf{E}\widetilde{\mathbf{\Phi}}\mathbf{\Lambda} \qquad \text{and} \qquad \mathbf{A}^{\mathbf{T}}\widetilde{\mathbf{\Psi}} = \mathbf{E}\widetilde{\mathbf{\Psi}}\mathbf{\Lambda}^*.$$

We choose $\widetilde{\mathbf{\Phi}}$ and $\widetilde{\mathbf{\Psi}}$ satisfying

$$\widetilde{\mathbf{\Phi}}^*\mathbf{E}\widetilde{\mathbf{\Psi}} = \mathbf{I}_{(K+1)N_h}.$$

# Real vectors

In order to work with real subspaces, we introduce

$$\mathbf{S} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \qquad \text{and} \qquad \mathbf{U} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \mathbf{S} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{S} \end{bmatrix} \in \mathbb{C}^{N \times N},$$

where $\mathbf{U}$ is a unitary matrix constituted of $(N-1)/2$ diagonal blocks of $S$. We then set

$$\mathbf{\Phi} = \widetilde{\mathbf{\Phi}} U^* \in \mathbb{R}^{(K+1)N_h \times N} \qquad \text{and} \qquad \mathbf{\Psi} = \widetilde{\mathbf{\Psi}} U^* \in \mathbb{R}^{(K+1)N_h \times N},$$

This is equivalent to $\widetilde{\mathbf{\Phi}} = \mathbf{\Phi U}$ and $\widetilde{\mathbf{\Psi}} = \mathbf{\Psi U}$, since $\mathbf{U}$ is a unitary matrix.

As immediate consequence, we have

$$\mathbf{\Phi}^{\mathbf{T}} \mathbf{E} \mathbf{\Psi} = \mathbf{Id}_{(K+1)N_h}.$$

# The projector on the unstable space

We define $P_+$ the projection onto the unstable subspace of the continuous operator $A$, parallel to the stable one and given by:

$$P_+ z = \sum_{m=1}^{N} (z, \psi_m)_X \phi_m,$$

where $(\phi_1, \cdots, \phi_m)$ (resp. $(\psi_1, \cdots, \psi_m)$) is a basis of the unstable subspace associated to the operator $A$ (resp. $A^*$).
We can show that a discrete version of this operator is given by

$$\mathbf{P}_+ = \mathbf{\Phi}\mathbf{\Psi}^{\mathbf{T}} \begin{bmatrix} \omega_0 \mathbb{M} & & \\ & \ddots & \\ & & \omega_K \mathbb{M} \end{bmatrix} \in \mathbb{R}^{(K+1)N_h \times (K+1)N_h},$$

in the sense that as $h$ and $\Delta a$ tend to zero, $P_+ z \simeq \mathbf{P}_+ \mathbf{Z}$, $\forall z \in X$, where

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}^0 \\ \vdots \\ \mathbf{Z}^K \end{bmatrix} \in \mathbb{R}^{(K+1)N_h \times 1}, \qquad \mathbf{Z}^k = \begin{bmatrix} z(a^k, x_1) \\ \vdots \\ z(a^k, x_{N_h}) \end{bmatrix} \in \mathbb{R}^{N_h \times 1}.$$

# Detectability of $(\mathbf{A}, \mathbf{C})$ from the detectability of a smaller finite dimensional system

The pair $(\mathbf{A}, \mathbf{C})$ is $\mathbf{E}$-detectable if there exists $\mathbf{L} \in R^{(K+1)N_h \times m}$ such that the solution $\mathbf{Z}$ of $\mathbf{E}\dot{\mathbf{Z}}(t) = \mathbf{A}\mathbf{Z}(t) + \mathbf{L}\mathbf{C}\mathbf{Z}(t)$ is exponentially stable. Such an operator $\mathbf{L}$ is called a stabilizing output injection operator for $(\mathbf{A}, \mathbf{C})$.

## Theorem

*Let*

$$\mathbf{C}_+ = \mathbf{C}\mathbf{\Phi} \in \mathbb{R}^{m \times N}$$

*and*

$$\mathbf{A}_+ = \mathbf{\Psi}^{\mathbf{T}}\mathbf{A}\mathbf{\Phi} \in \mathbb{R}^{N \times N}.$$

*Assume that the projected system $(\mathbf{A}_+, \mathbf{C}_+)$ of small dimension is detectable through a stabilizing output injection operator $\mathbf{L}_+ \in \mathbb{R}^{N \times m}$. Then, the high dimensional system $(\mathbf{A}, \mathbf{C})$ is $\mathbf{E}$-detectable through the stabilizing output injection operator*

$$\mathbf{L} = \mathbf{E}\mathbf{\Phi}\mathbf{L}_+ \in \mathbb{R}^{(K+1)N_h \times m}.$$

# Observer at the discrete level

Assume that $\mathbf{Y}(t) = \mathbf{CZ}(t)$, $(t \in (0,T))$ is known. Let $\hat{\mathbf{Z}}$ the observer defined by

$$\begin{cases} \mathbf{E}\dot{\hat{Z}}(t) = \mathbf{A}\hat{\mathbf{Z}}(t) + \mathbf{LC}\hat{\mathbf{Z}}(t) - \mathbf{LY}(t) \\ \hat{\mathbf{Z}}(0) = 0, \end{cases}$$

where $\mathbf{L} \in \mathbb{R}^{(K+1)N_h \times m}$. Then the error $\mathbf{e} = \hat{\mathbf{Z}} - \mathbf{Z}$ satisfies

$$\begin{cases} \mathbf{E}\dot{\mathbf{e}}(t) = (\mathbf{A} + \mathbf{LC})\mathbf{e}(t) \\ \mathbf{e}(0) = -\mathbf{Z}_0. \end{cases}$$

Consequently, if $(\mathbf{A}_+, \mathbf{C}_+)$ is detectable through $\mathbf{L}_+ \in \mathbb{R}^{N \times m}$, then $(\mathbf{A}, \mathbf{C})$ is detectable through $\mathbf{L} = \mathbf{E}\mathbf{\Phi}\mathbf{L}_+$ and

$$\left\| \hat{\mathbf{Z}}(t) - \mathbf{Z}(t) \right\| \leqslant K e^{-\omega t} \left\| \mathbf{Z}_0 \right\|.$$

# To go further

1. Approximation
   - Convergence analysis and error estimates, uniform exponential stability (with respect to $\Delta a$ et $h$)
   - Numerical validation
2. Other models
   - Age dependant coefficients $k = k(a)$
   - Nonlinear models