Remarks on the null controllability of elastic arch and shell

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Koiter type Shell Model :

$$\begin{cases} \rho \boldsymbol{y}_{\varepsilon}'' + \boldsymbol{A_M y_\epsilon} + \varepsilon^2 \boldsymbol{A_F y_\varepsilon} = 0, & \omega \times (0, T) \\ + \text{Boundary and Initial Conditions} \end{cases} \quad (1)$$

- $\varepsilon > 0$: Thickness of the shell
- $\boldsymbol{A_M}$: membranal operator of mixed order
- $\boldsymbol{A_F}$: flexural operator of order 4
- $\omega \subset \mathbb{R}^N$ - mid surface of the shell ($N = 2$) / arch ($N = 1$);
- $\boldsymbol{y_\varepsilon} = (y_{1,\varepsilon}, y_{2,\varepsilon}, y_{3,\varepsilon})$ (Shell) ; $\boldsymbol{y_\varepsilon} = (y_{1,\varepsilon}, y_{3,\varepsilon})$ (Arch)

- Boundary controllability of the system at time T large enough ?

$$\boldsymbol{y_\varepsilon}(T) = \boldsymbol{y}_\varepsilon'(T) = 0, \quad \omega \quad (2)$$

- Asymptotic behavior of the control as $\varepsilon \to 0$?

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Motivations

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Asymptotic behavior as $\varepsilon \to 0$: "$\boldsymbol{y}_{\varepsilon} \to \boldsymbol{y}$"

- $\rho(\varepsilon) = O(1)$, $Ker\,\boldsymbol{A_M} = \{\boldsymbol{0}\}$ (Inhibited shell)

$$\begin{cases} \boldsymbol{y}'' + \boldsymbol{A_M}\boldsymbol{y} = 0, & \omega \times (0,T) \\ + \text{Boundary and Initial Conditions} \end{cases} \tag{3}$$

- $\rho(\varepsilon) = O(\varepsilon^2)$, $Ker\,\boldsymbol{A_M} \neq \{\boldsymbol{0}\}$ (Non inhibited shell)

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$$\boldsymbol{A_M y} = \begin{pmatrix} -(y_{1,1} + r^{-1} y_3)_{,1} \\ r^{-1}(y_{1,1} + r^{-1} \boldsymbol{y_3}) \end{pmatrix} \quad (4)$$

Flexural Operator $\boldsymbol{A_F}$

$$\boldsymbol{A_F y} = \begin{pmatrix} 2r^{-1}(y_{3,111} - 2r^{-1} y_{1,11} - r^{-2} y_{3,1}) \\ \boldsymbol{y_{3,1111}} - 2r^{-1} y_{1,111} - 2r^{-2} y_{3,11} + 2r^{-3} y_{1,1} + r^{-4} y_3 \end{pmatrix} \quad (5)$$

- r^{-1} - curvature ; r - radius of curvature
- $y_{1,1} + r^{-1} y_3$ - longitudinal deformation of the arch
- y_1: tangential displacement; y_3: normal displacement
- $y_{i,1} = \frac{\partial y_i}{\partial \xi}$
- $\xi \in \omega = (0, 1)$: curvilinear abscissae

Remark

If $r^{-1} = 0$, the system reduces to

$$y_1'' - y_{1,11} = 0, \quad y_3'' + \varepsilon^2 y_{3,1111} = 0, \quad \omega \times (0, T)$$

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The case of a cylindrical arch - $N = 1$ - $\omega = (0,1)$

Membranal Operator $\boldsymbol{A_M}$

$$\boldsymbol{A_M y} = \begin{pmatrix} -(y_{1,1} + r^{-1} y_3)_{,1} \\ r^{-1}(y_{1,1} + r^{-1} \boldsymbol{y_3}) \end{pmatrix} \tag{4}$$

Flexural Operator $\boldsymbol{A_F}$

$$\boldsymbol{A_F y} = \begin{pmatrix} 2r^{-1}(y_{3,111} - 2r^{-1} y_{1,11} - r^{-2} y_{3,1}) \\ \boldsymbol{y_{3,1111}} - 2r^{-1} y_{1,111} - 2r^{-2} y_{3,11} + 2r^{-3} y_{1,1} + r^{-4} y_3 \end{pmatrix} \tag{5}$$

- r^{-1} - curvature ; r - radius of curvature
- $y_{1,1} + r^{-1} y_3$ - longitudinal deformation of the arch
- y_1: tangential displacement; y_3: normal displacement
- $y_{i,1} = \frac{\partial y_i}{\partial \xi}$
- $\xi \in \omega = (0,1)$: curvilinear abscissae

Remark

If $r^{-1} = 0$, the system reduces to

$$y_1'' - y_{1,11} = 0, \quad y_3'' + \varepsilon^2 y_{3,1111} = 0, \quad \omega \times (0,T)$$

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Step I - Controllability in the case $\varepsilon > 0$ fixed ?

$$\begin{cases} \boldsymbol{y}'' + \boldsymbol{A}^{\varepsilon}\boldsymbol{y} = \boldsymbol{f}, \quad \text{in} \quad \omega \times (0, T), \\ \boldsymbol{y_1 = v_1}, \quad y_3 = 0, \partial_{\nu}\boldsymbol{y_3} = \boldsymbol{v_3}, \quad \text{on} \quad \partial\omega, \\ (\boldsymbol{y}(\cdot, 0), \boldsymbol{y}'(\cdot, 0)) = (\boldsymbol{y^0}, \boldsymbol{y^1}), \quad \text{in} \quad \omega, \end{cases} \tag{6}$$

$$\forall(\boldsymbol{y^0}, \boldsymbol{y^1}), \exists v_1, v_3 \in L^2(\partial\omega \times (0, T))? \quad \text{s.t.} \quad \boldsymbol{y}(T) = \boldsymbol{y}'(T) = 0$$

The answer is yes because the resolvent of $\boldsymbol{A_M} + \varepsilon^2 \boldsymbol{A_F}$ is compact ... but ...

$$\boldsymbol{V} = H_0^1(\omega) \times H_0^2(\omega), \qquad \boldsymbol{H} = L^2(\omega) \times L^2(\omega)$$

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$\boldsymbol{y}(T) = \boldsymbol{y}'(T) = 0$ if and only (v_1, v_3) satisfies

$$\int_0^T \int_{\partial\omega} \left(\phi_{1,1} \boldsymbol{v_1} + \varepsilon^2 (\phi_{3,11} - 2r^{-1}\phi_{1,1})(\boldsymbol{v_3} - 2r^{-1}\boldsymbol{v_1}) \right) \nu d\sigma dt = < \boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{y}^{\mathbf{1}} >_{\boldsymbol{V},\boldsymbol{V}'} - < \boldsymbol{\phi}^{\mathbf{1}}, \boldsymbol{y}^{\mathbf{0}} >_{\boldsymbol{H},\boldsymbol{H}}$$

Let $\mathcal{J} : \boldsymbol{V} \times \boldsymbol{H} \rightarrow \mathbb{R}$

$$\mathcal{J}(\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}) = \frac{1}{2} \int_0^T \int_{\partial\omega} \left(\phi_{1,1}^2 + \varepsilon^2 (\phi_{3,11} - 2r^{-1}\phi_{1,1})^2 \right) d\sigma dt - < \boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{y}^{\mathbf{1}} >_{\boldsymbol{V},\boldsymbol{V}'} + < \boldsymbol{\phi}^{\mathbf{1}}, \boldsymbol{y}^{\mathbf{0}} >_{\boldsymbol{H},\boldsymbol{H}}$$

Coercivity of $\mathcal{J} \iff$ Observability inequality

$\Longrightarrow \exists C_1 = C_1(r^{-1}, T, \epsilon) > 0$? T large enough, s.t.

$$C_1 \|(\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}})\|^2_{\boldsymbol{V}\times\boldsymbol{H}} \leq \int_0^T \int_{\partial\omega} \left(\phi_{1,1}^2 + \varepsilon^2 (\phi_{3,11} - 2r^{-1}\phi_{1,1})^2 \right) d\sigma dt \tag{7}$$

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$$T^\star(r^{-1}, \varepsilon) = \frac{2C}{1 - r^{-1} D(\varepsilon)} \tag{8}$$

and let $b^\varepsilon(\phi, \phi)$ such that $\int_\omega b^\varepsilon(\phi, \psi) d\xi = \int_\omega \boldsymbol{A}^\varepsilon \phi \cdot \psi d\xi$ for all $\phi, \psi in \boldsymbol{V}$.

Theorem (Observability)

Let $\varepsilon > 0$ and $r^{-1} < 1/D(\varepsilon)$. For all $T > T^\star(r^{-1}, \varepsilon)$, there exists a constant $C_2 = (1 - r^{-1} D(\varepsilon))$ such that the weak solution of the adjoint system satisfies the inequality

$$\int_0^T \int_{\partial\omega} b^\varepsilon(\phi, \phi) d\sigma dt \geq C_2(r^{-1}, \varepsilon)(T - T^\star) E_0(\phi), \quad \forall (\phi^{\boldsymbol{0}}, \phi^{\boldsymbol{1}}) \in \boldsymbol{V} \times \boldsymbol{H}.$$

(From the Korn's inequality, $E_0(\phi)$ defines a norm over $\boldsymbol{V} \times \boldsymbol{H}$). ■

Theorem (Null controllability)

Let $\varepsilon > 0$, $r > D(\varepsilon)$ and $T > T^\star$. For any $(\boldsymbol{y^0}, \boldsymbol{y^1}) \in \boldsymbol{H} \times \boldsymbol{V'}$, there exists a control $\boldsymbol{v} = (v_1, v_3)$ with $v_1, \varepsilon v_3 \in L^2(\partial\omega \times (0, T))$ such that $E(T, \boldsymbol{y}) = 0$. ■

2

[2] B. Miara, V. Valente, *Exact controllability of a shallow Koiter Shell by a Boundary Action*, J. Elasticity (1999).

$\varepsilon > 0$: Observability and HUM (Hilbert Uniqueness Method)

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$\varepsilon > 0$: A numerical illustration

$$\boldsymbol{y}^0 = (\sin(\pi\xi), \sin^2(\pi\xi)), \quad \boldsymbol{y}^1 = (0,0), \quad r^{-1} = C = \pi/4, \quad \varepsilon = 1/10, \quad T = 3$$

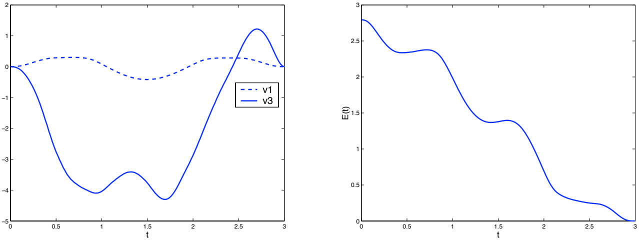

Figure: Control $(v_1, v_3) = (\partial_\nu u_1, \partial^2_{\nu\nu} u_3)$ at $\xi = 0$ **(Left)** - Energy $E_h(t, \boldsymbol{y})$ **(Right)**.

