Sparse stabilization and control of the Cucker-Smale model

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Cucker-Smale consensus model

Prototypical model for the interaction of N agents:

Cucker-Smale model (2007)

$$\dot{x}_i(t) = v_i(t)$$

$$\dot{v}_i(t) = \frac{1}{N}\sum_{j=1}^{N} \frac{v_j(t) - v_i(t)}{(1 + \|x_j(t) - x_i(t)\|^2)^\beta} \qquad i = 1, \dots, N$$

$\beta > 0$, $x_i \in \mathbb{R}^d$, $v_i \in \mathbb{R}^d$

x_i: main state of the agent i
v_i: consensus parameter of the agent i

- initially introduced to describe the formation and evolution of languages
- then for describing the flocking of a swarm of birds

Cucker-Smale consensus model

Prototypical model for the interaction of N agents:

General Cucker-Smale model

$$\dot{x}_i(t) = v_i(t),$$
$$\dot{v}_i(t) = \frac{1}{N}\sum_{j=1}^{N} a(\|x_j(t)-x_i(t)\|)(v_j(t)-v_i(t)), \qquad i=1,\dots,N$$

$x_i \in \mathbb{R}^d$, $v_i \in \mathbb{R}^d$
$a \in C^1([0,+\infty))$ nonincreasing positive function (interaction potential, rate of communication)

$a(s) = \frac{1}{(1+s^2)^\beta}$ in the classical Cucker-Smale model

There are other classes of models, cf social dynamics...

Bellomo, Bertozzi, Boudin, Carrillo, Couzin, Cristiani, Cucker, Degond, Desvillettes, Dong, Fornasier, Frasca, Ha, Haskovec, Kim, Lee, Leonard, Levy, Mordecki, Motsch, Parrish, Perthame, Piccoli, Salvarani, Smale, Tadmor, Toscani, Tosin, Vecil, Vicsek,........

Cucker-Smale consensus model

Prototypical model for the interaction of N agents:

General Cucker-Smale model

$$\dot{x}_i(t) = v_i(t),$$
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Objective:

- Model self-organisation and consensus emergence in a group of agents
- Organization via intervention (social forces), enforce or facilitate pattern formation or convergence to consensus

Cucker-Smale consensus model

$$\dot{x}_i(t) = v_i(t)$$

$$\dot{v}_i(t) = \frac{1}{N}\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) \qquad i = 1, \dots, N$$

Mean consensus vector: $\bar{v} = \frac{1}{N}\sum_{i=1}^{N} v_i(t) \quad \rightarrow$ constant

Cucker-Smale consensus model

$$\dot{x}_i(t) = v_i(t)$$
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With matrix notations:

$$\dot{v}_i = \frac{1}{N}\left(\sum_{j=1}^{N} a_{ij}(v_j - v_i)\right) = \frac{1}{N}\left((Av)_i - \left(\sum_{j=1}^{N} a_{ij}\right) v_i\right)$$

hence

$$\dot{x} = v, \qquad \dot{v} = -Lv$$

with

$$L = \frac{1}{N}(D-A) \quad \text{Laplacian}, \qquad D = \mathrm{diag}\left(\sum_{j=1}^{N} a_{ij}\right)$$

N.B.: $L \geq 0$ and $L(v,\ldots,v) = 0$

Cucker-Smale consensus model

$$\dot{x}_i(t) = v_i(t)$$
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$$(\mathbb{R}^d)^N = V_f \overset{\perp}{\oplus} V_\perp \text{ with } \left\{ \begin{array}{lcl} V_f & = & \{(v_1,\dots,v_N)\in(\mathbb{R}^d)^N \mid v_1=\dots=v_N\in\mathbb{R}^d\} \\ V_\perp & = & \{(v_1,\dots,v_N)\in(\mathbb{R}^d)^N \mid \sum_{i=1}^N v_i = 0\} \end{array}\right.$$

$\forall v\in(\mathbb{R}^d)^N, \quad v = v_f + v_\perp$ with $v_f = (\bar{v},\dots,\bar{v}) \in V_f$ and $v_\perp \in V_\perp$.

