Stabilité des systèmes hyperboliques avec des commutations

Christophe PRIEUR
CNRS, Gipsa-lab, Grenoble, France

Contrôles, Problèmes inverses et Applications
Clermont-Ferrand
Septembre 2014


ecc15
EUROPEAN CONTROL CONFERENCE
July 15-17, 2015 Linz
Call for Papers
Submission Deadline:
October 20, 2014
Final Submission and
Early Registration Deadline:
March 2015

Two goals

- Stability of hyperbolic systems containing some switches
- Stability of hyperbolic systems thanks to switching rules

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Outline

1 Stability analysis of switched hyperbolic systems
   Motivations using an open channel
   Related works
   Main results

2 Stabilization of hyperbolic systems by means of switching rules
   Motivations (same ones)
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3 Applications on the linear shallow water equations

4 Conclusion

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   - Motivations using an open channel
   - Related works
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3. Applications on the linear shallow water equations
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1 – Switched hyperbolic systems

Consider the following switched linear hyperbolic system

$$\partial_t y(x,t) + \Lambda_i \partial_x y(x,t) = 0 \; , \quad x \in [0,1], t \geq 0 \qquad (1)$$

where, for each $i \in I$, $\Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,n})$, with

$$\lambda_{i,1} < \ldots < \lambda_{i,m} < 0 < \lambda_{i,m+1} < \ldots < \lambda_{i,n}$$

and m does not depend on the index i in a given finite set I.

Use the notation:
$\Lambda_i^+ = \mathrm{diag}(|\lambda_{i,1}|, \ldots, |\lambda_{i,n}|)$, and $y = \left(y_-^\top, \; y_+^\top\right)^\top$

The boundary conditions are

$$\begin{pmatrix} y_+(0,t) \\ y_-(1,t) \end{pmatrix} = G_i \begin{pmatrix} y_+(1,t) \\ y_-(0,t) \end{pmatrix}, \qquad (2)$$

where, for each i in I, $G_i \in \mathbb{R}^{n \times n}$.

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This kind of models appear in many various applications such as

- the traffic flow control [Bressan, Han; 11], [Garavello, Piccoli; 06], [Gugat, Herty, Klar, Leugering; 06]
- the open-channel regulation [Bastin *et al*; 09]...

Example: Level and flow control in an horizontal reach of an open-channel. It will be considered in all this talk

Control = two gates operations:

The linearized shallow water equation may be used.

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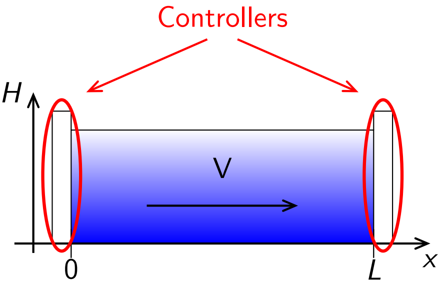

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Review of the literature on the unswitched case

Many technics exist for the unswitched case

Let us consider the following linear hyperbolic system:

$$\begin{array}{c} \partial_t y + \Lambda \partial_x y = 0 \ , \quad x \in [0,1], \ t \geq 0 \\ y(0,t) = G y(1,t) \ , \quad t \geq 0 \end{array} \tag{3}$$

There are sufficient conditions on G so that (3) is Locally Exponentially Stable in H^2, or in C^1...

[Coron, Bastin, d'Andréa-Novel; 08]

[CP, Winkin, Bastin; 08]

Notation:

$$\begin{array}{c} \|G\| = \max\{|Gy|, \ y \in \mathbb{R}^n, \ |y| = 1\} \\ \rho_1(G) = \inf\{\|\Delta G \Delta^{-1}\|, \ \Delta \in \mathcal{D}_{n,+}\} \end{array}$$

[Coron *et al*; 08]: if $\rho_1(G) < 1$ then the system (3) is Exp. Stable.

This sufficient condition is weaker that the one of [Li; 94].

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Problem under consideration

Stability analysis of (1) and (2) for different classes of switching signals.

Define a switching signal as a piecewise constant function $i : \mathbb{R}_+ \to I$.

Denote by $\mathcal{S}(\mathbb{R}_+, I)$ the set of switching signals such that, on each bounded interval of $\mathbb{R}_+$, there is only a finite number of discontinuity points.