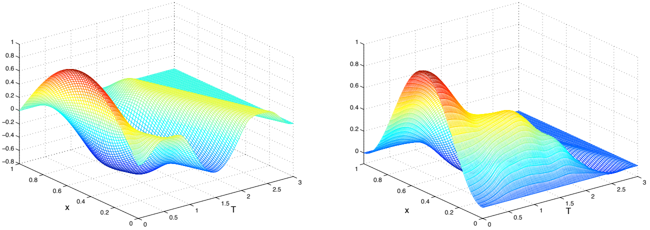

Figure: Controlled displacement y_1 (**Left**) and y_3 (**Right**) on $\omega \times (0, T)$.

$$\boldsymbol{y^0} = (\sin(\pi\xi), \sin^2(\pi\xi)), \qquad \boldsymbol{y^1} = (0,0), \quad C = \pi/4$$

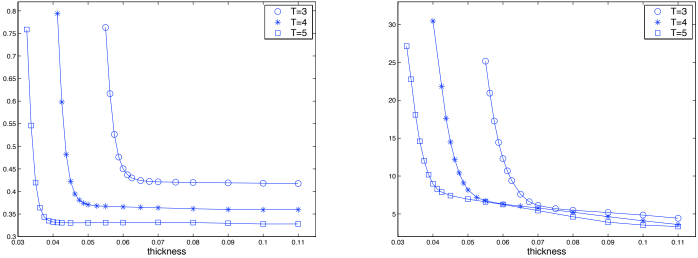

Figure: L^2-norm $||\boldsymbol{v_1}||_{L^2(0,T)}$ (**Left**) and $||\boldsymbol{v_3}||_{L^2(0,T)}$ (**Right**) vs. the thickness ε.

$$T^\star(r^{-1}, \varepsilon) = 2 + O\left(\frac{r^{-1}}{\varepsilon}\right) \tag{9}$$

[4]

$\Longrightarrow$ Non uniform controllability with respect to ε !!!!!!!!!!!!?!!

[4] Geymonat G., Loreti P., Valente V., *Exact controllability of thin elastic hemispherical shell via harmonic analysis*, (1993)

	$\varepsilon = 1/10$	$\varepsilon = 1/20$	$\varepsilon = 1/30$	$\varepsilon = 1/40$
Nb. iterations	8	9	16	n.c.
$\|v_{1h}\|_{L^2(\Sigma_T)}$	2.942×10^{-1}	2.952×10^{-1}	3.095×10^{-1}	n.c.
$\|\varepsilon v_{3h}\|_{L^2(\Sigma_T)}$	3.38×10^{-1}	3.56×10^{-1}	8.52×10^{-1}	n.c.
$\|b^\varepsilon(\phi,\phi)\|_{L^1(\Sigma_T)}$	1.24×10^{-1}	1.295×10^{-1}	3.37×10^{-1}	n.c.
$E_h(T)/E_h(0)$	7.29×10^{-6}	3.36×10^{-5}	1.90×10^{-3}	n.c.

Table: Approximation of the control vs. ε - $\boldsymbol{T = 3}$ - $C = \pi/32$ - $\Sigma_T = \partial\omega \times (0, T)$. n.c. stands for non controllability.

	$\varepsilon = 1/10$	$\varepsilon = 1/20$	$\varepsilon = 1/30$	$\varepsilon = 1/40$
Nb. iterations	8	8	8	8
$\|v_{1h}\|_{L^2(\Sigma_T)}$	2.942×10^{-1}	2.551×10^{-1}	2.302×10^{-1}	2.151×10^{-1}
$\|\varepsilon v_{3h}\|_{L^2(\Sigma_T)}$	3.38×10^{-1}	2.49×10^{-1}	2.41×10^{-1}	2.49×10^{-1}
$\|b^\varepsilon(\boldsymbol{u},\boldsymbol{u})\|_{L^1(\Sigma_T)}$	1.24×10^{-1}	8.57×10^{-2}	7.20×10^{-2}	6.70×10^{-2}
$E_h(T)/E_h(0)$	7.29×10^{-6}	1.04×10^{-5}	2.14×10^{-4}	1.38×10^{-4}

Table: Approximation of the control vs. ε - $\boldsymbol{T(\varepsilon) = 3/(10\varepsilon)}$ - $C = \pi/32$ - $\Sigma_T = \partial\omega \times (0, T)$.

Step 1: Non uniform controllability in $\boldsymbol{H} \times \boldsymbol{V'}$ w.r.t. ε

Step 2: What is about the case $\varepsilon = 0$?

$$\boldsymbol{V} = H_0^1(\omega) \times L^2(\omega), \boldsymbol{H} = L^2(\omega) \times L^2(\omega)$$

$\omega = (0,1),\ T > 0, r > 0,\ (\boldsymbol{y^0}, \boldsymbol{y^1}) \in \boldsymbol{H'} \times \boldsymbol{V'}$

$$\begin{cases} \boldsymbol{y''} + \boldsymbol{A_M y} = \boldsymbol{0}, & \text{in} \quad \omega \times (0,T), \\ y_1(0,t) = 0, \quad \boldsymbol{y_1(1,t) = v(t)}, & t \in (0,T), \\ (\boldsymbol{y}(\cdot,0), \boldsymbol{y'}(\cdot,0)) = (\boldsymbol{y^0}, \boldsymbol{y^1}), & \text{in} \quad \omega, \end{cases} \tag{10}$$

$$\boldsymbol{v} \in L^2(0,T)? \quad \text{s.t.} \quad \boldsymbol{y}(\xi,T) = \boldsymbol{y'}(\xi,T) = 0, \quad \forall \xi \in \omega.$$

$\Longrightarrow$ Exact controllability of y_1 and y_3 by only one control ! [5]

[5] F. Ammar-Khodja, Geymonat G., AM, *On the exact controllability of a system of mixed order with essential spectrum*, C.R.Acad. Sci. Paris Série I, (2008).

Adjoint system -

$$\begin{cases} \phi'' + A_M \phi = \mathbf{0}, & \text{in} \quad \omega \times (0,T), \\ \phi_1(0,\cdot) = \phi_1(1,\cdot) = 0, & t \in (0,T) \\ (\phi(\cdot,0), \phi'(\cdot,0)) = (\phi^0, \phi^1), & \text{in} \quad \omega, \end{cases} \tag{11}$$

Weak formulation - For all $(\phi^0, \phi^1) \in V \times H$, there exists a unique weak solution $\phi \in C(0,T;V) \cap C^1(0,T;H)$ that satisfies the variational problem

$$\int_\omega \phi'' \cdot v \, d\xi + \int_\omega (\phi_{1,1} + r^{-1}\phi_3)(v_{1,1} + r^{-1} v_3) \, d\xi = 0, \quad \forall v \in V.$$

Energy -

$$E(t,\phi) = \frac{1}{2} \int_\omega (|\phi_1'|^2 + |\phi_3'|^2 + (\phi_{1,1} + r^{-1}\phi_3)^2) \, d\xi = E(0,\phi) \quad \forall t \in (0,T)$$

$$A_M\psi = \lambda\psi, \quad \xi \in \omega, \qquad \psi_1 = 0, \quad \xi \in \partial\omega$$

$$\sigma(A_M) = \{0, \lambda_0 = r^{-2}, \lambda_k = r^{-2} + (k\pi)^2, k \geq 1\}, \quad \sigma_{ess}(A_M) = \{0\}. \tag{12}$$

$$Ker A_M = \{v_\zeta = (-r^{-1}\zeta, \zeta_{,1}) \in H, \zeta \in H_0^1(\omega)\}$$

and the eigenfunctions associated with λ_0 and λ_k are respectively :

$$v_0 = (0, 1), \quad v_k = \left(\sin(k\pi\zeta), \frac{r^{-1}}{k\pi}\cos(k\pi\zeta)\right),$$

An orthogonal basis in H of $Ker A_M$ is

$$w_k = \left(-\frac{r^{-1}}{k\pi}\sin(k\pi\zeta), \cos(k\pi\zeta)\right), \quad k = 1, 2, \ldots$$

and $\{w_k, v_0, v_k\}$ is an orthogonal basis in H. This permits to expanded the weak solution of (11) in term of a Fourier serie (setting $\mu_0 = r^{-1}$ and $\mu_k = \sqrt{\lambda_k}$):

$$\phi(\cdot, t) = \sum_{k=1}^{\infty}(a_k + b_k t)w_k + (A_0\cos(\mu_0 t) + B_0\sin(\mu_0 t))v_0 + \sum_{k=1}^{\infty}(A_k\cos(\mu_k t) + B_k\sin(\mu_k t))v_k$$

$$\phi^0 = \sum_{k=1}^{\infty} a_k w_k + A_0 v_0 + \sum_{k=1}^{\infty} A_k v_k, \quad \phi^1 = \sum_{k=1}^{\infty} b_k w_k + \mu_0 B_0 v_0 + \sum_{k=1}^{\infty} \mu_k B_k v_k.$$

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Membranal case - Spectrum of $\boldsymbol{A_M}$

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$C_1, C_2 > 0$? such that

$$C_1 ||(\phi^0, \phi^1)||^2_{V \times H} \leq \int_0^T (\phi_{1,1} + r^{-1}\phi_3)^2(1,t)dt \leq C_2 ||(\phi^0, \phi^1)||^2_{V \times H} \quad \forall (\phi^0, \phi^1) \in V \times H. \qquad (13)$$

The observability does not hold in $Ker A_M$:

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We note $\boldsymbol{H}^{\perp}$ and $\boldsymbol{V}^{\perp}$ the orthogonal of $Ker\, \boldsymbol{A_M}$ in $\boldsymbol{H}$ and $\boldsymbol{V}$

Remark

From the Korn's inequality, the energy $E(t,\phi)$ defines a norm over $\boldsymbol{V}^{\perp} \times \boldsymbol{H}^{\perp}$.