$$X(t) = \frac{1}{2N^2}\sum_{i,j=1}^{N}\|x_i(t)-x_j(t)\|^2$$
$$V(t) = \frac{1}{2N^2}\sum_{i,j=1}^{N}\|v_i(t)-v_j(t)\|^2 = \frac{1}{N}\sum_{i=1}^{N}\|v(t)_{\perp_i}\|^2$$

indeed, $\frac{1}{2N^2}\sum_{i,j}\|v_i-v_j\|^2 = \frac{1}{N}\sum_i\|v_i\|^2 - \left\|\frac{1}{N}\sum_i v_i\right\|^2 = \frac{1}{N}\sum_i\left\|v_i-\frac{1}{N}\sum_j v_j\right\|^2$

Cucker-Smale consensus model

$$\dot{x}_i(t) = v_i(t)$$
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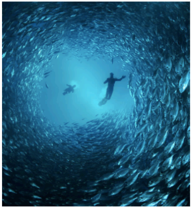
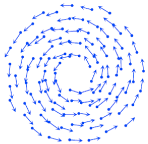
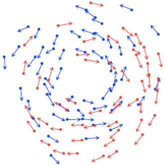
Definitions

- Consensus point = steady configuration = $(x, v) \in (\mathbb{R}^d)^N \times V_f$.
- Consensus manifold = $(\mathbb{R}^d)^N \times V_f$.
- A solution $(x(t), v(t))$ tends to *consensus* if $v_i(t) \underset{t\to+\infty}{\longrightarrow} \bar{v}$ for every $i = 1, \ldots, N$, or, equivalently, if $V(t) \underset{t\to+\infty}{\longrightarrow} 0$.

Cucker-Smale consensus model

$$\begin{aligned}\dot{x}_i(t) &= v_i(t)\\ \dot{v}_i(t) &= \frac{1}{N}\sum_{j=1}^{N} a(\|x_j(t)-x_i(t)\|)(v_j(t)-v_i(t))\end{aligned} \qquad i=1,\ldots,N$$

Remark: by a computation,

$$\begin{aligned}\dot{V}(t) &= -\frac{1}{N}\sum_{i,j=1}^{N} a(\|x_j(t)-x_i(t)\|)\|v_i(t)-v_j(t)\|^2\\ &\leq -2a\left(\sqrt{2NX(t)}\right)V(t)\end{aligned}$$

and hence $\underbrace{\textit{if } X(t) \textit{ remains bounded}}_{??}$ then $\dot{V} \leq -\alpha V$, whence flocking.

Cucker-Smale consensus model

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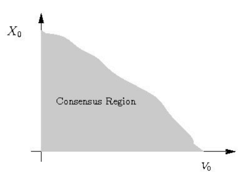
Proposition (Ha-Ha-Kim, 2010)

Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ be such that

$$\int_{\sqrt{X(0)}}^{+\infty} a(\sqrt{2N}r)\, dr \geq \sqrt{V(0)}.$$

Then the solution with initial data (x_0, v_0) tends to consensus.



$\longrightarrow$ self-organization of the group in this so-called *consensus region* (region of natural asymptotic stability to consensus). Sharp estimate.

Example: $a(x) = \frac{1}{(1+x^2)^\beta}$ with $0 < \beta \leq 1/2$ (initial model by Cucker-Smale, 2007)
$\longrightarrow$ globally stable

Control of the Cucker-Smale model

When consensus is not achieved by self-organization: is it possible to control the group to consensus by means of an external action?

$\longrightarrow$ organization via intervention

Controlled Cucker-Smale model

$$\dot{x}_i(t) = v_i(t)$$

$$\dot{v}_i(t) = \frac{1}{N}\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) + u_i(t) \qquad i = 1, \ldots, N$$

with

$$\sum_{i=1}^{N} \|u_i(t)\| \leq M,$$

for $M > 0$ fixed.

(Caponigro - Fornasier - Piccoli - Trélat, 2013)

Control of the Cucker-Smale model

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with $\sum_{i=1}^{N} \|u_i(t)\| \leq M$.

Objectives:

- Design a *feedback control* steering ”optimally” the system to consensus, with:
  - (i) a minimal amount of components active at each time
  - (ii) a minimal amount of switchings in time
- Control the system to any prescribed consensus.

Control of the Cucker-Smale model

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$\longrightarrow$ concept of sparse control

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with $\sum_{i=1}^{N} \|u_i(t)\| \leq M$.

