Polynomial decay rate for a wave equation with general acoustic boundary feedback laws

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Outline

1. Introduction
2. Well Posedness
3. Asymptotic stability
4. Polynomial Energy decay
5. Optimality of the energy decay rate
6. Example

Motivation

Consider the evolution system:

$$\begin{cases} y_{tt}(x,t) - c^2 \Delta y(x,t) = 0 & , x \in \Omega, t > 0, \\ m(x)\delta_{tt}(x,t) + d(x)\delta_t(x,t) + k(x)\delta(x,t) = -\rho y_t(x,t) & , x \in \partial\Omega, t > 0, \\ \frac{\partial y}{\partial \nu}(x,t) = \delta_t(x,t) & , x \in \partial\Omega, t > 0, \end{cases}$$

y is the velocity potential,
δ is the normal displacement.
The acoustic boundary condition was first introduced in

J.T. Beale, S.I. Rosencrans
Acoustic boundary conditions.
Bull. Amer.Math., Vol. 80 (1974), pp. 1276-1278.

Questions

- Existence and Uniqueness of solution in appropriate space.
- Strong stability of the solution.
- Uniform Stability.
- Polynomial Stability.

Non Uniform Stability was proved in

J.T. Beale.
Spectral Properties of an acoustic boundary condition.
Indiana Univ. Math.J., Vol. 25 (1976), pp. 895-917.

A first stability result

$$\begin{cases} y_{tt}(x,t) - c^2 \Delta y(x,t) = 0 & , x \in \Omega, t > 0, \\ y(x,t) = 0 & , x \in \Gamma_1, t > 0, \\ m(x)\delta_{tt}(x,t) + d(x)\delta_t(x,t) + k(x)\delta(x,t) = -\rho y_t(x,t) & , x \in \Gamma_0, t > 0, \\ \frac{\partial y}{\partial \nu}(x,t) = \delta_t(x,t) & , x \in \Gamma_0, t > 0, \end{cases}$$

Polynomial stability:

$$E(t) \leq \frac{C}{t}(E(0) + E_1(0)),$$

Method:Introduce a Lyapounov functional and multiplier method. optimality was not discussed.

J.E. Munoz Rivera, Y. Qin.
Polynomial decay for the energy with an acoustic boundary condition.
Appl. Math.Lett., Vol 16 No. 2 (2003), pp. 249-256.

An extension in 1-d

Generalization of that system:

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) & = & 0, & 0 < x < 1, t > 0, \\ y(0,t) & = & 0, & t > 0, \\ y_x(1,t) + (\eta(t), C) & = & 0, & t > 0, \\ \eta_t(t) - B\eta(t) - Cy_t(1,t) & = & 0, & t > 0, \end{cases} \tag{1}$$

with the following initial conditions:

$$y(x,0) = y_0(x), y_t(x,0) = y_1(x), \ 0 < x < 1, \eta(0) = \eta_0. \tag{2}$$

- n a fixed positive integer and $\eta \in \mathbb{C}^n$ is the control variable,
- $C \in \mathbb{C}^n$ and $B \in M_n(\mathbb{C})$ such that $\Re(BX, X) \leq 0$, for all $X \in \mathbb{C}^n$ with an appropriate i. p. $(X, Y) = \bar{Y}^\top MX$ of $\mathbb{C}^n$.

Main question: Kind of energy decay of this system?

The energy space

Consider the energy space

$$\mathcal{H} = V \times L^2(0,1) \times \mathbb{C}^n,$$

endowed with the inner product

$$((y,z,\eta),(y_1,z_1,\eta_1))_{\mathcal{H}} = \int_0^1 y_x \bar{y}_{1x} dx + \int_0^1 z\bar{z}_1 dx + (\eta,\eta_1),$$

with

$$V = \{y \in H^1(0,1) : y(0) = 0\}.$$

Cauchy problem/Maximal dissipativity

- Cauchy problem:

$$\dot{u} = \mathcal{A}u, \quad u(0) = u_0, \tag{3}$$

$$D(\mathcal{A}) = \{(y, z, \eta) \in H^2(0,1) \cap V \times V \times \mathbb{C}^n : y_x(1) = -(\eta, C)\},$$

and

$$\mathcal{A}\begin{pmatrix} y \\ z \\ \eta \end{pmatrix} = \begin{pmatrix} z \\ y_{xx} \\ B\eta + Cz(1) \end{pmatrix}, \forall \begin{pmatrix} y \\ z \\ \eta \end{pmatrix} \in D(\mathcal{A}).$$

- $\mathcal{A}$ is m-dissipative, i.e., $\lambda I - \mathcal{A}$ is surjective for some $\lambda > 0$ and

$$\Re(\mathcal{A}U, U) = \Re(B\eta, \eta) \leq 0,$$

using Lumer-Phillips' thm, $\mathcal{A}$ generates a C_0-semigroup of contraction.