Proposition

[Observability] Let $r > 0$. For all time T such that

$$T > T^{\star}(r) \equiv \frac{2\pi}{\gamma}, \quad \gamma = \min\left(2r^{-1}, \sqrt{r^{-2} + \pi^2} - r^{-1}\right)$$

there exist two strictly positive constants $C_1(r)$ and $C_2(r)$ such that

$$C_1(r)E(0,\phi) \leq \int_0^T (\phi_{1,1} + r^{-1}\phi_3)^2(1,t)dt \leq C_2(r)E(0,\phi)$$

$\forall(\phi^{\mathbf{0}}, \phi^{\mathbf{1}}) \in \boldsymbol{V}^{\perp} \times \boldsymbol{H}^{\perp}$.

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We obtain

$$C_1(r)E(0,\phi) \leq \int_0^T (\phi_{1,1} + r^{-1}\phi_3)^2(1,t)dt \leq C_2(r)E(0,\phi)$$

using

Theorem (Ingham, 1936)

Let $K \in \mathbb{Z}$ and $(w_k)_{k\in K}$ be a family of real numbers satisfying the uniform gap condition

$$\gamma = \inf_{k\neq n} |w_k - w_n| > 0. \tag{14}$$

If I is a bounded interval of length $|I| > 2\pi/\gamma$, then there exist two positives constants C_1 and C_2 such that

$$C_1 \sum_{k\in K} |x_k|^2 \leq \int_I |x(t)|^2 dt \leq C_2 \sum_{k\in K} |x_k|^2$$

for all functions given by the sum

$$x(t) = \sum_{k\in K} x_k e^{iw_k t}$$

with square-summable complex coefficients x_k. ■

with $x(t) = (\phi_{1,1} + r^{-1}\phi_3)(1,t)$ and $I = (0,T)$.

[6]

[6] Ingham A.E., *Some trigonometrical inequalities with applications to the theory of series*, Math. Z., (1936).

We compute that

$$E(0,\phi) = \frac{\mu_0^2}{2}(A_0^2 + B_0^2) + \frac{1}{4}\sum_{k=1}^{\infty}\frac{\mu_k^4}{k^2\pi^2}(A_k^2 + B_k^2).$$

On the other hand, we have

$$\phi_{1,1}(1,t) + r^{-1}\phi_3(1,t) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k\mu_k^2}{k\pi}(A_k + iB_k)e^{-i\mu_k t} + \frac{r^{-1}}{2}(A_0 + iB_0)e^{-i\mu_0 t}$$
$$+ \frac{r^{-1}}{2}(A_0 - iB_0)e^{i\mu_0 t} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k\mu_k^2}{k\pi}(A_k - iB_k)e^{i\mu_k t}.$$

We then apply Ingham's theorem with $I = (0,T)$ and the sequence

$$w = (\cdots, -\mu_2, -\mu_1, -\mu_0, \mu_0, \mu_1, \mu_2, \cdots)$$

to obtain that there exists a positive constant C_1 such that

$$C_1\left(\frac{\mu_0^2}{2}(A_0^2 + B_0^2) + \frac{1}{2}\sum_{k=1}^{\infty}\frac{\mu_k^4}{(k\pi)^2}(A_k^2 + B_k^2)\right) \le \int_0^T (\phi_{1,1}(1,t) + r^{-1}\phi_3(1,t))^2\, dt$$

under the condition $T > 2\pi/\gamma$ with $\gamma = \min(\mu_0 - (-\mu_0), \inf_{k\in\mathbb{N}}|\mu_k - \mu_{k-1}|)$ leading to $T > T^\star(r)$.

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$$T^\star(r) = \frac{\pi}{r^{-1}} \mathcal{X}_{(r^{-1} \leq \pi^2/8)} + \frac{2\pi}{\sqrt{r^{-2} + \pi^2} - r^{-1}} \mathcal{X}_{(r^{-1} > \pi^2/8)}$$

$$\sqrt{8} = T^\star(\frac{\pi^2}{8}) \leq T^\star(r), \quad \forall r > 0$$

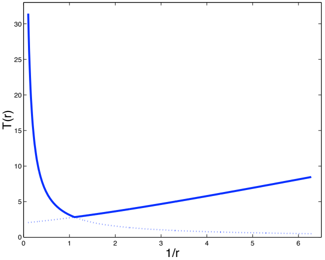

Figure: Lower bound $T^\star$ of the time of controllability T with respect to the curvature r^{-1}.

$\forall (\phi^{\mathbf{0}}, \phi^{\mathbf{1}}) \in \boldsymbol{v_0}$

$$2\min(T, \frac{T^3}{3} r^{-2}) E_0(\phi) \leq \int_0^T (\phi_{1,1} + r^{-1}\phi_3)^2(1,t)\, dt \leq 2\max(T, \frac{T^3}{3} r^{-2}) E_0(\phi). \tag{15}$$

We denote by $\boldsymbol{H_K}$, $\boldsymbol{V_K}$ the closed subspace of $\boldsymbol{H}$ and $\boldsymbol{V}$ generated by $\boldsymbol{v_k}$, for all $k \geq 1$.

Proposition

Let $r > 0$. For all $T > 0$ such that

$$T > T^{\star\star}(r) \equiv \frac{2\pi}{\gamma}, \quad \gamma = \sqrt{r^{-2} + 4\pi^2} - \sqrt{r^{-2} + \pi^2}$$

there exist two positive constants C_1 and C_2 independent of r such that

$$C_1 ||(\phi^{\mathbf{0}}, \phi^{\mathbf{1}})||^2_{\boldsymbol{V}\times\boldsymbol{H}} \leq \int_0^T (\phi_{1,1} + r^{-1}\phi_3)^2(1,t)dt \leq C_2 ||(\phi^{\mathbf{0}}, \phi^{\mathbf{1}})||^2_{\boldsymbol{V}\times\boldsymbol{H}}$$

$\forall (\phi^{\mathbf{0}}, \phi^{\mathbf{1}}) \in \boldsymbol{V_K} \times \boldsymbol{H_K}$. ■

$$T^{\star\star}(r) < T^{\star}(r), \quad \forall r > 0; \qquad lim_{r^{-1}\to 0} T^{\star\star}(r) = 2.$$

We note $\boldsymbol{V}^{\perp\prime}$ the closure of $\boldsymbol{V}^{\perp}$ in $\boldsymbol{V'}$. The control v is characterized by

$$\int_0^T (\phi_{1,1} + r^{-1}\phi_3)(1,t)v(t)dt = <\boldsymbol{\phi}^0, \boldsymbol{y}^1>_{\boldsymbol{V},\boldsymbol{V'}} - <\boldsymbol{\phi}^1, \boldsymbol{y}^0>_{\boldsymbol{H},\boldsymbol{H}} \tag{16}$$

We introduce the continuous and convex functional $\mathcal{J} : \boldsymbol{V} \times \boldsymbol{H} \to \mathbb{R}$ defined by

$$\mathcal{J}(\boldsymbol{\phi}^0, \boldsymbol{\phi}^1) = \frac{1}{2}\int_0^T (\phi_{1,1} + r^{-1}\phi_3)^2(1,t)dt - <(\boldsymbol{\phi}^0, \boldsymbol{\phi}^1), (\boldsymbol{y}^1, -\boldsymbol{y}^0)>_{\boldsymbol{V}\times\boldsymbol{H},\boldsymbol{V'}\times\boldsymbol{H}} \tag{17}$$

Moreover, We then have to enforce that $\boldsymbol{y}^0$ be in the dual of $\boldsymbol{H}^{\perp}$ and that $\boldsymbol{y}^1$ be in the dual of $\boldsymbol{V}^{\perp}$ in order that

$$<\boldsymbol{\phi}^0, \boldsymbol{y}^1>_{\boldsymbol{V},\boldsymbol{V'}} - <\boldsymbol{\phi}^1, \boldsymbol{y}^0>_{\boldsymbol{H},\boldsymbol{H}} \neq 0, \quad \forall(\boldsymbol{\phi}^0, \boldsymbol{\phi}^1) \in \boldsymbol{V}^{\perp} \times \boldsymbol{H}^{\perp}.$$

Let $r > 0$. For any $T > T^\star(r)$ and any initial data $(\boldsymbol{y}^0, \boldsymbol{y}^1) \in \boldsymbol{H}^{\perp} \times \boldsymbol{V}^{\perp\prime}$, there exists a control function $v \in L^2(0,T)$ which drives to rest at time T the solution $\boldsymbol{y}$ of (10) associated with $(\boldsymbol{y}^0, \boldsymbol{y}^1)$. Moreover, the control of minimal L^2-norm is given by $v = (\phi_{1,1} + r^{-1}\phi_3)(1,\cdot)$ where ϕ is solution of (11) and associated with $(\boldsymbol{\phi}^0, \boldsymbol{\phi}^1)$ minimum of $\mathcal{J}$ defined by (17) over $\boldsymbol{V}^{\perp} \times \boldsymbol{H}^{\perp}$. ■

The non controllable modes $\boldsymbol{w_k} \in Ker\,\boldsymbol{A_M}$, $k \geq 1$, do not correspond to modes of arbitrarily small energy. For $(\boldsymbol{y}^0, \boldsymbol{y}^1) = \sum_{k\geq 1}(a_k, b_k)\boldsymbol{w_k}$, the norm of the solution at time $t = T$ is such that (since the control v has no effect on $\boldsymbol{w_k}$)

$$\|\boldsymbol{y}(T)\|^2_{\boldsymbol{V}} = \sum_{k\geq 1}(a_k + b_k T)^2\left(r^{-2} + \frac{\lambda_k}{(k\pi)^2}\right), \quad \|\boldsymbol{y}'(T)\|^2_{\boldsymbol{H}} = \frac{1}{2}\sum_{k\geq 1} b_k^2 \frac{\lambda_k}{(k\pi)^2} \tag{18}$$

Consequently, approximate controllability for the system (10) does not hold anymore. ■

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Moreover, We then have to enforce that $\boldsymbol{y}^{\mathbf{0}}$ be in the dual of $\boldsymbol{H}^{\perp}$ and that $\boldsymbol{y}^{\mathbf{1}}$ be in the dual of $\boldsymbol{V}^{\perp}$ in order that

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Theorem

Let $r > 0$. For any $T > T^{\star}(r)$ and any initial data $(\boldsymbol{y}^{\mathbf{0}}, \boldsymbol{y}^{\mathbf{1}}) \in \boldsymbol{H}^{\perp} \times \boldsymbol{V}^{\perp\prime}$, there exists a control function $v \in L^2(0,T)$ which drives to rest at time T the solution $\boldsymbol{y}$ of (10) associated with $(\boldsymbol{y}^{\mathbf{0}}, \boldsymbol{y}^{\mathbf{1}})$. Moreover, the control of minimal L^2-norm is given by $v = (\phi_{1,1} + r^{-1}\phi_3)(1,\cdot)$ where ϕ is solution of (11) and associated with $(\phi^{\mathbf{0}}, \phi^{\mathbf{1}})$ minimum of $\mathcal{J}$ defined by (17) over $\boldsymbol{V}^{\perp} \times \boldsymbol{H}^{\perp}$. ■