Objectives:

- Design a *feedback control* steering "optimally" the system to consensus, with:
  - (i) a minimal amount of components active at each time $\to$ componentwise sparse control
  - (ii) a minimal amount of switchings in time $\to$ time sparse control
- Control the system to any prescribed consensus.

$\to$ concept of sparse (feedback) control

Sparse stabilization of the Cucker-Smale model

$$\dot{x}_i(t) = v_i(t)$$
$$\dot{v}_i(t) = \frac{1}{N}\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) + u_i(t) \qquad i = 1, \dots, N$$

with $\sum_{i=1}^{N} \|u_i(t)\| \leq M$.

A first remark

The feedback control defined by

$$u(t) = -\alpha v_\perp(t),$$

with $\alpha > 0$ small enough, stabilizes the system to consensus (in infinite time).

Indeed: $\dot{V} \leq -\frac{2}{N}\sum_i \langle v_{\perp i}, u_i \rangle = -2\alpha V$

BUT this control is far from sparse!

Sparse stabilization of the Cucker-Smale model

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$$\dot{v}_i(t) = \frac{1}{N}\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) + u_i(t) \qquad i = 1, \dots, N$$

with $\sum_{i=1}^{N} \|u_i(t)\| \leq M$.

Idea:

$$\min_{\sum_{i=1}^{N} \|u_i\| \leq M} \left(\frac{1}{2N^2} \sum_{i,j=1}^{N} \langle v_i - v_j, u_i - u_j \rangle \right) = \min_{\sum_{i=1}^{N} \|u_i\| \leq M} \left(\frac{1}{N} \sum_{i=1}^{N} \langle v_{\perp i}, u_{\perp i} \rangle \right)$$

$$\Rightarrow u = -\alpha v_{\perp}$$

Sparse stabilization of the Cucker-Smale model

$$\dot{x}_i(t) = v_i(t)$$
$$\dot{v}_i(t) = \frac{1}{N}\sum_{j=1}^{N} a(\|x_j(t)-x_i(t)\|)(v_j(t)-v_i(t)) + u_i(t) \qquad i=1,\dots,N$$

with $\sum_{i=1}^N \|u_i(t)\| \leq M$.

<u>Idea:</u>

$$\min_{\sum_{i=1}^N \|u_i\|\leq M} \left(\frac{1}{2N^2}\sum_{i,j=1}^{N} \langle v_i - v_j, u_i - u_j\rangle + \gamma(X)\frac{1}{N}\sum_{i=1}^{N}\|u_i\| \right),$$

where

$$\gamma(X) = \int_{\sqrt{X}}^{+\infty} a(\sqrt{2N}r)\,dr$$

- ℓ^1 norm $\Rightarrow$ enforce sparsity
- $\gamma(X)$ threshold $\Rightarrow$ the control switches off when entering the consensus region

Sparse stabilization of the Cucker-Smale model

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with $\sum_{i=1}^{N} \|u_i(t)\| \leq M$.

This leads to the **componentwise sparse feedback control** u:

- if $\max_{1\leq i\leq N} \|v_{\perp_i}(t)\| \leq \gamma(X(t))^2$, then $\boxed{u(t) = 0}$
- if $\|v_{\perp_j}(t)\| = \max_{1\leq i\leq N} \|v_{\perp_i}(t)\| > \gamma(X(t))^2$ (with j be the smallest index) then

$$\boxed{u_j(t) = -M\frac{v_{\perp_j}(t)}{\|v_{\perp_j}(t)\|}, \quad \text{and} \quad u_i(t) = 0 \quad \text{for every } i \neq j}$$

Theorem

This feedback control stabilizes the system to consensus.

$$\dot{x}_i(t) = v_i(t)$$
$$\dot{v}_i(t) = \frac{1}{N}\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) + u_i(t) \qquad i = 1, \dots, N$$

with $\sum_{i=1}^{N} \|u_i(t)\| \leq M$.

Theorem

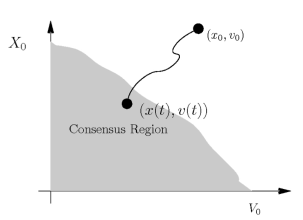
This feedback control stabilizes the system to consensus.