Denote by $\mathcal{S}_\tau(\mathbb{R}_+, I)$ the set of switching signals with a *dwell time* larger than τ.

Given a piecewise continuous function $y^0 : [0, 1] \to \mathbb{R}^n$, the initial condition is

$$y(x, 0) = y^0(x) \ , \quad x \in [0, 1] \ . \tag{4}$$

Given a switching signal i, the existence of a solution for the switched hyperbolic (1) with the boundary conditions (2) and the initial condition (4) is established in [Hante, Leugering, Seidman; 09] and [Amin, Hante, Bayen; 12].

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Stability condition under a dwell time property

Assumption 1

For all i in I, the following holds $\rho_1(G_i) < 1$.

Theorem 1 [CP, Girard, Witrant; 14]

Under Assumption 1, there exists $\bar{\tau} > 0$ such that for all $0 < \bar{\tau} < \tau$, the switched system

$$\partial_t y(x,t) + \Lambda_i \partial_x y(x,t) = 0 \ , \quad x \in [0,1], t \geq 0 \qquad (5)$$

with the boundary conditions

$$\begin{pmatrix} y_+(0,t) \\ y_-(1,t) \end{pmatrix} = G_i \begin{pmatrix} y_+(1,t) \\ y_-(0,t) \end{pmatrix}, \qquad (6)$$

is exponentially stable uniformly for all switching signals in $\mathcal{S}_\tau(\mathbb{R}_+, I)$.

Some ideas on the proof

Assumption 1

For all i in I, the following holds $\rho_1(G_i) < 1$.

For the unswitched case If $i(t) = \bar{i}$, for all $t \geq 0$, then Assumption 1 implies with [Coron *et al*; 08], that system $(5)_{\bar{i}}$ with boundary conditions $(6)_{\bar{i}}$ is asymptotically stable.
More precisely, due to the definition of ρ_1 norm, there exists a diagonal positive definite matrix $\Delta_{\bar{i}}$ such that $\|\Delta_{\bar{i}} G_{\bar{i}} \Delta_{\bar{i}}^{-1}\| < 1$.
Then, letting $Q_{\bar{i}} = \Delta_{\bar{i}}^2 (\Lambda_{\bar{i}}^+)^{-1}$, we have

$$\Lambda_{\bar{i}}^+ Q_{\bar{i}} - G_{\bar{i}}^\top Q_{\bar{i}} \Lambda_{\bar{i}}^+ G_{\bar{i}} > 0_n$$

and, then with a suitable $\mu > 0$, the function $V : y \mapsto \int_0^1 e^{-\mu x} y(x)^\top Q_{\bar{i}} y(x) dx$ is a Lyapunov function for the unswitched system $(1)_{\bar{i}}$ and $(2)_{\bar{i}}$. ■

For the switched case Combine these technics with finite-dimensional Lyapunov theory for switched linear systems. ■

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Stability condition for all switching signals

Assumption 1 implies, for each i in I,

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Assumption 2

There exists a diagonal positive definite matrix Q in $\mathbb{R}^{n\times n}$ such that, for all i in I, $\Lambda_i^+ Q - G_i^\top Q \Lambda_i^+ G_i > 0_n$, and $\Lambda_i^+ Q > 0_n$ hold.

Theorem 2

Under Assumption 2, the switched system

$$\partial_t y(x,t) + \Lambda_i \partial_x y(x,t) = 0\ , \quad x \in [0,1], t \geq 0$$

with the boundary conditions $\begin{pmatrix} y_+(0,t) \\ y_-(1,t) \end{pmatrix} = G_i \begin{pmatrix} y_+(1,t) \\ y_-(0,t) \end{pmatrix}$, is exponentially stable uniformly for all switching signals in $\mathcal{S}(\mathbb{R}_+, I)$.