Remark

The non controllable modes $\boldsymbol{w_k} \in Ker\,\boldsymbol{A_M}$, $k \geq 1$, do not correspond to modes of arbitrarily small energy. For $(\boldsymbol{y}^{\mathbf{0}}, \boldsymbol{y}^{\mathbf{1}}) = \sum_{k\geq 1}(a_k, b_k)\boldsymbol{w_k}$, the norm of the solution at time $t = T$ is such that (since the control v has no effect on $\boldsymbol{w_k}$)

$$\|\boldsymbol{y}(T)\|_{\boldsymbol{V}}^2 = \sum_{k\geq 1}(a_k + b_k T)^2\left(r^{-2} + \frac{\lambda_k}{(k\pi)^2}\right), \quad \|\boldsymbol{y}'(T)\|_{\boldsymbol{H}}^2 = \frac{1}{2}\sum_{k\geq 1} b_k^2 \frac{\lambda_k}{(k\pi)^2} \tag{18}$$

Consequently, approximate controllability for the system (10) does not hold anymore. ■

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We introduce the continuous and convex functional $\mathcal{J} : \boldsymbol{V} \times \boldsymbol{H} \rightarrow \mathbb{R}$ defined by

$$\mathcal{J}(\boldsymbol{\phi^0}, \boldsymbol{\phi^1}) = \frac{1}{2}\int_0^T (\phi_{1,1} + r^{-1}\phi_3)^2(1,t)dt - < (\boldsymbol{\phi^0}, \boldsymbol{\phi^1}), (\boldsymbol{y^1}, -\boldsymbol{y^0}) >_{\boldsymbol{V}\times\boldsymbol{H},\boldsymbol{V}'\times\boldsymbol{H}} \tag{17}$$

Moreover, We then have to enforce that $\boldsymbol{y^0}$ be in the dual of $\boldsymbol{H}^{\perp}$ and that $\boldsymbol{y^1}$ be in the dual of $\boldsymbol{V}^{\perp}$ in order that

$$< \boldsymbol{\phi^0}, \boldsymbol{y^1} >_{\boldsymbol{V},\boldsymbol{V}'} - < \boldsymbol{\phi^1}, \boldsymbol{y^0} >_{\boldsymbol{H},\boldsymbol{H}} \neq 0, \quad \forall (\boldsymbol{\phi^0}, \boldsymbol{\phi^1}) \in \boldsymbol{V}^{\perp} \times \boldsymbol{H}^{\perp}.$$

Theorem

Let $r > 0$. For any $T > T^\star(r)$ and any initial data $(\boldsymbol{y^0}, \boldsymbol{y^1}) \in \boldsymbol{H}^{\perp} \times \boldsymbol{V}^{\perp\prime}$, there exists a control function $v \in L^2(0,T)$ which drives to rest at time T the solution $\boldsymbol{y}$ of (10) associated with $(\boldsymbol{y^0}, \boldsymbol{y^1})$. Moreover, the control of minimal L^2-norm is given by $v = (\phi_{1,1} + r^{-1}\phi_3)(1,\cdot)$ where ϕ is solution of (11) and associated with $(\boldsymbol{\phi^0}, \boldsymbol{\phi^1})$ minimum of $\mathcal{J}$ defined by (17) over $\boldsymbol{V}^{\perp} \times \boldsymbol{H}^{\perp}$. ■

Remark

The non controllable modes $\boldsymbol{w_k} \in Ker \boldsymbol{A_M}$, $k \geq 1$, do not correspond to modes of arbitrarily small energy. For $(\boldsymbol{y^0}, \boldsymbol{y^1}) = \sum_{k\geq 1}(a_k, b_k)\boldsymbol{w_k}$, the norm of the solution at time $t = T$ is such that (since the control v has no effect on $\boldsymbol{w_k}$)

$$\|\boldsymbol{y}(T)\|^2_{\boldsymbol{V}} = \sum_{k\geq 1}(a_k + b_k T)^2\left(r^{-2} + \frac{\lambda_k}{(k\pi)^2}\right), \quad \|\boldsymbol{y}'(T)\|^2_{\boldsymbol{H}} = \frac{1}{2}\sum_{k\geq 1} b_k^2 \frac{\lambda_k}{(k\pi)^2} \tag{18}$$

Consequently, approximate controllability for the system (10) does not hold anymore. ■

We assume that the initial condition $(\boldsymbol{y^0}, \boldsymbol{y^1})$ are generated by $\{\boldsymbol{v_k}\}$ and that $(\boldsymbol{y^0}, \boldsymbol{y^1}) \rightharpoonup (\widetilde{\boldsymbol{y^0}}, \widetilde{\boldsymbol{y^1}})$ in $\boldsymbol{H} \times \boldsymbol{V'}$ as r^{-1} goes to zero.
Let y_1 solution of

$$\begin{cases} y_1'' - y_{1,11} = 0, \quad \omega \times (0, T), \\ y_1(0, t) = 0, \quad y_1(1, t) = \widetilde{v}, \quad (0, T), \\ (y_1(\xi, 0), y_1'(\xi, 0)) = (\widetilde{y_1^0}, \widetilde{y_1^1}). \end{cases} \tag{19}$$

Theorem

Let $r > 0$. For any $T > T^{\star\star}(r)$ and any initial data $(\boldsymbol{y^0}, \boldsymbol{y^1}) \in (\boldsymbol{H_K'} \times \boldsymbol{V_K'}$, there exists a control function $v \in L^2(0, T)$ which drives to rest at time T the solution $\boldsymbol{y}$ of (10) associated with $(\boldsymbol{y^0}, \boldsymbol{y^1})$. Moreover, the control of minimal L^2-norm is given by $v = (\phi_{1,1} + r^{-1}\phi_3)(1, \cdot)$ where ϕ is solution of (11) and associated with $(\boldsymbol{\phi^0}, \boldsymbol{\phi^1})$ minimum of $\mathcal{J}$ defined by (17) over $\boldsymbol{V_K} \times \boldsymbol{H_K}$. Finally, this control converges weakly in $L^2(0, T)$ as r^{-1} goes to zero toward the control of minimal L^2-norm which drives to rest the solution y_1 of (19). ■

$$\begin{cases} \boldsymbol{y''} + \boldsymbol{A_M y} = \boldsymbol{0}, & \text{in} \quad \omega \times (0,T), \\ \boldsymbol{r^{-1} y_1 + y_{3,1} = 0}, & \text{in} \quad \omega \times (0,T) \\ y_1(0,t) = 0, y_1(1,t) = v(t), & t \in (0,T) \\ (\boldsymbol{y}(\cdot,0), \boldsymbol{y'}(\cdot,0)) = (\boldsymbol{y^0}, \boldsymbol{y^1}), & \text{in} \quad \omega, \end{cases} \tag{20}$$

If $v \in L^2(0,T)$ is the HUM control, i.e. of minimal L^2-norm for (20), then, using formally the relation $r^{-1}y_1 + y_{3,1} = 0$, v is also a control (but *a priori* not the control of minimal L^2-norm, except at the limit as r^{-1} goes to zero) for y_1 solution of

$$\begin{cases} y_1'' - y_{1,11} + r^{-2} y_1 = 0, & \text{in} \quad \omega \times (0,T), \\ y_1(0,t) = 0, y_1(1,t) = v(t), & t \in (0,T) \\ (y_1(\cdot,0), y_1'(\cdot,0)) = (y_1^0, y_1^1) \in L^2(\omega) \times H^{-1}(\omega) \end{cases}$$

and also a control for y_3 solution of

$$\begin{cases} y_3'' - y_{3,11} + r^{-2} y_3 = 0, & \text{in} \quad \omega \times (0,T), \\ y_{3,1}(0,t) = 0, y_{3,1}(1,t) = -r^{-1} v(t), & t \in (0,T) \\ (y_3(\cdot,0), y_3'(\cdot,0)) = (y_3^0, y_3^1) \in L^2(\omega) \times L^2(\omega). \end{cases}$$

assuming a compatibility condition at time $t = 0$ such that:

$$y_1(\xi,0) = -r y_{3,1}(\xi,0), \quad y_1'(\xi,0) = -r y_{3,1}'(\xi,0), \quad \text{in} \quad \omega.$$

$$\begin{cases} \boldsymbol{y''} + \boldsymbol{A_M y} = \boldsymbol{0}, & \text{in} \quad \omega \times (0, T), \\ \boldsymbol{r^{-1} y_1 + y_{3,1} = 0}, & \text{in} \quad \omega \times (0, T) \\ y_1(0, t) = 0, y_1(1, t) = v(t), & t \in (0, T) \\ (\boldsymbol{y}(\cdot, 0), \boldsymbol{y'}(\cdot, 0)) = (\boldsymbol{y^0}, \boldsymbol{y^1}), & \text{in} \quad \omega, \end{cases} \tag{20}$$

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$$\begin{cases} y_1'' - y_{1,11} + r^{-2} y_1 = 0, & \text{in} \quad \omega \times (0, T), \\ y_1(0, t) = 0, y_1(1, t) = v(t), & t \in (0, T) \\ (y_1(\cdot, 0), y_1'(\cdot, 0)) = (y_1^0, y_1^1) \in L^2(\omega) \times H^{-1}(\omega) \end{cases}$$

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assuming $\mathbf{y^0} \in H^1(\omega) \times L^2(\omega)$, and $\mathbf{y^1} \in L^2(\omega) \times H^{-1}(\omega)$. The variable $z = y_{1,1} + r^{-1} y_3$ solves

$$\begin{cases} z'' - z_{,11} + r^{-2} z = 0, & \text{in} \quad \omega \times (0,T), \\ z(0,t) = 0, \quad z(1,t) = v(t), & t \in (0,T), \\ (z(\cdot,0), z'(\cdot,0)) = (y^0_{1,1} + r^{-1} y^0_3, y^1_{1,1} + r^{-1} y^1_3), & \text{in} \quad \omega; \end{cases} \tag{22}$$

There exists $v \in L^2(0,T)$ such that $z(\xi,T) = (y_{1,1} + r^{-1} y_3)(\xi,T) = 0$ and $z'(\xi,T) = 0$ for T large enough.