- $\mathcal{A}$ has a compact resolvent.

Eigenvalue problem

$\lambda \in \sigma(\mathcal{A})$ iff λ satisfies the characteristic equation :

$$C_{\mathcal{A}}(\lambda) = \det \begin{pmatrix} \lambda I - B & -C \sinh \lambda \\ C^* M & \cosh \lambda \end{pmatrix} = 0.$$

Note that each $\lambda \in \sigma(\mathcal{A}) \setminus \sigma(B)$ satisfies:

$$\cosh \lambda + \left((\lambda I - B)^{-1} C, C \right) \sinh \lambda = 0, \quad (4)$$

with the associated eigenvector

$$\alpha(\sinh(\lambda x), \lambda \sinh(\lambda x), (\lambda I - B)^{-1} C \lambda \sinh \lambda).$$

Let B^* be the adjoint matrix of B with respect to $(\cdot,\cdot)$, write
$B = B_0 + R$

- $B_0 = \frac{B-B^*}{2}$ skew-adjoint.
- $R = \frac{B+B^*}{2}$ self-adjoint.

Define

- $A\begin{pmatrix} y \\ z \\ \eta \end{pmatrix} = \begin{pmatrix} z \\ y_{xx} \\ B_0\eta + Cz(1) \end{pmatrix}, \begin{pmatrix} y \\ z \\ \eta \end{pmatrix} \in D(A) = D(\mathcal{A}).$
- $P : \mathbb{C}^n \to W$: the projection map from $\mathbb{C}^n$ onto $W = (\ker R)^\perp$.

Remarks:

$$(-R\kappa, \kappa)^{\frac{1}{2}} \sim \|P\kappa\|, \forall \kappa \in \mathbb{C}^n,$$

$$\frac{d}{dt}E(t) = (R\eta, \eta) \sim -\|P\eta\|^2.$$

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Asymptotic+Non Exp.stability of $\mathcal{A}$

Suppose

(A_1) $\sigma(A) \cap \sigma(B_0) \subset \{ik\pi : k \in \mathbb{Z}\}$,

(A_2) The eigenvector ν_k of B_0 associated with $ik\pi \in \sigma(B_0)$ satisfies $P\nu_k \neq 0$,

(A_3) $P\left((i\mu I - B_0)^{-1} C\right) \neq 0, \forall i\mu \in \sigma(A) \backslash \sigma(B_0)$.

then

- $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$, hence $\mathcal{A}$ is asymptotically stable.
- $P\eta_\mu \neq 0$, for all $i\mu \in \sigma(A)$.

Proposition (Prop 1)

The system (3) is not uniformly stable in the energy space $\mathcal{H}$.

Proof. $(\mathcal{A} - A)(y, z, \eta)^\top = (0, 0, R\eta)^\top$ is a compact perturbation of a generator of a conservative problem (cfr. [Russell, 75]).

Decomposition of solution

- Let u be a solution of the original system

$$\begin{cases} \frac{d}{dt}u(t) &= \mathcal{A}u(t), \quad t > 0, \\ u(0) &= u_0, \end{cases}$$

- Let u_1 be the solution of the conservative problem,

$$\begin{cases} \frac{d}{dt}u_1(t) &= Au_1(t), \quad t > 0, \\ u_1(0) &= u_0. \end{cases} \tag{5}$$

- $u_2 = u - u_1$ fulfills

$$\begin{cases} \frac{d}{dt}u_2(t) &= Au_2(t) + f(t), \quad t > 0, \\ u_2(0) &= 0, \end{cases}$$

where $f = (0, 0, R\eta)$ with η the last component of u.

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An a priori estimate+observability inequality

The proof of our polynomial stability result is based on the following a priori estimate:

Proposition (Prop 2)

For all $T > 0$, there exists $c > 0$ depending on T such that

$$\int_0^T (-R\eta_1(t), \eta_1(t))dt \leq c \int_0^T (-R\eta(t), \eta(t))dt. \tag{6}$$

and on the observability inequality:

Proposition (Prop 3)

There exists $m \in \mathbb{N} := \{0, 1, 2, \ldots\}$, $T > 0$ and $C > 0$ such that

$$\|u_0\|^2_{D(A^{-(m+1)})} \leq C \int_0^T (-R\eta_1(t), \eta_1(t))dt \leq c \int_0^T (-R\eta(t), \eta(t))dt. \tag{7}$$