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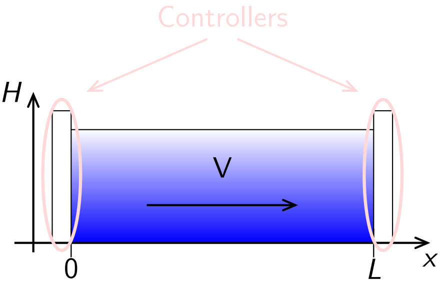

$$\partial_t \begin{pmatrix} H \\ V \end{pmatrix} + \partial_x \begin{pmatrix} HV \\ \frac{V^2}{2} + gH \end{pmatrix} = 0 \qquad (7)$$

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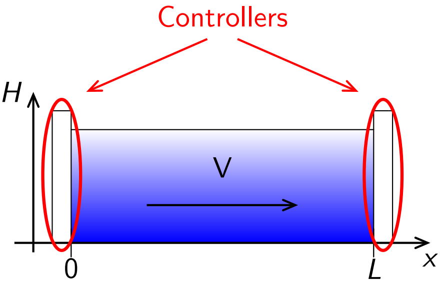

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Linearization around a steady state $(H^\star, V^\star)$

Consider a steady state $(H^\star, V^\star)$, constant not only in time but also in space.
Defining the deviations of the state $H(x,t)$, $V(x,t)$ w.r.t. a steady state as

$$h(x,t) = H(x,t) - H^\star$$
$$v(x,t) = V(x,t) - V^\star$$

it gives

$$\partial_t \begin{pmatrix} h \\ v \end{pmatrix} + \begin{pmatrix} V^\star & H^\star \\ g & V^\star \end{pmatrix} \partial_x \begin{pmatrix} h \\ v \end{pmatrix} = 0. \qquad (8)$$

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Change of variables

Making the diagonalization of the system (8) we define the Riemann coordinates y_i, $i = 1, 2$:

$$y_1 = h + \sqrt{\frac{H^\star}{g}} v \ , \quad y_2 = h - \sqrt{\frac{H^\star}{g}} v$$

To be sure that the matrix of the system in (8) is diagonalizable, the following assumption holds:

$$gH^\star - V^{\star 2} > 0$$

This assumption is called *fluviality assumption*.
Thus the system is rewritten:

$$\partial_t \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \partial_x \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0. \qquad (9)$$

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Boundary conditions

The channel is provided with actuators in such a way that we can prescribe the inflow rate at the beginning and the outflow rate at the end of the channel:

$$u_1(t) = H(0,t)V(0,t) \tag{10a}$$

$$u_2(t) = H(L,t)V(L,t) \tag{10b}$$

It can be shown that the following boundary conditions

$$y_1(0,t) = k_1 y_2(0,t)$$
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are equivalent to (10) for a suitable choice of $u_1(t)$ and $u_2(t)$:

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Motivation for switching and relative questions

- Considering discontinuous controllers may add some degrees of freedom w.r.t. continuous controllers.
- Improve the convergence of the system to the desired behavior

Drawback: the question of existence of solutions in the case of switched system can be hard to solve.

See e.g. [Hante, Sigalotti; 11],
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We are concerned with $n \times n$ linear hyperbolic systems of conservation laws of the form:

$$\partial_t y(x,t) + \Lambda \partial_x y(x,t) = 0,\ t \geq 0,\ x \in [0,1], \tag{11a}$$

$$y(0,t) = G_{\sigma(t)} y(1,t),\ t \geq 0, \tag{11b}$$

$$y(x,0) = y^0(x),\ x \in [0,1], \tag{11c}$$

where $y : [0,1] \times [0,\infty) \to \mathbb{R}^n$, $\sigma : \mathbb{R}_+ \to I$ is a piecewise constant function, and is the controlled switching signal. $I = \{1,\ldots,N\}$ is a finite set of index. $y^0 \in L^2(0,1)$.

Λ is a positive diagonal matrix.

Remark Instead of Λ, it could be considered $\Lambda_{s(t)}$ where $s(t)$ is an (uncontrolled) switching signal.

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Stabilizability

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Problem statement

Stabilization by means of a switching rule

Design $\sigma : [0, \infty) \to I$ satisfying the existence of $c > 0$ and $\alpha > 0$ such that all y^0 in $L^2(0,1)$, the solution of (11) satisfies

$$|y(.,t)|_{L^2(0,1)} \leq ce^{-\alpha t} \left|y^0\right|_{L^2(0,1)} , \qquad (12)$$

for all $t \geq 0$.