$$\phi(\xi,t) = \sum_{k \geq 0} \begin{pmatrix} r^{-1}(a_k(\xi) + b_k(\xi)t) \\ -(a_k(\xi) + b_k(\xi)t)_{,1} \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$+ \sum_{k=1}^{\infty} (A_k \cos(\mu_k t) + B_k \sin(\mu_k t)) \begin{pmatrix} \cos(k\pi\xi) \\ -\frac{r^{-1}}{k\pi} \sin(k\pi\xi) \end{pmatrix}$$

$$C_1 \sum_{k=0}^{\infty} (A_k^2 + B_k^2) \leq \int_0^T \phi_1(1,t)^2 dt \leq C_2 \sum_{k=0}^{\infty} (A_k^2 + B_k^2)$$

provided

$$T > \frac{2\pi}{\sqrt{\pi^2 + r^{-2}}}$$

$$\begin{cases} \boldsymbol{y}'' + \boldsymbol{A_M y} = \boldsymbol{0}, & \text{in} \quad \omega \times (0,T), \\ (y_{1,1} + r^{-1} y_3)(0,t) = 0, \quad \boldsymbol{(y_{1,1} + r^{-1} y_3)(1,t) = v(t)}, & t \in (0,T) \\ (\boldsymbol{y}(\cdot,0), \boldsymbol{y}'(\cdot,0)) = (\boldsymbol{y^0}, \boldsymbol{y^1}), & \text{in} \quad \omega, \end{cases} \tag{21}$$

assuming $\boldsymbol{y^0} \in H^1(\omega) \times L^2(\omega)$, and $\boldsymbol{y^1} \in L^2(\omega) \times H^{-1}(\omega)$. The variable $z = y_{1,1} + r^{-1} y_3$ solves

$$\begin{cases} z'' - z_{,11} + r^{-2} z = 0, & \text{in} \quad \omega \times (0,T), \\ z(0,t) = 0, \quad z(1,t) = v(t), & t \in (0,T), \\ (z(\cdot,0), z'(\cdot,0)) = (y^0_{1,1} + r^{-1} y^0_3, y^1_{1,1} + r^{-1} y^1_3), & \text{in} \quad \omega; \end{cases} \tag{22}$$

There exists $v \in L^2(0,T)$ such that $z(\xi,T) = (y_{1,1} + r^{-1} y_3)(\xi,T) = 0$ and $z'(\xi,T) = 0$ for T large enough.

$$\phi(\xi,t) = \sum_{k \geq 0} \begin{pmatrix} r^{-1}(a_k(\xi) + b_k(\xi)t) \\ -(a_k(\xi) + b_k(\xi)t)_{,1} \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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There exists $v \in L^2(0,T)$ such that $z(\xi,T) = (y_{1,1} + r^{-1} y_3)(\xi,T) = 0$ and $z'(\xi,T) = 0$ for T large enough.

$$\phi(\xi,t) = \sum_{k \geq 0} \begin{pmatrix} r^{-1}(a_k(\xi) + b_k(\xi)t) \\ -(a_k(\xi) + b_k(\xi)t)_{,1} \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$+ \sum_{k=1}^{\infty} (A_k \cos(\mu_k t) + B_k \sin(\mu_k t)) \begin{pmatrix} \cos(k\pi\xi) \\ -\frac{r^{-1}}{k\pi} \sin(k\pi\xi) \end{pmatrix}$$

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assuming $\boldsymbol{y^0} \in H^1(\omega) \times L^2(\omega)$, and $\boldsymbol{y^1} \in L^2(\omega) \times H^{-1}(\omega)$. The variable $z = y_{1,1} + r^{-1} y_3$ solves

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provided

$$T > \frac{2\pi}{\sqrt{\pi^2 + r^{-2}}}$$

For the circular arch, the map is $\phi(\xi) = \Big(r\sin(r^{-1}\xi), r\cos(r^{-1}\xi)\Big)$.

In the general case $\phi(\xi) = (\phi_1(\xi), \phi_2(\xi))$, we have

$$\boldsymbol{A_M y} = \begin{pmatrix} -\gamma_{11,1}(y) - 2\gamma_{11}(y)\Gamma^1_{11} \\ -\gamma_{11}(y)b_{11} \end{pmatrix}. \tag{23}$$

with

$$\begin{cases} t = \phi^2_{1,1} + \phi^2_{2,1}; \Gamma^1_{11} = t^{-1}(\phi_{1,1}\phi_{1,11} + \phi_{2,1}\phi_{2,11}); \\ \gamma_{11}(y) = y_{1,1} - \Gamma^1_{11}y_1 - b_{11}y_3; \\ b_{11} = t^{-3/2}(-\phi_{2,1}\phi_{1,11} + \phi_{1,1}\phi_{2,11}) \end{cases}$$

$$Ker\,\boldsymbol{A_M} = \Big\{(y_1, y_3) \in H^1_0(\omega) \times L^2(\omega), \gamma_{11}(y) = 0\Big\} \neq \emptyset$$
$$= \{v = (\zeta, b^{-1}_{11}(\zeta_{,1} - \Gamma^1_{11}\zeta)) \in \boldsymbol{H}, \zeta \in H^1_0(\omega)\}$$

Once again, $\sigma_{ess}(\boldsymbol{A_M}) = \{0\}$. But, in that case, the discrete spectrum is not explicit.

For the circular arch, the map is $\phi(\xi) = \Big(r\sin(r^{-1}\xi), r\cos(r^{-1}\xi)\Big)$.

In the general case $\phi(\xi) = (\phi_1(\xi), \phi_2(\xi))$, we have

$$\boldsymbol{A_M y} = \begin{pmatrix} -\gamma_{11,1}(y) - 2\gamma_{11}(y)\Gamma^1_{11} \\ -\gamma_{11}(y)b_{11} \end{pmatrix}. \tag{23}$$

with

$$\begin{cases} t = \phi^2_{1,1} + \phi^2_{2,1}; \Gamma^1_{11} = t^{-1}(\phi_{1,1}\phi_{1,11} + \phi_{2,1}\phi_{2,11}); \\ \gamma_{11}(y) = y_{1,1} - \Gamma^1_{11}y_1 - b_{11}y_3; \\ b_{11} = t^{-3/2}(-\phi_{2,1}\phi_{1,11} + \phi_{1,1}\phi_{2,11}) \end{cases}$$

$$\begin{aligned} Ker\boldsymbol{A_M} =& \Big\{(y_1, y_3) \in H^1_0(\omega) \times L^2(\omega), \gamma_{11}(y) = 0\Big\} \neq \emptyset \\ =& \{v = (\zeta, b^{-1}_{11}(\zeta_{,1} - \Gamma^1_{11}\zeta)) \in \boldsymbol{H}, \zeta \in H^1_0(\omega)\} \end{aligned}$$

Once again, $\sigma_{ess}(\boldsymbol{A_M}) = \{0\}$. But, in that case, the discrete spectrum is not explicit.

$(\boldsymbol{y^0}, \boldsymbol{y^1}) \in \boldsymbol{H} \times \boldsymbol{V'}$, $(y_0^T, y_1^T) \in L^2(\omega) \times H^{-1}(\omega)$. Is there exists a control $v \in L^2(0,T)$, such that

$$(y_1(\cdot, T), y_1'(\cdot, T)) = (y_0^T, y_1^T)$$

$$\begin{cases} y_1''(\xi,t) - y_{1,11}(\xi,t) + r^{-1}\int_0^t \sin(r^{-1}(t-u))y_{1,11}(\xi,u)du = 0 & \text{in} \quad \omega \times (0,T) \\ y_1(0,t) = 0; y_1(1,t) = v(t) & (0,T) \end{cases} \tag{24}$$

$$\begin{cases} z''(\xi,t) - z_{,11}(\xi,t) + r^{-1}\int_t^T \sin(r^{-1}(u-t))z_{,11}(x,u)du = 0, & (\xi,t) \in \omega \times (0,T), \\ z(0,t) = z(1,t) = 0, & t \in (0,T), \\ (z(\cdot,T), z'(\cdot,T)) = (z^0, z^1), & \xi \in \omega. \end{cases}$$

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From Ingham theorem, we get the observability (uniform w.r.t. r^{-1}): if $T \geq T^{\star\star}(r)$

$$\exists C > 0, \|z^0, z^1\|^2_{H_0^1 \times L^2} \leq C\int_0^T \Big(z_{,1}(1,t) - r^{-1}\int_t^T \sin(r^{-1}(u-t))z_{,1}(1,u)du\Big)^2 dt, \tag{25}$$

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$$r^{-1} = \pi/5, T = 3.5 > T^\star(r) \approx 3.2552, \quad \mathbf{y^0} = \mathbf{v_0} + \mathbf{v_1} \quad \mathbf{y^1} = \mu_0 \mathbf{v_0} + \mu_2 \mathbf{v_2} \tag{26}$$

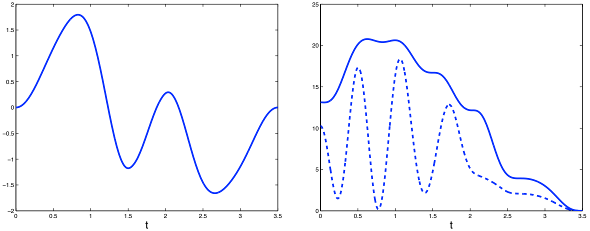

Figure: **Left**: HUM control v; **Right**: Energy (solid line) and kinetic energy (dashed line)

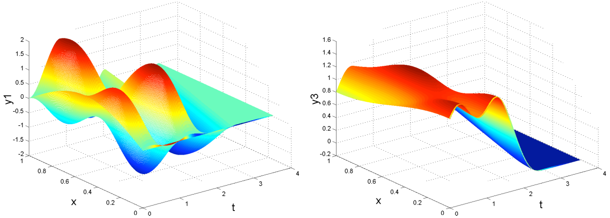

Figure: Controlled solution $\mathbf{y} = (y_1, y_3)$ in $(0, T) \times \omega$.

$$y(x_1, x_3) = y_1(\xi)\boldsymbol{\tau}(\xi) + y_3(\xi)\boldsymbol{\nu}(\xi), \quad \xi \in \omega \quad (x_1, x_3) = \phi(\xi) \tag{27}$$

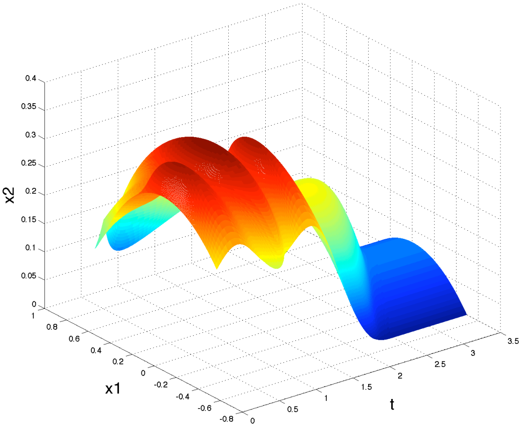

Figure: Evolution in the cartesian plane (O, x_1, x_3) of the controlled arch vs. $t \in (0, T)$.

