Robust Shape Optimization for Elliptic PDEs with random input data

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(joint work with Jesús Martínez Frutos and Mathieu Kessler)
to appear in ESAIM:COCV

Un peu de contrôle dans le Puy-de-Dôme

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Outline

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Therefore, the issue of (UQ) or uncertainty mathematical modelling is of a major importance in real-world problems.

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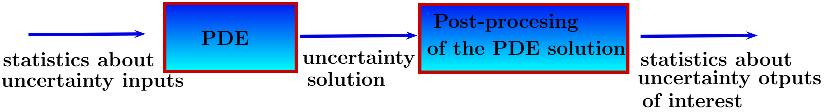

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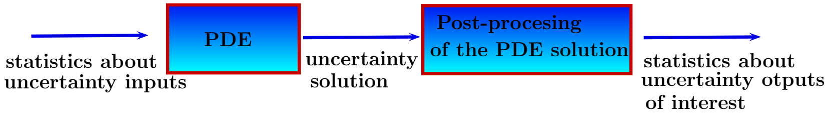

There are many different types of possible input noise. In most of applications Gaussian fields are the model of choice. Thus, next we focus on the mathematical and numerical analysis of them.

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- An $\mathbb{R}^n$-valued random variable $X = (X_1, \cdots, X_n)$ is said to be multivariate Gaussian if for every $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$, the real valued random variable $\sum_{j=1}^n \alpha_j X_j$ is Gaussian. In this case, there exist a mean vector $\mu \in \mathbb{R}^n$, with $\mu_j = E(X_j)$, and a positive definite $n \times n$ covariance matrix C, with elements $C_{ij} = E((X_i - \mu_i)(X_j - \mu_j))$ such that the PDF of X is given by

$$f(x) = \frac{1}{(2\pi)^{d/2}|C|^{1/2}} e^{-\frac{1}{2}(x-\mu)C^{-1}(x-\mu)'}, \quad x \in \mathbb{R}^n.$$

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- A real-valued Gaussian random field defined in a subset $D \subset \mathbb{R}^d$ is a parametrized family of random variables $\{f_x \equiv f(x)\}_{x \in D}$ for which $(f(x^1), \cdots, f(x^m))$ is a multivariate Gaussian random variable for each $1 \leq m < \infty$ and each $(x^1, \cdots, x^m) \in D^m$.

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  The functions
  $$\mu(x) = E(f_x), \quad C(x, x') = E((f_x - \mu(x))(f_{x'} - \mu(x'))), \quad x, x' \in D$$
  are called the mean and covariance of f.

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- The spatially-dependent centred (mean equals zero) random field with covariance function
  $$C(x, x') = \sigma^2 \exp(-\frac{|x - x'|^2}{L^2}), \quad x, x' \in D \subset \mathbb{R}^d,$$
  with σ^2 the variance and L a correlation length parameter, is widely used to model, e.g., uncertainty in the force term of elliptic PDEs.

Approximating Gaussian random fields. The Karhunen-Loève expansion

Since we are interested in spatially-dependent random fields, from now on we shall use the notation $f(x, \omega)$, with x in the spatial domain D and ω in the underlying space of random outcomes Ω.

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Let $C(x,x') \in L^2(D \times D)$ be the covariance function of a Gaussian random field. Consider the compact and self-adjoint operator

$$\varphi \mapsto \int_D C(x,x')\varphi(x')\,dx', \quad \varphi \in L^2(D)$$

and denote by $\{\lambda_n, \varphi_n(x)\}_{n=1}^{\infty}$ its associated eigenpairs.

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Then, the Gaussian random field $f(x,\omega)$ admits the KL expansion

$$f(x,\omega) = \mu(x) + \sum_{n=1}^{\infty} \sqrt{\lambda_n}\varphi_n(x) Y_n(\omega)$$

where $\{Y_n(\omega)\}_{n=1}^{\infty}$ are i.i.d. standard Gaussian (zero mean and unit variance).

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The convergence of the KL expansion is in $L^2(D \times \Omega)$.