Indeed:

$$\dot{V} \leq \frac{2}{N}\sum_i \langle v_{\perp i}, u_i \rangle = -2\frac{M}{N}\|v_{\perp j}\|$$

with $\|v_{\perp_j}\| = \max_{1\leq i\leq N} \|v_{\perp_i}\| \geq \sqrt{V} \quad \Rightarrow \quad \dot{V} \leq -2\frac{M}{N}\sqrt{V}$

hence in finite time we enter the consensus region, and then $u = 0$ (forever).



Sparse stabilization of the Cucker-Smale model

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with $\sum_{i=1}^{N} \|u_i(t)\| \leq M$.

Remarks

- This feedback control is componentwise sparse.
- This feedback control is however not necessarily time sparse: it may *chatter*.

$\longrightarrow$ To avoid possible chattering in time: sampling in time (sample-and-hold)

$\Rightarrow$ we obtain a time sparse and componentwise sparse feedback control.

Short digression on sampling

Closed-loop system

$$\dot{x}(t) = f(x(t), u(x(t))) \quad (S)$$

sample-and-hold $\longrightarrow$

Sampled closed-loop system

$$\dot{y}(t) = f\Big(y(t), u\Big(y\Big(T\Big[\frac{t}{T}\Big]\Big)\Big)\Big) \quad (S_{\text{samp}})$$

Assumptions

There exist a Lyapunov function for (S) and $\alpha, \beta, \gamma, \delta > 0$ such that

$$\begin{aligned} (1) \qquad & \alpha\|x\|^2 \leq V(x) \leq \beta\|x\|^2, \\ (2) \qquad & \|\nabla V(x)\| \leq \gamma\|x\|, \\ (3) \qquad & \langle \nabla V(x), f(x, u(x))\rangle \leq -\delta V(x). \end{aligned}$$

Let us then prove that (S_{samp}) is exponentially stable.

Short digression on sampling

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$$\dot{x}(t) = f(x(t), u(x(t))) \quad (S)$$

sample-and-hold
$\longrightarrow$

Sampled closed-loop system

$$\dot{y}(t) = f\left(y(t), u\left(y\left(T\left[\frac{t}{T}\right]\right)\right)\right) \quad (S_{\text{samp}})$$

$$\frac{d}{dt}V(y(t)) = \left\langle \nabla V(y(t)), f\left(y(t), u\left(y\left(T\left[\frac{t}{T}\right]\right)\right)\right)\right\rangle$$

$$= \langle \nabla V(y(t)), f(y(t), u(y(t)))\rangle + \left\langle \nabla V(y(t)), f\left(y(t), u\left(y\left(T\left[\frac{t}{T}\right]\right)\right)\right) - f(y(t), u(y(t)))\right\rangle$$

$$\leq -\delta V(y(t)) + \gamma \|y(t)\| \left\| f\left(y(t), u\left(y\left(T\left[\frac{t}{T}\right]\right)\right)\right) - f(y(t), u(y(t)))\right\|$$

and then essentially one has to ensure that the term

$$\left\| f\left(y(t), u\left(y\left(T\left[\frac{t}{T}\right]\right)\right)\right) - f(y(t), u(y(t)))\right\|$$

is ”small enough” whenever the sampling time T is small enough.

Short digression on sampling

Closed-loop system

$$\dot{x}(t) = f(x(t), u(x(t))) \quad (S)$$

sample-and-hold
$\longrightarrow$

Sampled closed-loop system

$$\dot{y}(t) = f\left(y(t), u\left(y\left(T\left[\frac{t}{T}\right]\right)\right)\right) \quad (S_{\text{samp}})$$

Assume that f is uniformly Lipschitz w.r.t. u and that u is uniformly Lipschitz w.r.t. x.
Then,

$$\left\| f\left(y(t), u\left(y\left(T\left[\frac{t}{T}\right]\right)\right)\right) - f(y(t), u(y(t))) \right\| \leq C_1 T \|\dot{y}(t)\|,$$

for some uniform $C_1 > 0$.
Then,

$$\|\dot{y}(t)\| = \left\| f\left(y(t), u\left(y\left(T\left[\frac{t}{T}\right]\right)\right)\right) \right\| \leq C_2 \|y(t)\|,$$

which is ok is f is uniformly Lipschitz w.r.t. x and the controls are bounded.