1-D Membranal case: Numerical experiments

$$y^{\mathbf{0}}(\xi) = \alpha \mathbf{v_0} + \mathbf{v_1}, \quad y^{\mathbf{1}}(\xi) = \alpha\mu_0 \mathbf{v_0} + \mu_2 \mathbf{v_2} \tag{28}$$

	$r^{-1} = \pi$	$r^{-1} = \pi/4$	$r^{-1} = \pi/16$	$r^{-1} = \pi/64$	$r^{-1} = \pi/256$
Nb. Iterations	8	8	9	7	8
$\|\|v\|\|_{L^2(0,T)}$	1.415	1.601	4.823	15.791	59.220
$E(0)/E(T)$	2.51×10^{-5}	4.47×10^{-7}	3.12×10^{-6}	3.26×10^{-5}	4.49×10^{-4}
$\frac{\|\|v\|\|^2_{L^2(0,T)}}{\|\|(\phi^{\mathbf{0}},\phi^{\mathbf{1}})\|\|^2_{\mathbf{V}\times\mathbf{H}}}$	5.33×10^{-1}	7.95×10^{-2}	1.67×10^{-3}	9.86×10^{-5}	6.1804×10^{-6}

Table: $\alpha = 1$ - Evolution of the L^2-norm of the control vs. the curvature r^{-1}.

	$r^{-1} = \pi$	$r^{-1} = \pi/4$	$r^{-1} = \pi/16$	$r^{-1} = \pi/64$	$r^{-1} = \pi/256$
Nb. Iterations	5	4	4	4	4
$\|\|v\|\|_{L^2(0,T)}$	0.823	0.692	0.679	0.678	0.678
$E(0)/E(T)$	1.12×10^{-6}	8.47×10^{-8}	4.28×10^{-6}	2.83×10^{-7}	1.05×10^{-7}
$\frac{\|\|v\|\|^2_{L^2(0,T)}}{\|\|(\phi^{\mathbf{0}},\phi^{\mathbf{1}})\|\|^2_{\mathbf{V}\times\mathbf{H}}}$	2.019	1.565	1.515	1.5126	1.5124
$\frac{\|\|v-v_{r=\infty}\|\|_{L^2(0,T)}}{\|\|v_{r=\infty}\|\|_{L^2(0,T)}}$	1.679	1.24×10^{-1}	7.87×10^{-3}	5.77×10^{-4}	4.28×10^{-5}

Table: $\alpha = 0$ - Evolution of the L^2-norm of the control vs. the curvature r^{-1}.

$$r^{-1} = \pi/5,\ T = 3.5 > T^\star(r) \approx 3.2552, \quad (\mathbf{y^0}, \mathbf{y^1}) = (\mathbf{0}, \mathbf{0}), \quad (y_T^0(\xi), y_T^1(\xi)) = (\sin(\pi\xi), \mu_2 \sin(2\pi\xi)) \tag{29}$$

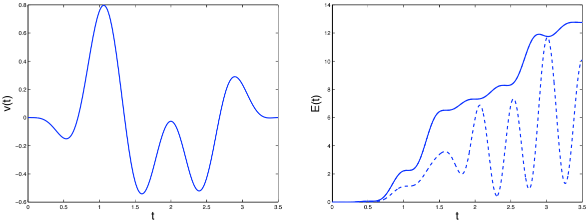

Figure: **Left**: RHUM control v; **Right**: Energy (solid line) and kinetic energy (dashed line)

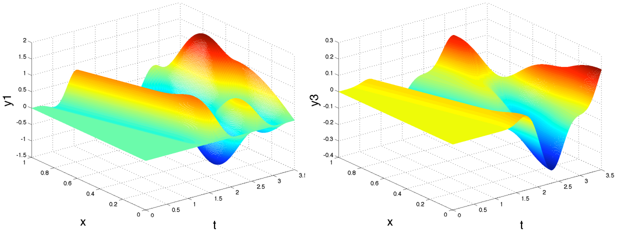

Figure: Evolution of the controlled component y_1 vs. t (**Left**) and corresponding evolution of the component y_3 (**Right**), starting from $(\mathbf{0}, \mathbf{0})$.

Step 1: $\varepsilon > 0$ - Non uniform controllability in $\boldsymbol{H} \times \boldsymbol{V'}$ w.r.t. ε

Step 2: $\varepsilon = 0$ - Controllability only in $(Ker\,\boldsymbol{A_M})^{\perp}$

Step 3: $\varepsilon \rightarrow 0$ in $(Ker\,\boldsymbol{A_M})^{\perp}$

Let us assume that $(\boldsymbol{y_\varepsilon^0}, \boldsymbol{y_\varepsilon^1}) \rightarrow (\boldsymbol{y^0}, \boldsymbol{y^1}) \in (Ker\,\boldsymbol{A_M})^{\perp}$.

Question: Is the observability uniform w.r.t. ε in that case ?

The answer is .... no !!!!!!?!!

(From now,

$$\boldsymbol{A_F y} = \begin{pmatrix} 2r^{-1}(y_{3,111} - 2r^{-1}y_{1,11} - r^{-2}y_{3,1}) \\ y_{3,1111} - 2r^{-1}y_{1,111} - 2r^{-2}y_{3,11} + 2r^{-3}y_{1,1} + r^{-4}y_3 \end{pmatrix} \rightarrow \boldsymbol{A_F^\star y} = \begin{pmatrix} 0 \\ y_{3,1111} \end{pmatrix}$$

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Let us assume that $(\boldsymbol{y^0_\varepsilon}, \boldsymbol{y^1_\varepsilon}) \to (\boldsymbol{y^0}, \boldsymbol{y^1}) \in (Ker\, \boldsymbol{A_M})^{\perp}$.

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- $$\begin{cases} -\psi_{1,11} - r^{-1}\psi_{3,1} = \lambda^\varepsilon \psi_1 \\ r^{-1}\psi_1 + r^{-2}\psi_3 + \varepsilon^2 \psi_{3,1111} = \lambda^\varepsilon \psi_3 \\ \psi_1 = \psi_3 = \psi_{3,1} = 0 \end{cases} \quad (30)$$

$$\varepsilon^2 \psi_i^{(6)} + \lambda \varepsilon^2 \psi_i^{(4)} - \lambda \psi_i^{(2)} - \lambda(\lambda - r^{-2})\psi_i = 0, \quad i = 1, 3$$

- Introducing the corresponding characteristic equation $\varepsilon^2 X^6 + \lambda \varepsilon^2 X^4 - \lambda X^2 - \lambda(\lambda - r^{-2}) = 0$, we have to solve the third order polynomial

$$p(m) = m^3 + \lambda m^2 - \frac{\lambda}{\varepsilon^2} m - \frac{\lambda}{\varepsilon^2}(\lambda - r^{-2}), \quad m = X^2$$

If $\lambda > r^{-2}$ three real roots: $\quad R_1(\lambda) < R_2(\lambda) < 0 < R_3(\lambda)$

- Moreover, with respect to ε, we have

$$R_1^\varepsilon(\lambda) = -\frac{\sqrt{\lambda}}{\varepsilon} - \frac{r^{-2}}{2} + O(\varepsilon), \quad R_3^\varepsilon(\lambda) = \frac{\sqrt{\lambda}}{\varepsilon} - \frac{r^{-2}}{2} + O(\varepsilon),$$
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- $$\begin{cases} -\psi_{1,11} - r^{-1}\psi_{3,1} = \lambda^\varepsilon \psi_1 \\ r^{-1}\psi_1 + r^{-2}\psi_3 + \varepsilon^2\psi_{3,1111} = \lambda^\varepsilon\psi_3 \\ \psi_1 = \psi_3 = \psi_{3,1} = 0 \end{cases} \qquad (30)$$

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$$R_2^\varepsilon(\lambda) = -(\lambda - r^{-2}) + O(\varepsilon^2)$$

- $r_1(\lambda) = \sqrt{-R_1(\lambda)}$, $r_2(\lambda) = \sqrt{-R_2(\lambda)}$ and $r_3(\lambda) = \sqrt{R_3(\lambda)}$
- $$K_1 = \frac{r_1(\lambda)^2 - \lambda}{r^{-1} r_1(\lambda)}, \quad K_2 = \frac{r_2(\lambda)^2 - \lambda}{r^{-1} r_2(\lambda)}, \quad K_3 = \frac{r_3(\lambda)^2 + \lambda}{r^{-1} r_3(\lambda)}$$
- $$\begin{pmatrix} \psi_1(\xi) \\ \psi_3(\xi) \end{pmatrix} = A_1 \begin{pmatrix} \cos(r_1(\lambda)\xi) \\ K_1 \sin(r_1(\lambda)\xi) \end{pmatrix} + A_2 \begin{pmatrix} \sin(r_1(\lambda)\xi) \\ -K_1 \cos(r_1(\lambda)\xi) \end{pmatrix} + A_3 \begin{pmatrix} \cos(r_2(\lambda)\xi) \\ K_2 \sin(r_2(\lambda)\xi) \end{pmatrix}$$
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- Taking into account the 6 boundary conditions, we get that λ^ε solves the eigenvalue problem if

$$det(D^\varepsilon(\lambda)) = 0 \quad (32)$$

with

$$D^\varepsilon(\lambda) = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ \cos(r_1(\lambda)) & \sin(r_1(\lambda)) & \cos(r_2(\lambda)) & \sin(r_2(\lambda)) & e^{r_3(\lambda)} & e^{-r_3(\lambda)} \\ 0 & -K_1 & 0 & -K_2 & -K_3 & K_3 \\ K_1 \sin(r_1(\lambda)) & -K_1 \cos(r_1(\lambda)) & K_2 \sin(r_2(\lambda)) & -K_2 \cos(r_2(\lambda)) & -K_3 e^{r_3(\lambda)} & K_3 e^{-r_3(\lambda)} \\ K_1 r_1(\lambda) & 0 & K_2 r_2(\lambda) & 0 & -K_3 r_3(\lambda) & -K_3 r_3(\lambda) \\ K_1 r_1 \cos(r_1(\lambda)) & K_1 r_1 \sin(r_1(\lambda)) & K_2 r_2 \cos(r_2(\lambda)) & r_2 K_2 \sin(r_2(\lambda)) & -K_3 r_3 e^{r_3(\lambda)} & -K_3 r_3 e^{-r_3(\lambda)} \end{pmatrix}$$