The Karhunen-Loève expansion. An example: the Brownian motion

Recall that for Brownian motion B_t

$$\mu(t) = 0 \quad \text{and} \quad C(t, t') = \min(t, t'), \quad t \in D = [0, 1]$$

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Let $D \subset \mathbb{R}^d$ a bounded domain. If the covariance function $C(x, x')$ is

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Given a tolerance $0 < \delta < 1$, the truncation order N of a KL expansion is chosen as the smallest integer N such that

$$\frac{\sum_{n=N+1}^{\infty} \lambda_n}{\sum_{n=1}^{\infty} \lambda_n} \leq \delta \quad \text{or, equivalently} \quad \frac{\sum_{n=1}^{N} \lambda_n}{\sum_{n=1}^{\infty} \lambda_n} \geq 1 - \delta$$

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The associated eigenfunctions $\varphi_m(x)$ enjoy the same regularity as the kernel $C(x, x')$. Moreover, if $C(x, x')$ is piecewise analytic, then for any $s > 0$

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R. A. Todor, *Robust eigenvalue computation for smoothing operators.* SIAM J. Numer. Anal. 44 (2006), no. 2, 865 - 878

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A physical interpretation: D is a two-dimensional membrane and f a vertical force acting on D. We want to reinforce a part of the membrane, i.e. a sub-domain $\mathcal{O}$ of given measure with stiffness equal to 1. In this case, $a = Eh$, with E the modulus of elasticity and h the thickness. These input data (also the force f) show a stochastic probabilistic character.

State law: Assumptions

(A1) There exists a positive random variable $a_{min}(\omega)$ such that

$$a(x,\omega) \geq a_{min}(\omega) > 0 \quad a.s. \quad \omega \in \Omega \quad a.e. \quad x \in D,$$

with $1/a_{min} \in L^2_P(\Omega)$.

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A straightforward application of the Lax-Milgram lemma allows one to state the well posedness of (VP).

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- G. Buttazzo, N. Varchon and H. Zoubairi, *Optimal measures for elliptic problems,* Ann. Mat. Pura Appl. 185(2) (2006) 207-221.
- P. Villaggio, *Mathematical models for elastic structures*, Cambridge University Press, 1997.

Preliminary results I: existence of relaxed optimal shapes

Consider the relaxed problem

$$(ROP)\left\{\begin{array}{ll} \text{Minimize in } s: & J(s) = \alpha E(compl.) + (1-\alpha)Var(compl.) \\ \text{subject to} & \\ & u = u_s \text{ satisfies } (VP), \text{ and} \\ & s \in L^\infty(D;[0,1]), \quad \|s\|_{L^1(D)} = L|D|. \end{array}\right.$$

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Theorem

$J(s)$ is continuous in $L^\infty(D)$ w.r.t. the weak-$\star$ topology. In particular, there exists $s^\star$, admissible for (ROP), such that

$$\inf J(1_{\mathcal{O}}) = \min J(s) = J(s^\star).$$

Moreover, if $\alpha = 1$, then $J(s)$ is convex.

Main ingredients of the proof

- A priori estimates for the solution of the PDE

$$\|u(\cdot,\omega)\|^2_{H^1_0(D)} \leq C_P \frac{\|f(\cdot,\omega)\|_{L^2(D)}}{a_{min}(\omega)} \quad a.s.$$

and

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- Dominated convergence to pass to the limit in the integrals over the probability space

Algorithm of optimization

A gradient-based minimization algorithm is used. Precisely, (an improvement of) the Method of Moving Asymptotes (MMA) as proposed in

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The key point is the computation of the continuous gradient of the objective function.