(there are many other possible assumptions...)

Short digression on sampling

Closed-loop system

$$\dot{x}(t) = f(x(t), u(x(t))) \quad (S)$$

sample-and-hold
$\longrightarrow$

Sampled closed-loop system

$$\dot{y}(t) = f\left(y(t), u\left(y\left(T\left[\frac{t}{T}\right]\right)\right)\right) \quad (S_{\text{samp}})$$

Then,

$$\begin{aligned}\frac{d}{dt}V(y(t)) &\leq -\delta V(y(t)) + \gamma\|y(t)\| \left\| f\left(y(t), u\left(y\left(T\left[\frac{t}{T}\right]\right)\right)\right) - f(y(t), u(y(t)))\right\| \\ &\leq -\delta V(y(t)) + C_1 C_2 T \gamma \|y(t)\|^2 \\ &\leq \left(-\delta + \frac{C_1 C_2 \gamma}{\alpha} T\right) V(y(t))\end{aligned}$$

and hence if T is small enough so that $\delta - \frac{C_1 C_2 \gamma}{\alpha} T \geq \delta/2$ then

$$\frac{d}{dt}V(y(t)) \leq -\frac{\delta}{2} V(y(t))$$

and the exponential decreasing follows.

Sparse is better

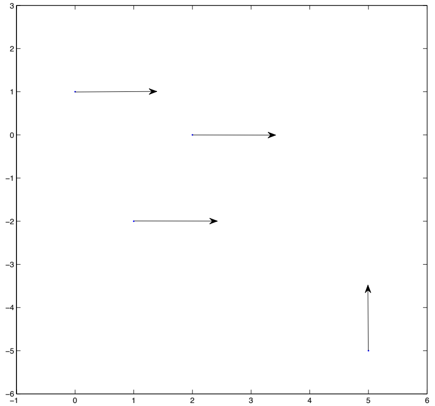
Proposition

For every time t, $u(t)$ minimizes $\frac{d}{dt}V(t)$ over all possible feedbacks controls.

In other words, the feedback control $u(t)$ is the best choice in terms of the rate of convergence to consensus.

→ "Sparse is better":

A policy maker, who is not allowed to have prediction on future developments, should always consider more favorable to intervene with stronger actions on the fewest possible instantaneous optimal leaders than trying to control more agents with minor strength.



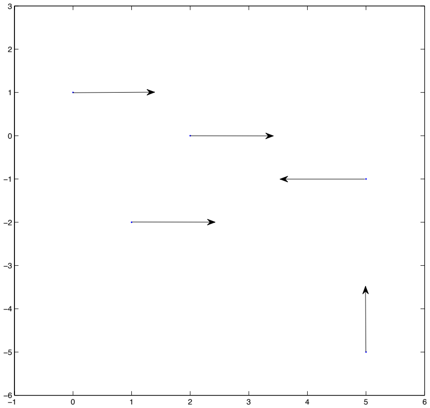
- 3 agents in flocking position:

$$\begin{aligned} x_1 &= (2,0), & v_1 &= (5,0) \\ x_2 &= (1,-2), & v_2 &= (5,0) \\ x_3 &= (0,1), & v_3 &= (5,0) \end{aligned}$$

- plus one agent to be controlled:

$$x_4 = (5,-5), \qquad v_4 = (0,5)$$

—— uncontrolled trajectories
—— controlled trajectories
**** active control



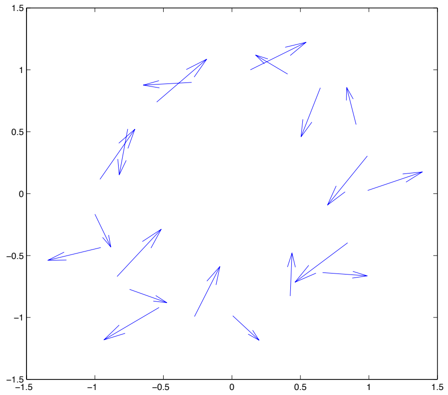
- 3 agents in flocking position:

$$x_1 = (2,0), \qquad v_1 = (5,0)$$
$$x_2 = (1,-2), \qquad v_2 = (5,0)$$
$$x_3 = (0,1), \qquad v_3 = (5,0)$$