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The adjoint time dependent solution may then be expended as follows :

$$\phi(\xi,t) = \sum_{\lambda_k^{\varepsilon} \in \Lambda^+} \left(a_k \cos(\sqrt{\lambda_k^{\varepsilon}} t) + \frac{b_k}{\sqrt{\lambda_k^{\varepsilon}}} \sin(\sqrt{\lambda_k^{\varepsilon}} t) \right) \psi(\xi,\lambda_k), \quad \Lambda^+ = \{\lambda^{\varepsilon} > r^{-2}, det(\boldsymbol{D}(\boldsymbol{\lambda}^{\varepsilon})) = 0\}$$

For $\lambda_k = r^{-2} + (k\pi)^2 \in \sigma(\boldsymbol{A_M})$,

$$det(D^{\varepsilon}(\lambda_k)) = O(\sqrt{\epsilon}) \longrightarrow \lambda_k^{\varepsilon} = \lambda_k + O(\varepsilon^{\alpha}), \text{ for some } \alpha > 0 \quad (33)$$

and

$$\sin(r_2(\lambda_k)) = O(\sqrt{\varepsilon})$$

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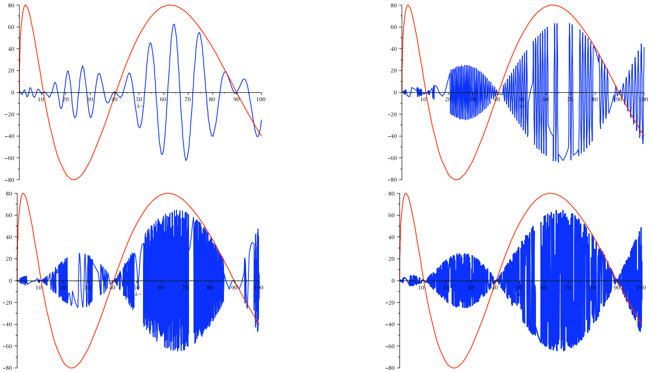

Figure: $\boldsymbol{det(D(\lambda^\varepsilon))}$ and $\mathbf{sin(r_2(\lambda^\varepsilon))}$- $\lambda^\varepsilon > r^{-2}$ - $\varepsilon = 10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}$ - $r^{-1} = 1$

$$\lim_{\varepsilon \longrightarrow 0} \overline{\sigma(\boldsymbol{A^\varepsilon})} = \mathbb{R}^+ \quad \text{(convex hull of } \sigma_{ess}(\boldsymbol{A_M})) \tag{34}$$

The roots of $\sin(r_2(\lambda^\varepsilon))$ converge to $\lambda_k = r^{-2} + (k\pi)^2 \in \sigma(\boldsymbol{A_M})$. The spurious eigenvalues are related to $\sin(r_1(\lambda^\varepsilon)) \approx \sin(\frac{1}{\sqrt{\varepsilon}})$,[7]

[7] Sanchez-Hubert, Sanchez-Palencia, *Coques elastiques minces: Propriétés asymptotique*, 1997.

$$\{\lambda^\varepsilon \in \sigma(\boldsymbol{A}^{\boldsymbol{\varepsilon}}), \lambda^\varepsilon > r^{-2}\} = \sigma^{\varepsilon,+} \cup \sigma^{\varepsilon,-}$$

with

$$\begin{aligned}
\sigma^{\varepsilon,+} &= \{\lambda_k^\varepsilon > r^{-2}, \boldsymbol{A}^{\boldsymbol{\varepsilon}}\psi_k^\varepsilon = \lambda_k^\varepsilon \psi_k^\varepsilon, \lambda_k^\varepsilon \rightarrow r^{-2} + (k\pi)^2 \in \sigma(\boldsymbol{A_M}), \psi_k^{\varepsilon,+} \rightharpoonup \boldsymbol{v_k} \text{ in } \boldsymbol{H}\},\\
\sigma^{\varepsilon,-} &= \{\lambda_k^\varepsilon > r^{-2}, \boldsymbol{A}^{\boldsymbol{\varepsilon}}\psi_k^{\varepsilon,-} = \lambda_k^\varepsilon \psi_k^{\varepsilon,-}, \lambda_k^\varepsilon \rightarrow \delta_k \notin \sigma(\boldsymbol{A_M}), \psi_k^{\varepsilon,-} \rightharpoonup \boldsymbol{0} \text{ in } \boldsymbol{H}\}
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In

$$\begin{pmatrix}\psi_1(\xi)\\ \psi_3(\xi)\end{pmatrix} = A_1\begin{pmatrix}\cos(r_1(\lambda)\xi)\\ K_1\sin(r_1(\lambda)\xi)\end{pmatrix} + A_2\begin{pmatrix}\sin(r_1(\lambda)\xi)\\ -K_1\cos(r_1(\lambda)\xi)\end{pmatrix} + A_3\begin{pmatrix}\cos(r_2(\lambda)\xi)\\ K_2\sin(r_2(\lambda)\xi)\end{pmatrix}$$
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we take $A_4 = 1$ and solve

$$D(\lambda^\varepsilon)(A_1, A_2, A_3, 1, A_5, A_6)^T = 0$$

leading to

$$A_1^\varepsilon, A_2^\varepsilon, A_3^\varepsilon, A_4^\varepsilon = O(\varepsilon^{\alpha/2}), \quad A_5^\varepsilon = O(e^{-1/\sqrt{\varepsilon}}) \quad A_6^\varepsilon = O(\varepsilon^{-\alpha/2})$$

$$\phi(\xi,t) = \sum_{\lambda_k\in\sigma^{\varepsilon,+}}\left(a_k\cos(\sqrt{\lambda_k}t) + \frac{b_k}{\sqrt{\lambda_k}}\sin(\sqrt{\lambda_k}t)\right)\psi(\xi,\lambda_k)$$

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$$\phi_{1,1}(0,t) = \sum_{\lambda_k^\varepsilon \in \sigma^{\varepsilon,+}} \left(a_k \cos(\sqrt{\lambda_k^\varepsilon} t) + \frac{b_k}{\sqrt{\lambda_k^\varepsilon}} \sin(\sqrt{\lambda_k^\varepsilon} t) \right) [r_{1,k}^\varepsilon A_{2,k}^\varepsilon + r_{2,k}^\varepsilon + A_{5,k}^\varepsilon r_{3,k}^\varepsilon - A_{6,k}^\varepsilon r_{3,k}^\varepsilon]$$

$$\phi_{3,11}(0,t) = \sum_{\lambda_k^\varepsilon \in \sigma^{\varepsilon,+}} \left(a_k \cos(\sqrt{\lambda_k^\varepsilon} t) + \frac{b_k}{\sqrt{\lambda_k^\varepsilon}} \sin(\sqrt{\lambda_k^\varepsilon} t) \right) [K_1 (r_{1,k}^\varepsilon)^2 A_{2,k}^\varepsilon + (r_{2,k}^\varepsilon)^2 K_2 + A_{5,k}^\varepsilon K_3 (r_{3,k}^\varepsilon)^2 + A_{6,k}^\varepsilon K_3 (r_{3,k}^\varepsilon)^2]$$

Applying Ingham on $\phi_{1,1}(0,t)$ and on $\phi_{3,11}(0,t)$ plus tedious computations, we get that

Proposition (Uniform observability w.r.t. ε (and r^{-1}))

For any $\varepsilon > 0$, there exist a $T^{\star\star\star}$ and $C_1 > 0$ independent of ε, such that for any $T \geq T^{\star\star\star}(r^{-1})$, the following inequality holds :

$$C_1 E^\varepsilon(0,\phi) \leq \int_0^T (\phi_{1,1}^2 + \varepsilon^2 \phi_{3,11}^2)(0,t)dt, \qquad (36)$$

where ϕ is the solution of the adjoint system with initial condition

$$\phi(\xi,0) = \sum_{\lambda_k^\varepsilon \in \sigma^{\varepsilon,+}} a_k \psi(\xi, \lambda_k^\varepsilon) \in H_0^1(\omega) \times H^2(\omega),$$

$$\phi'(\xi,0) = \sum_{\lambda_k^\varepsilon \in \sigma^{\varepsilon,+}} b_k \psi(\xi, \lambda_k^\varepsilon) \in L^2(\omega) \times L^2(\omega)$$

- A priori, IMPOSSIBLE, because $V^{\varepsilon,+}$ is unknown.
- A way is to compute the first N $\lambda_k^{\varepsilon} \in \sigma^{\varepsilon,+}$ and corresponding ψ_k^{ε} (duable but not straightforward due to the densification).
- However, the approximate controllability is very likely possible because the spurious modes converge to zero with ε.

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Work in progress by Karine Mauffrey (PhD, Besancon).

$\Omega = (0,1)^2$

$$\begin{cases} u'' = \Delta u + \alpha v_x, & \Omega \times (0,T), \\ v'' = -\alpha u_x - av, & \Omega \times (0,T), \\ u = f, & \partial\Omega \times (0,T), \\ + \text{Initial conditions} \end{cases}$$

$$\lambda^{\pm}_{p,q} = \frac{1}{2}\left((p^2+q^2)\pi^2 + a \pm \sqrt{((p^2+q^2)\pi^2 - a)^2 + 4\alpha^2 p^2 \pi^2}\right)$$

$$e^{\pm}_{p,q}(x,y) = \left(\frac{2(\lambda^{\pm}_{p,q} - a)}{\sqrt{(\lambda^{\pm}_{p,q} - a)^2 + \alpha^2 p^2 \pi^2}} \sin(p\pi x)\sin(q\pi y), \frac{2\alpha p \pi}{\sqrt{(\lambda^{\pm}_{p,q} - a)^2 + \alpha^2 p^2 \pi^2}} \cos(p\pi x)\sin(q\pi y)\right)$$

$$\sigma_{ess} = [a - \alpha^2, a]$$

$$E(u,v,0) \le C(T)\int_0^T \int_\Gamma \left((u_x + \alpha v)^2 + u_y^2\right) d\sigma dt \quad (37)$$

$\Gamma = \{(x,y) \in [0,1]^2, xy = 0\}$ [8]

[8] M. Mehrenberger, An Ingham type proof for the boundary observability of a N-d wave equation, C.R.A.S 2009.