Computation of the gradient

Theorem

$J(s)$ is Gâteaux differentiable at each admissible s. Moreover, the steepest decent direction of $J(s)$ is given by

$$\begin{aligned} -J'(s)(\cdot) \quad &= \alpha \int_\Omega u_s^2(\cdot,\omega)\,dP(\omega) \\ &-(1-\alpha)\int_\Omega u_s(\cdot,\omega)p_s(\cdot,\omega)\,dP(\omega) \\ &-2(1-\alpha)\left(\int_{\Omega\times D} fu_s\,dxdP(\omega)\right)\int_\Omega u_s^2(\cdot,\omega)\,dP(\omega) \end{aligned}$$

where u_s solves the direct problem and p_s the adjoint problem
Find $p_s \in V_{P,a}$ such that $\forall v \in V_{P,a}$

$$\int_{\Omega\times D}\left[a\nabla p_s\cdot\nabla v + sp_s v\right]dxdP(\omega) = -2\int_\Omega\left[\int_D fu_s\,dx\int_D fv\,dx\right]dP(\omega)$$

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Assumption (finite dimensional noise)

$$a(x,\omega) = a(x, Y_1(\omega), \cdots, Y_N(\omega)), \quad f(x,\omega) = f(x, Y_1(\omega), \cdots, Y_N(\omega))$$

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Denote by $\Gamma_n \equiv Y_n(\Omega)$ the image of Ω by Y_n, $\Gamma = \prod_{n=1}^{N} \Gamma_n$ and assume that $[Y_1, \cdots, Y_N]$ have a joint probability density function $\rho : \Gamma \to \mathbb{R}_+$, with $\rho \in L^\infty(\Gamma)$.

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We then may use Dood-Dynkin's lemma to conclude that $u(x, \omega) = u(x, Y_1(\omega), \cdots, Y_N(\omega))$. Thus our goal is to compute $u(x, y)$ with $x \in D$ and $y \in \Gamma$.

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(VP) has the equivalent form: find $u \in V_{\rho,a}$ such that

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It can be shown that (VP) is equivalent to

$$\int_D [a(y)\nabla u \cdot \nabla\phi + su\phi]\, dx = \int_D f(y)\phi\, dx, \quad \forall \phi \in H_0^1(D), \quad \rho - a.e. \text{ in } \Gamma.$$

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Step 1. Proyection onto $H_h(D)$:

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Step 3. Compute an approximation of the statistical parameters of interest (mean and variance of compliance in our case) by using the fully discrete solution $u_{N,h,p}$.

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if we project onto the finite element space $H_h(D)$ at each node y_k of the random space, then

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Thus,

$$p_h(y_k)=c_ku_h(y_k),\quad \text{with } c_k=-2\int_D f(y_k)u_h(y_k)\,dx.$$

Numerical resolution of SEPDE: a remark on computational costs

Since the number N of random parameters can be much bigger than 2 or 3, the computational cost increases very, very quickly. This phenomenon is known as **the curse of dimensionality** (la malédiction de la dimension).

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Isotropic and anisotropic sparse grids seems to be very efficient (see works by F. Nobile, R. Tempone, .. MOX group at Politecnico di Milano)

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Numerical resolution of SEPDE: some comments on error analysis

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Error truncation due to KL expansion

If $C(x,x')$ is piecewise analytic, then

$$\|u - u_N\| \leq c_1 \exp(-c_2(1/2 - s)N^{1/d}) \quad \text{for any } s > 0.$$

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Error due to finite elements approximation in the physical domain D

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$$\|u_N - u_{N,h}\| \leq \text{ standard bounds}$$

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Error due to stochastic collocation in the random domain Γ_N

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Assume that $a(x, \omega)$ has a truncated *exponential* KL expansion and $f(x, \omega)$ a truncated KL expansion. Then,

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$$\beta_n = O(\sqrt{p_n}) \quad \text{and} \quad r_n = r_n(\sqrt{\lambda_n}, \|\varphi_n\|_{L^\infty(D)})$$

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I. Babuška, F. Novile and R. Tempone, *A Stochastic collocation method for elliptic partial differential equations with random input data*, SIAM Review Vol. 52, n^o 2 (2010), 317 - 355.