Using again a Lyapunov function

The candidate Lyapunov function considered is defined by

$$V(y) = \int_0^1 y^\top Q y e^{-\mu x} dx \,, \tag{13}$$

for all $y \in L^2(0,1)$. Q is a diagonal positive definite matrix $Q \in \mathbb{R}^{n\times n}$.
Along the pde (11a) and using the bound. cond. (11b), it holds

$$\begin{aligned}
\dot V =& -2\int_0^1 y(x,t)^\top Q\Lambda \partial_x y(x,t) e^{-\mu x} dx \\
=& -\left[y(x,t)^\top Q\Lambda y(x,t)e^{-\mu x}\right]_0^1 - \mu \int_0^1 y(x,t)^\top Q\Lambda y(x,t) e^{-\mu x} dx \\
\leq & \, y(1,t)^\top \left[G_i^\top Q\Lambda G_i - Q\Lambda e^{-\mu}\right] y(1,t) - \mu\lambda V \,.
\end{aligned}$$

where $\lambda = \min_{j\in\{1,\ldots,n\}} \lambda_j$.

Let the output be defined as $w(t) = y(1,t)$ and $\alpha = \frac{1}{2}\lambda\mu$ we obtain:

Lemma

$$\dot{V} \leq -2\alpha V + q_i(w(t))$$

where $q_i(w(t)) = q_i(w(t)) = w(t)^\top \left[G_i^\top Q\Lambda G_i - Q\Lambda e^{-\mu}\right] w(t)$.

Thanks to this lemma it is natural to consider the following optimal switching controller [Geromel, Colaneri; 06]

$$\sigma[w](t) = \underset{i \in \{1,\ldots,N\}}{\operatorname{argmin}} \; q_i(w(t)) \tag{14}$$

Let us define the simplex:

$$\Gamma := \left\{ \gamma \in \mathbb{R}^N \,\middle|\, \sum_{i=1}^{N} \gamma_i = 1, \gamma_i \geq 0 \right\}. \tag{15}$$

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Assumption 3

There exist $\gamma \in \Gamma$, a diagonal definite positive matrix Q and a coefficient $\mu > 0$ such that

$$\sum_{i=1}^{N} \gamma_i \left(G_i^\top Q \Lambda G_i - e^{-\mu} Q \Lambda \right) \leq 0_n \, . \qquad (16)$$

Theorem 3 [Lamare, Girard, CP; 14]

Under Assumption 3, system (11a) with switching control (14) is globally exponentially stable. Letting V as proposed, there exist $c > 0$ such that all solutions of (11a) satisfy the inequality

$$|y(., t)|_{L^2(0,1)} \leq c e^{-\mu \lambda t} \left| y^0 \right|_{L^2(0,1)} \, , \qquad (17)$$

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Proof

With the switching control (14):

$$\dot{V} \leq -2\alpha V + q_{\sigma[w](t)}(w(t)) = -2\alpha V + \min_{i\in\{1,\dots,N\}} q_i(w(t)) \,.$$

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Hence $\forall t > 0\,,\ \exists i \in I\,:\ q_i(w(t)) \leq 0$, which gives $\dot{V} \leq -2\alpha V$.

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- $\Lambda_{i\in\{1,2\}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$
- $G_1 = \begin{pmatrix} 1.1 & 0 \\ -0.3 & 0.1 \end{pmatrix}$
- $G_2 = \begin{pmatrix} 0 & 0.2 \\ 0.1 & -1 \end{pmatrix}$
- $Q = I_2$
- $\mu = 0.1$
- $\gamma_1 = \frac{3}{4}$ and $\gamma_2 = \frac{1}{4}$
- $y^0(x) = \begin{pmatrix} \sqrt{2}\sin(3\pi x) \\ \sqrt{2}\sin(4\pi x) \end{pmatrix}$

Data of the linear hyperbolic system

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Assumption 3 holds:

$$A = \sum_{i=1}^{2} \gamma_i \left(G_i^\top Q \Lambda_i G_i - e^{-\mu} Q \Lambda_i \right)$$
$$= \begin{pmatrix} -1.1747 & -0.0825 \\ -0.0825 & -0.0923 \end{pmatrix}$$

$$\mathrm{spec}\,(A) = \{-1.1809; -0.0861\}$$

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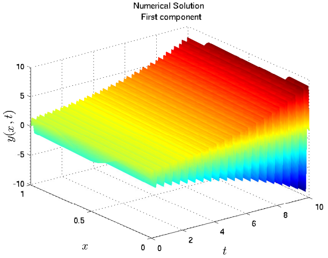
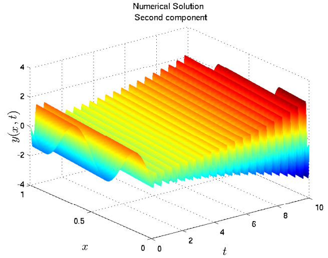
Unstability when applying only G_1