- plus 2 agents to be controlled:

$$x_4 = (5,-5), \qquad v_4 = (0,5)$$
$$x_5 = (5,-1), \qquad v_5 = (-5,0)$$

uncontrolled trajectories
controlled trajectories
**** active control



- 20 agents
- random initial positions (on a circle)
- random initial speeds (sufficiently large so that the uncontrolled system does not flock)

red line	uncontrolled trajectories
blue line	controlled trajectories
****	active control

Other videos: 10 agents, 50 agents

Controllability near the consensus manifold

Proposition

Local controllability holds at almost every point of the consensus manifold. Moreover, controllability can be realized with time sparse and componentwise sparse controls.

Proof: linearization around a consensus point $\Rightarrow$ $\dot{x} = v, \ \dot{v} = Av + Bu$ with $B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

and A Laplacian matrix (note that $(1, \ldots, 1) \in \ker A$). Hence $\exists P$ orthogonal s.t.

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \qquad P^{-1}AP = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_N \end{pmatrix}, \qquad B_1 = P^{-1}B = \begin{pmatrix} 1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix}.$$

Controllability near the consensus manifold

Proposition

Local controllability holds at almost every point of the consensus manifold. Moreover, controllability can be realized with time sparse and componentwise sparse controls.

Then, with the Kalman matrix $K(A, B) = (B, AB, \dots, A^{N-1}B)$, the matrix

$$P^{-1}K(A,B) = K(P^{-1}AP, P^{-1}B) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_2 & \lambda_2\alpha_2 & \lambda_2^2\alpha_2 & \cdots & \lambda_2^{N-1}\alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_N & \lambda_N\alpha_N & \lambda_N^2\alpha_N & \cdots & \lambda_N^{N-1}\alpha_N \end{pmatrix}$$

is invertible if and only if all eigenvalues $0, \lambda_2, \dots, \lambda_N$ are pairwise distinct, and all coefficients $\alpha_2, \dots, \alpha_N$ are nonzero.

$\longrightarrow$ Algebraic conditions, whence the "almost".

Controllability near the consensus manifold

Proposition

Local controllability holds at almost every point of the consensus manifold. Moreover, controllability can be realized with time sparse and componentwise sparse controls.

Corollary

Any point of $(\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ can be steered to almost any point of the consensus manifold in finite time by means of a time sparse and componentwise sparse control.

(by stabilization and then iterated local controllability along a path of consensus points)

Sparse optimal control

Another way of designing sparse controls: by optimal control

Optimal control problem with a fixed initial point and free final point, where the cost to be minimized is

$$\int_0^T \sum_{i=1}^N \Big(v_i(t) - \frac{1}{N}\sum_{j=1}^N v_j(t) \Big)^2 dt + \gamma \sum_{i=1}^N \int_0^T \|u_i(t)\| dt$$

where $\gamma > 0$ is fixed, under the constraint $\sum_{i=1}^N \|u_i(t)\| \leq M$.

The ℓ^1-norm in the red term implies componentwise sparsity features of the optimal control.

(proof by applying the Pontryagin maximum principle + genericity arguments)

Generalizations

There are many generalizations of the Cucker-Smale model:

- general potentials (friction, attraction/repulsion, ...):
  Cucker, Dong, Ha, Ha, Kim, Leonard, Motsch, Slemrod, Tadmor
- stochastic aspects (adding noise): Carrillo, Cucker, Fornasier, Ha, Lee, Levy, Mordecki, Toscani
- delay: ongoing work with Cristina Pignotti
- Models in infinite dimension (hydrodynamic, kinetic, mean field limit):
  Carrillo, Degond, Fornasier, Ha, Hascovek, Motsch, Rosado, Tadmor, Toscani

Example of controlled infinite dimensional model:

$$\frac{\partial f}{\partial t} + v.\nabla_x f = \nabla_v.(\xi(f)f + \chi_\omega uf)$$

where $\xi(f)(x,v,t) = \displaystyle\int \frac{v-w}{(1+\|x-y\|^2)^\beta} f(y,w,t)\,dy\,dw$

with the control $\chi_\omega u$.

(Piccoli - Rossi - Trélat, ongoing)

Other open problems:

- cluster control
- control of opinion formation
- black swan
- cancerology