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$$\begin{cases} u'' = \Delta u + \alpha v_x, & \Omega \times (0,T), \\ v'' = -\alpha u_x - av, & \Omega \times (0,T), \\ u = f, & \partial\Omega \times (0,T), \\ + \text{Initial conditions} & \end{cases}$$

$$\lambda^{\pm}_{p,q} = \frac{1}{2}\left((p^2+q^2)\pi^2 + a \pm \sqrt{((p^2+q^2)\pi^2 - a)^2 + 4\alpha^2 p^2 \pi^2}\right)$$

$$e^{\pm}_{p,q}(x,y) = \left(\frac{2(\lambda^{\pm}_{p,q} - a)}{\sqrt{(\lambda^{\pm}_{p,q} - a)^2 + \alpha^2 p^2 \pi^2}} \sin(p\pi x)\sin(q\pi y), \frac{2\alpha p \pi}{\sqrt{(\lambda^{\pm}_{p,q} - a)^2 + \alpha^2 p^2 \pi^2}} \cos(p\pi x)\sin(q\pi y)\right)$$

$$\sigma_{ess} = [a - \alpha^2, a]$$

$$E(u,v,0) \leq C(T) \int_0^T \int_\Gamma \left((u_x + \alpha v)^2 + u_y^2\right) d\sigma dt \tag{37}$$

$\Gamma = \{(x,y) \in [0,1]^2, xy = 0\}$ [8]

[8] M. Mehrenberger, An Ingham type proof for the boundary observability of a N-d wave equation, C.R.A.S 2009.

$\omega = (0,1)^2$, $(\xi_1, \xi_2) \in \omega$ - Two controls $y_\alpha = v_\alpha$ for the three components $\boldsymbol{y} = (y_1, y_2, y_3)$.

$$\boldsymbol{A_M y} = \begin{pmatrix} -a(y_{1,11} + \boldsymbol{r^{-1} y_{3,1}}) - cy_{1,22} - (b+c)y_{2,21} \\ -cy_{2,11} - ay_{2,22} - (b+c)y_{1,12} - b\boldsymbol{r^{-1} y_{3,2}} \\ \boldsymbol{r^{-1}}(a(y_{1,1} + \boldsymbol{r^{-1} y_3}) + by_{2,2}) \end{pmatrix}; \quad \boldsymbol{A_M} = \begin{pmatrix} \boldsymbol{A} & B \\ B^\star & ar^{-2} I \end{pmatrix}. \tag{38}$$

$$a = \frac{8\mu(\lambda+\mu)}{\lambda + 2\mu}; \; b = \frac{4\lambda\mu}{\lambda+2\mu}, \; c = 2\mu \tag{39}$$

- $Ker\boldsymbol{A_M} = \{\phi \in \boldsymbol{V} = H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega), \phi_{1,1} + r^{-1}\phi_3 = 0, \phi_{2,2} = 0, \phi_{1,2} + \phi_{2,1} = 0\} = \{\boldsymbol{0}\}$
- $\sigma_{ess}(\boldsymbol{A_M}) = [0, \frac{2(3\lambda+2\mu)\mu}{\lambda+\mu} r^{-2}] \neq \emptyset \Longrightarrow$ Non uniform controllability in $(L^2(\omega))^3 \times \boldsymbol{V'}$
- $ar^{-2} \in \sigma(\boldsymbol{A_M}) \setminus \sigma_{ess}(\boldsymbol{A_M})$ corresponding to $(0,0,1)$
- $\lambda \in \sigma(\boldsymbol{A_M})$ s.t. $\lambda \neq ar^{-2}$ fulfills

$$\left(\boldsymbol{A} + \frac{r^{-2}}{\lambda - ar^{-2}} BB^\star\right)(\phi_1, \phi_2) = \lambda(\phi_1, \phi_2), \quad (\phi_1, \phi_2) \in H_0^1(\omega) \times H_0^1(\omega) \tag{40}$$

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$$\left(A + \frac{r^{-2}}{\lambda - ar^{-2}} BB^{\star}\right)(\phi_1, \phi_2)^T \equiv A_\lambda(\phi_1, \phi_2)^T = \lambda(\phi_1, \phi_2)^T \tag{41}$$

The analytical computation of the other eigenvalues seems more difficult. However, according to [9], the asymptotic spectrum (which approximate the highest eigenvalues) is derived from the study of A.

Lemma

Let $s = b/a$.Assume that $\lambda > ar^{-2}$. Then A_λ is a positive selfadjoint operator and

$$\forall \phi = (\phi_1, \phi_2) \in D(A_\lambda),\ c \int_\omega |\nabla \phi|^2\, dx \leq \int_\omega A_\lambda(\phi).\phi\, dx \leq d_+(\lambda) \int_\omega |\nabla \phi|^2\, dx,$$

with

$$d_+(\lambda) = a + \frac{\left(1+s^2\right)}{2} \frac{a^2}{r^2\lambda - a} + \frac{1}{2}\left((1+s^2)^2 \left(\frac{a^2}{r^2\lambda - a}\right)^2 + 8\frac{s(a-c)a^2}{r^2\lambda - a} + 4(a-c)^2\right)^{1/2}.$$

Asymptotically, the spectrum of A_λ is included in the spectrum of $-[c, 2a - c]\Delta$.

[9] Grubb-Geymonat, Eigenvalue asymptotics for self-adjoint elliptic mixed order systems with nonempty essential spectrum (1979)

In order to compute the essential spectrum, we consider the determinant of

$$A(\zeta_1,\zeta_2,\alpha) = \begin{pmatrix} a\zeta_1^2 + c\zeta_2^2 & (b+c)\zeta_1\zeta_2 & -iar^{-1}\zeta_1 \\ (b+c)\zeta_1\zeta_2 & c\zeta_1^2 + a\zeta_2^2 & -ibr^{-1}\zeta_2 \\ iar^{-1}\zeta_1 & ibr^{-1}\zeta_2 & ar^{-2} - \alpha \end{pmatrix}$$

The essential spectrum is defined by [10]

$$\sigma_{ess}(\boldsymbol{A_M}) = \{\alpha \in \mathbb{R}, det(A(\zeta_1,\zeta_2,\alpha)) = 0, (\zeta_1,\zeta_2) \in \mathbb{R}^2, (\xi_1,\xi_2) \neq (0,0)\} \quad (42)$$

After computation, we obtain that $det(A(\zeta_1,\zeta_2,\alpha)) = 0$ for α solution of

$$c\left(ac\zeta_1^4 + (a^2 - b^2 - 2cb)\zeta_1^2\zeta_2^2 + ac\zeta_2^4\right)\alpha = c^2\zeta_2^4 r^{-2}(a^2 - b^2)$$

so that

$$\sigma_{ess}(\boldsymbol{A_M}) = \begin{cases} \left[0, \dfrac{4c(a-c)}{a} r^{-2}\right], & \text{if} \quad c \neq 0, \\ \mathbb{R} & \text{if} \quad c = 0 \end{cases}$$

Therefore, the essential spectrum is not empty and is equal to

$$\sigma_{ess}(\boldsymbol{A_M}) = \left[0, \frac{2(3\lambda + 2\mu)\mu}{\lambda + \mu} r^{-2}\right] (c \neq 0)$$

$$\sigma_{ess}(\boldsymbol{A_M}) \neq \emptyset \iff \text{Lack of controllability}\ [11].$$

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[10] Grubb-Geymonat, The essential spectrum of elliptic boundary value problem, Math. Ann. (1977)

[11] Geymonat G., Valente V., *A noncontrollability result for systems of mixed order*, SIAM J. Control Optim, (2000)

$r = 1$, $\sigma_{ess}(A_M) \approx [0, 11.529441]$; known eigenvalue $\lambda_0 = ar^{-2} \approx 14.31578$, $\omega = (0, 1)^2$)

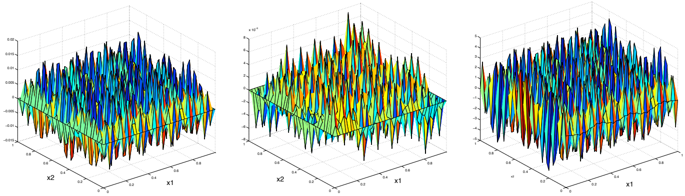

Figure: Approximation of the mode (ϕ_1, ϕ_2, ϕ_3) associated with $\lambda \approx 11.5294$

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[12] Pellerin, Sanchez-Palencia, Local and global vibrations of shells in membrane approximation, (1993)

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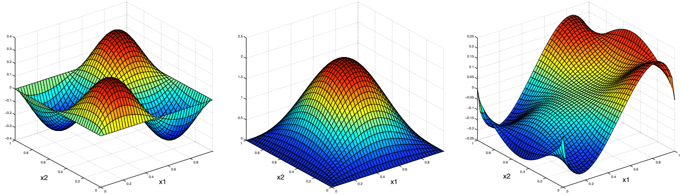

Figure: Approximation of the mode (ϕ_1, ϕ_2, ϕ_3) associated with $\lambda \approx 169.58 > \lambda_0$

Is it possible to eliminate the essential spectrum by optimizing the distribution of (λ, μ) along ω ?

$(\lambda(\boldsymbol{\xi}), \mu(\boldsymbol{\xi})) = (\lambda_\alpha, \mu_\alpha)\mathcal{X}_{\mathcal{O}}(\boldsymbol{\xi}) + (\lambda_\beta, \mu_\beta)(1 - \mathcal{X}_{\mathcal{O}}(\boldsymbol{\xi})), \quad \boldsymbol{\xi} \in \omega, \quad \mathcal{O} \subset \omega$

$$\inf_{\mathcal{O}\subset\omega} \sup_{\boldsymbol{\phi^0},\boldsymbol{\phi^1}} \frac{\|\boldsymbol{\phi^0}, \boldsymbol{\phi^1}\|^2_{\boldsymbol{V}\times\boldsymbol{H}}}{\int_0^T \int_{\partial\omega} b_M(\phi, \phi) d\sigma dt} \tag{43}$$

- Exact controllability and asymptotic in ε do NOT commute
- Similar remarks for parabolic system

$$y' + \boldsymbol{A_M} y = 0 \tag{44}$$

- Multipliers or Carleman do not see the essential spectrum
- Spectral analysis out of the non-controlable space (Spectral compensation method due to Loreti-Komornik)
- Micro-local analysis ?
- Controllability of pure bending shell

$$\begin{aligned} &\boldsymbol{y}'' + \boldsymbol{A_F}\boldsymbol{y} = 0, \quad \Omega \times (0,T), \\ &\boldsymbol{A_M}\boldsymbol{y} = 0, \quad \Omega \times (0,T) \end{aligned} \tag{45}$$

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Final comments

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Thank you for your attention