Experiment 1: Uncertainty in the force term

$$\begin{cases} -\nabla \cdot [a(x,\omega)\nabla u(x,\omega)] + 1_{\mathcal{O}} u(x,\omega) = f(x,\omega) & \text{in } D \times \Omega \\ u = 0, & \text{on } \partial D \times \Omega \end{cases}$$

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$$f(x,y,\omega) = \begin{cases} 20 & 0 \le x \le 0.4, 0 \le y \le 1, \\ 0.001 & 0.4 < x \le 0.6, 0 \le y \le 1, \\ f_2(x,y,\omega) & 0.6 < x \le 1, 0 \le y \le 1, \end{cases}$$

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$$f_2(x,y,\omega) = \exp(2.9 + 0.2U(x,\omega)), \quad U(x,\omega) = \sum_{n=1}^{10} \lambda_n^{1/2} \varphi_n(x) Y_n(\omega)$$

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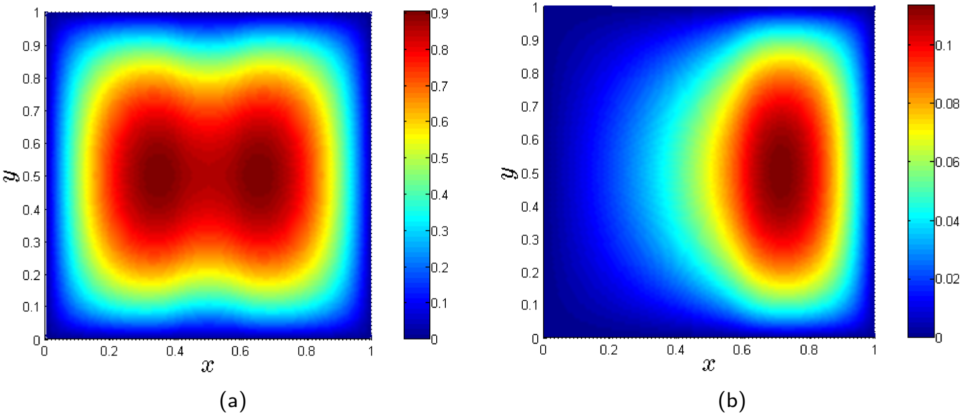

Figure: Experiment 1. Expectation (a) and standard deviation (b) of $u(x, y, \omega)$ without optimization, i.e. $s = 0$.

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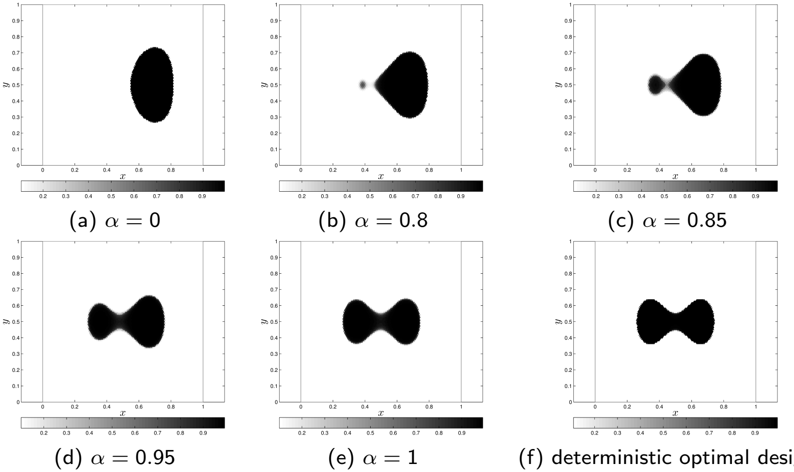

(a) $\alpha = 0$ (b) $\alpha = 0.8$ (c) $\alpha = 0.85$

(d) $\alpha = 0.95$ (e) $\alpha = 1$ (f) deterministic optimal design

Figure: Experiment 1. Optimal design $s(x, y)$ for different values of α. Case (a) corresponds to minimal variance, case (e) minimal expectation. $L = 0.1$