3 – Numerical simulations: academic example

Unstability when applying only G_2

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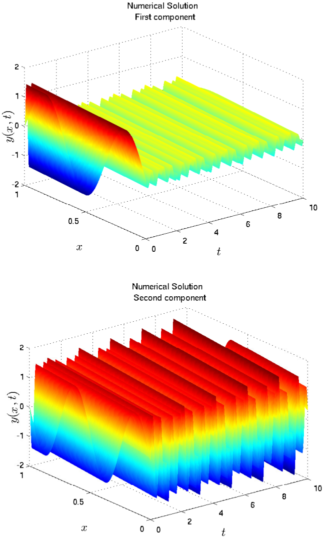

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Stability with argmin control

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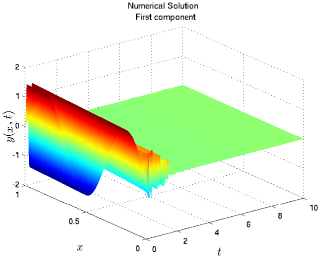

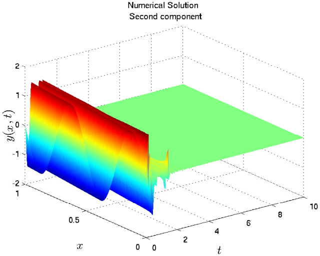

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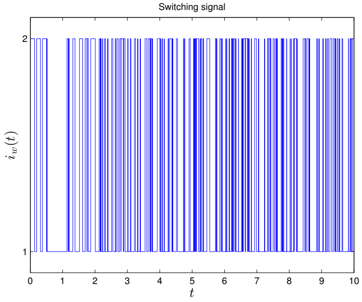
Switching signal:



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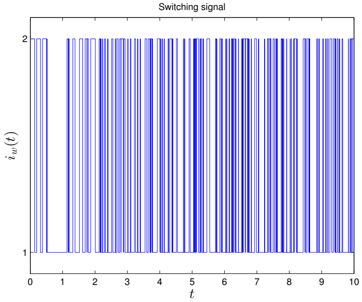

$\Rightarrow$ Fast switching, this behavior is most of the time undesirable!

Control by avoiding this...

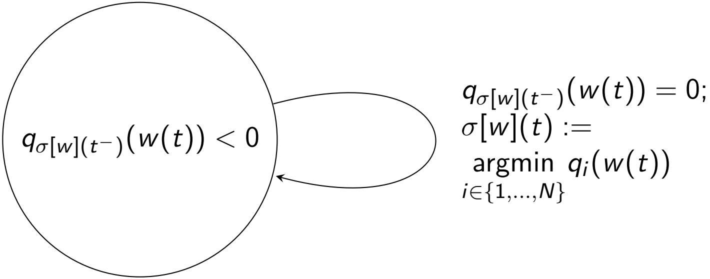

Figure: Argmin with hysteresis automaton.

$$\sigma[w](t) = \begin{cases} \sigma[w](t^-) & \text{if } q_{\sigma[w](t^-)}(w[t]) < 0 \\ \underset{i \in \{1,\ldots,N\}}{\text{argmin}}\ q_i(w(t)), & \text{if } q_{\sigma[w](t^-)}(w(t)) = 0 \end{cases} \tag{18}$$

Theorem 4 (Lamare, Girard, CP; 14)

Under Assumption 3, system (11a) with hysteresis control (18) is globally exponentially stable, i.e. with the same V, there exists $c > 0$ such that all solutions of (11a) satisfy the inequality

$$|y(.,t)|_{L^2(0,1)} \leq c e^{-\mu\lambda t} \left|y^0\right|_{L^2(0,1)} , \tag{19}$$

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Hysteresis controller yields a robustness issue

[CP and Trélat; 05 and 06]: design of hysteresis controller for the stabilization of finite dimensional systems in presence of measurement noise.