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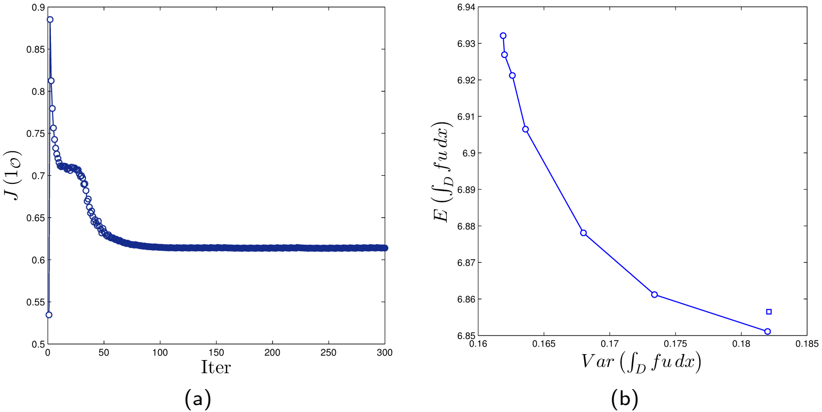

Figure: Experiment 1. Convergence history of the algorithm for $\alpha = 0$ (left), and Pareto front of optimal solutions in circles (right).

Experiment 2: Uncertainty in the boundary conditions

$$\begin{cases} -\nabla \cdot \nabla u(\cdot,\omega) + s(\cdot)u(\cdot,\omega) & = 1 \quad \text{in } D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \\ u(\cdot,\omega) & = 0 \quad \text{on } \Gamma_D(\omega), \\ \dfrac{\partial u}{\partial \vec{n}}(\cdot,\omega) & = 0 \quad \text{on } \Gamma_N(\omega), \end{cases}$$

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$\Gamma_D(\omega)$ is parametrized by $\gamma_D(\omega) : I(\omega) \to \mathbb{R}^2$ with $\gamma_D(\omega,\theta) = (\cos\theta, \sin\theta)$,

$$I(\omega) = [\frac{\omega}{2}, \frac{\pi}{2} - \frac{\omega}{2}] \bigcup [\frac{\pi}{2} + \frac{\omega}{2}, \pi - \frac{\omega}{2}] \bigcup [\pi + \frac{\omega}{2}, \frac{3\pi}{2} - \frac{\omega}{2}] \bigcup [\frac{3\pi}{2} + \frac{\omega}{2}, 2\pi - \frac{\omega}{2}]$$

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ω is a random variable that follows a truncated normal distribution with zero mean and $\sigma = \pi/12$ standard deviation. The lower/upper truncation point of the distribution is $-2\sigma/+2\sigma$.

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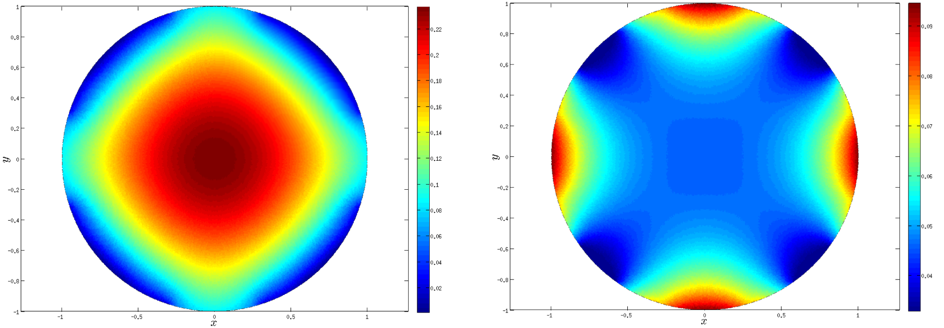

(a) (b)

Figure: Experiment 2. Expectation (a) and standard deviation (b) of the uncontrolled ($s = 0$) solution.

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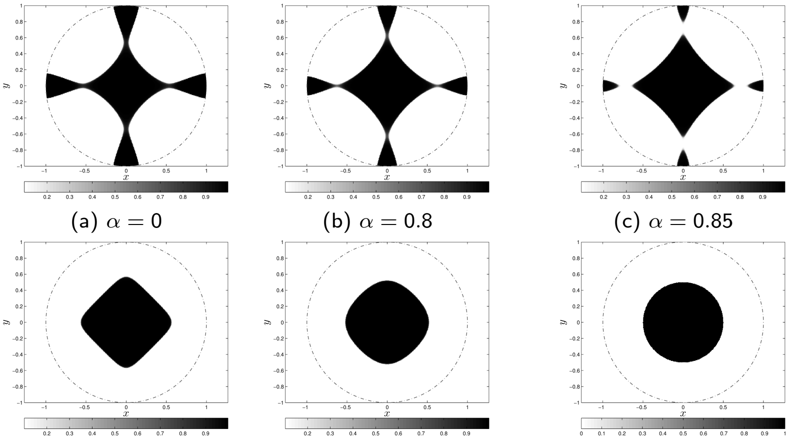

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Other functional costs: risk averse shape optimization

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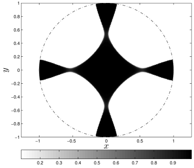
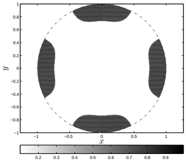
$$J_\varepsilon(1_\mathcal{O}) = \frac{1}{2}\int_\Omega \left[\sqrt{\varepsilon + (X(\omega) - E(X))^2} + (X(\omega) - E(X))\right] dP(\omega), \quad 0 < \varepsilon \ll 1.$$



(a)



(b)

Figure: Experiment 2. Minimal variance (a). Minimal semi-variance (b) ($\varepsilon = 0.01$).

Conclusions and related open problems

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$$\begin{cases} y_t - \nabla \cdot [a(x,\omega)\nabla y(x,\omega)] = u(x,t)1_{\mathcal{O}} & \text{in} \quad (0,T) \times D \times \Omega \\ y = 0, & \text{on} \quad (0,T) \times \partial D \times \Omega \\ y(x,0) = y^0(x) & \text{in} \quad D \end{cases}$$

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Averaged null controllability: Given $T > 0$, find a control $u(x,t)$ such that

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Averaged null controllability is equivalent to the averaged observability inequality

$$\| \int_\Omega \varphi(x,0,\omega)\, dP(\omega)\|^2_{L^2(D)} \le C \int_0^T \int_{\mathcal{O}} |\int_\Omega \varphi(x,t,\omega)\, dP(\omega)|^2\, dxdt$$

for all adjoint solution φ. This inequality is **open**.

Conclusions and related open problems

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M. Lazar and E. Zuazua, *Averaged control and observation of parameter-depending wave equations* , C. R. acad. Sci. Paris, Ser. I 352 (2014) 497-502.

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In order to increase the effectiveness of the control for each realization of the random parameter ω it is natural to look for the averaged null control that minimizes the variance of the null controllability condition.

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An Optimal Control approach of the **Robust Averaged null controllability problem** is: given $T > 0$, find a control $u(x,t)$ that minimizes the functional

$$J_{\alpha,\gamma}(u) = \alpha E(X) + (1-\alpha) Var(X) + \gamma \|u\|^2_{L^2}, \quad 0 \leq \alpha \leq 1, \gamma > 0$$

where

$$X(\omega) = \int_D y^2(x,T,\omega)\, dx$$

and

$$\begin{cases} y_t - \nabla \cdot [a(x,\omega)\nabla y(x,\omega)] = u(x,t)1_{\mathcal{O}} & \text{in} \quad (0,T)\times D \times \Omega \\ y = 0, & \text{on} \quad (0,T)\times \partial D \times \Omega \\ y(x,0) = y^0(x,\omega) & \text{in} \quad D \end{cases}$$

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$$J_\alpha(u) = \alpha \int_D \left(\int_\Omega y(x, T, \omega)\, dP(\omega) \right)^2 dx + (1-\alpha) \| Var(y(T)) \|^2_{L^2(D)}, \quad 0 \leq \alpha \leq 1,$$

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- it is also very natural impose a constraint on the size of $\overline{u}$, namely,

$$|\overline{u}(x,t)| \le C \quad x \in D, t > 0.$$

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People - groups - Links...

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Merci beaucoup!