In this context:

Instead of the previous hysteresis controller (18), consider the following:

$$\sigma[w](t) = \begin{cases} \sigma[w](t^-) & \text{if } q_{\sigma[w](t^-)}(w[t]) \leq -\gamma\|w(t)\|^2 \\ \underset{i\in\{1,\dots,N\}}{\operatorname{argmin}}\ q_i(w(t)), & \text{if } q_{\sigma[w](t^-)}(w(t)) \geq -\gamma\|w(t)\|^2 \end{cases}$$

then consider $q_{\sigma[w+\delta](t)}(w(t))$ instead of $q_{\sigma[w](t)}(w(t))$ where δ is a small measurement noise wrt $w(t)$, that is (for a given $\rho > 0$),

$$\|\delta(t)\| \leq \rho\|w(t)\|$$

Under actual investigation with Lamare and Girard.

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Under actual investigation with Lamare and Girard.

Thanks to the Lemma it holds:

$$\dot{V} \leq -2\alpha V + q_{\sigma[w](t)}(w(t)) \,.$$

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With Gronwall Lemma:

$$V \leq e^{-2\alpha t} V\left(y^0\right) + e^{-2\alpha t} \int_0^t e^{2\alpha s} q_{\sigma[w](s)}(w(s)) ds \qquad (20)$$

If $e^{-2\alpha t} \int_0^t e^{2\alpha s} q_{\sigma[w](s)}(w(s)) ds \leq 0$ then

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Let us define $m(t) = e^{-2\alpha t} \int_0^t e^{2\alpha s} q_{\sigma[w](s)}(w(s))ds$.

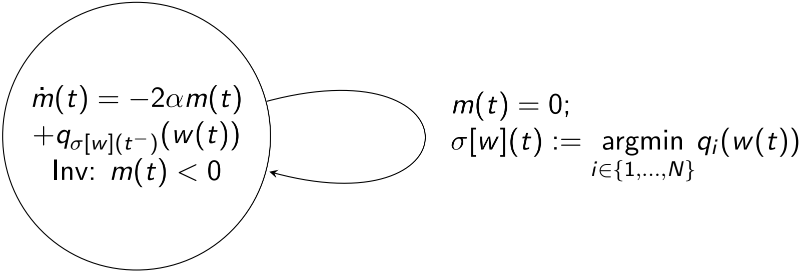

Figure: Argmin, hysteresis and filter automaton.

$$\sigma[w](t) = \begin{cases} \sigma[w](t^-) & \text{if } m(t) < 0 \\ \underset{i\in\{1,\dots,N\}}{\text{argmin}}\; q_i(w(t)) & \text{if } m(t) = 0\,. \end{cases} \tag{21}$$

Theorem 5

Under Assumption 3, system (11a) with the switching rule (21) is globally exponentially stable, i.e. with the same V, there exists $c > 0$ such that all solutions of (11a) satisfy the inequality

$$|y(.,t)|_{L^2(0,1)} \leq ce^{-\mu\lambda t} \left|y^0\right|_{L^2(0,1)} , \tag{22}$$

for all $t \geq 0$.

The proof follows the line than with the *argmin* rule.

Comparison of the strategies

$$y_k^0(x) = \begin{pmatrix} \sqrt{2}\sin((2k-1)\pi x) & \sqrt{2}\sin(2k\pi x) \end{pmatrix}^\top, \; k = 1, 2, 3$$

Initial cond. y_k^0	Argmin	Hysteresis	Low-pass filter
Theoretical bound on the speed of convergence: $\mu = 0.05$			
Number of switches by time unit			
k = 1	4.2	2.8	0.2
k = 2	21.1	13.7	0.1
k = 3	18.5	17.1	0.1
Speed of convergence			
k = 1	1.4278	1.3393	0.0960
k = 2	1.6332	1.5466	0.1279
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Table: Comparison of the different switching strategies for the example with three initial conditions in $L^2([0,1])$ basis. With 10 units of time.

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The simulation parameters are:

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- Width: $l = 1m$
- $H^\star = 0.8\text{m}$
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- $u_1^\star = u_2^\star = 0.72\text{m}^3.\text{s}^{-1}$
- $h(x, t = 0) = 0.1$ m
- $v(x, t = 0) = 0.6$ m.s^{-1}
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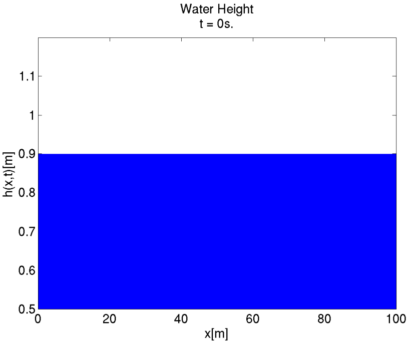

3 – Back to the channel!

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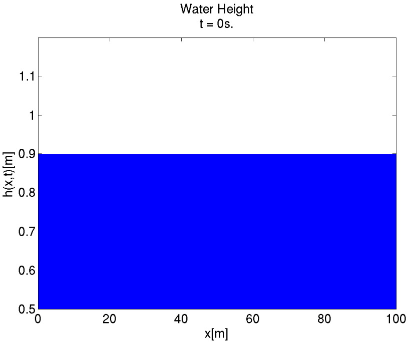

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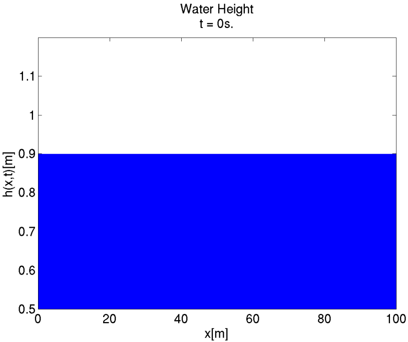

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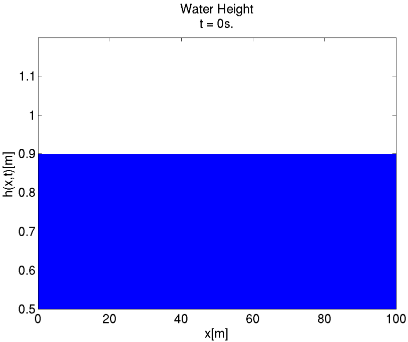

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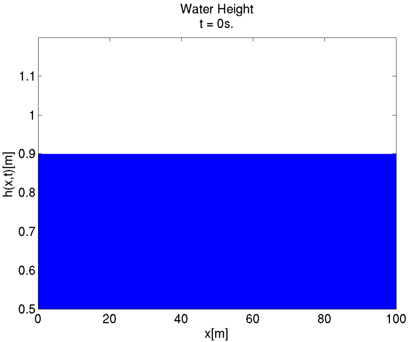

Comparison of the three strategies

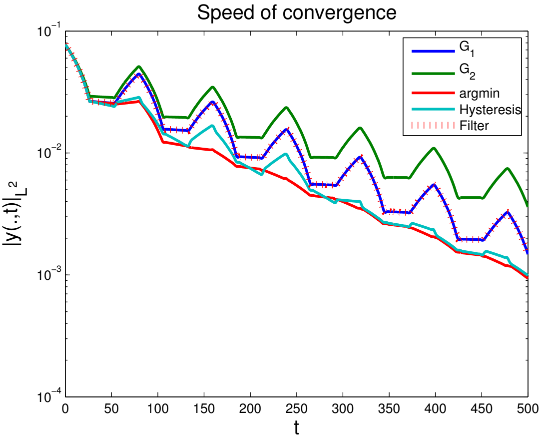

Figure: Convergence in L^2-norm for different control strategies.

Conclusion

- Lyapunov functions have been computed for a class of linear hyperbolic systems
- It gives sufficient stability condition under a dwell time assumption, and without any dwell time

  It generalizes what has been done in [CP, Winkin, Bastin; 08] and [Coron, Bastin, d'Andréa-Novel; 08] and parallels [Amin, Hante, Bayen; 12].

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Open questions

- Improvement of Theorem 4 by considering source terms, and systems of balance laws:

$$\partial_t y + \Lambda_i \partial_x y = F_i y$$

- Generalization to quasilinear hyperbolic systems or quasilinear systems of balance laws (using [Drici, Coron, *preprint*]

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- Applications and stability/performance improvement using switches?

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