These two propositions $\Rightarrow$

$$E(0) - E(T) = \int_0^T (-R\eta(t), \eta(t))dt \geq c\|u_0\|^2_{D(A^{-(m+1)})}.$$

We will see later on that this implies that

$$E(t) \leq \frac{M}{(1+t)^{\frac{1}{m+1}}}\|u_0\|^2_{D(\mathcal{A})}.$$

Proof of the a priori estimate

$$
\begin{aligned}
(u_2(t), u_2(t))_{\mathcal{H}} &= \int_0^t \left(e^{A(t-s)} \begin{pmatrix} 0 \\ 0 \\ R\eta(s) \end{pmatrix}, u_2(t) \right)_{\mathcal{H}} ds \\
&= \int_0^t \left(\begin{pmatrix} 0 \\ 0 \\ R\eta(s) \end{pmatrix}, e^{-A(t-s)} u_2(t) \right)_{\mathcal{H}} ds \\
&= \int_0^t \left(R\eta(s), p_3(e^{-A(t-s)} u_2(t)) \right) ds
\end{aligned}
$$

$$
\begin{aligned}
\|u_2(t)\|^2_{\mathcal{H}} &\lesssim \left(\int_0^t (-R\eta(s), \eta(s)) ds \right)^{\frac{1}{2}} \left(\int_0^t \|p_3(e^{-A(t-s)} u_2(t))\|^2_{\mathbb{C}^n} ds \right)^{\frac{1}{2}} \\
&\lesssim \left(\int_0^t (-R\eta(s), \eta(s)) ds \right)^{\frac{1}{2}} \left(\int_0^t \|u_2(t)\|^2_{\mathcal{H}} ds \right)^{\frac{1}{2}} .
\end{aligned}
$$

Proof of the a priori estimate ctd

Hence integrating this estimate between 0 and T

$$\begin{aligned}\int_0^T (-R\eta_2(t), \eta_2(t))dt &\lesssim \int_0^T \|u_2(t)\|_{\mathcal{H}}^2 dt \\ &\lesssim \int_0^T (-R\eta(s), \eta(s))ds.\end{aligned}$$

The conclusion follows by noticing that $\eta_1 = \eta - \eta_2$.

Asymptotic behavior of the spectrum of A

As A is skew-adjoint, all the eigenvalues of A are purely imaginary. Denote by $\lambda = i\mu$ the eigenvalues of A and by $\phi_\mu = (y_\mu, z_\mu, \eta_\mu)$ the associated eigenvectors. An associated eigenvector with $\lambda = i\mu$, where $|\mu| > \|B_0\|$, is given by

$$\phi_\mu = \frac{1}{\sqrt{N(\mu)}}(i\sin(\mu x), -\mu\sin(\mu x), -(i\mu I - B_0)^{-1}C\mu\sin\mu).$$

Assume that there exists $p \in \mathbb{N}$ such that $P(B_0^p C) \neq 0$. Let

$$m = \min\{p \in \mathbb{N} \cup \{0\} : P(B_0^p C) \neq 0\}. \tag{8}$$

The expansions of μ_k (with $\mu_k \in (k\pi, (k+1)\pi)$) and $P\eta_{\mu_k}$ are

$$\mu_k = k\pi + \frac{\pi}{2} + \frac{\|C\|^2}{k\pi} - \frac{\|C\|^2}{2k^2\pi} + \frac{(B_0 C, C)}{ik^2\pi^2} + o\left(\frac{1}{k^2}\right),$$

$$P\eta_{\mu_k} = (-1)^k \frac{1}{i^{m-1}} \left(\frac{P(B_0^m C)}{k^{m+1}\pi^{m+1}} + o\left(\frac{1}{k^{m+1}}\right)\right).$$

An Observability Inequality

Now we suppose that

$$\text{all } \lambda \in \sigma(A) \cap \sigma(B_0) \subset \{in\pi : n \in \mathbb{Z}\} \text{ are simple eigenvalues of } B_0. \quad (9)$$

This implies that A has simple eigenvalues $(i\mu_n)_{n\in I}$ with the gap condition

$$\mu_{n+1} - \mu_n \geq \gamma_0 > 0, \forall n \in I.$$

Proposition

Assume (A_2), (8) and (9) and let $u_1 = (y_1, z_1, \eta_1)^T$ be the solution of the conservative problem (5) with initial datum $u_0 \in D(\mathcal{A})$. Then there exist $T > 0$ and $c > 0$ depending on T such that

$$\int_0^T \|P\eta_1(t)\|^2 dt \geq c\|u_0\|^2_{D(A^{-(m+1)})}. \quad (10)$$

Proof

Since the sequence $(\phi_{0,n})_{n\in I}$ of eigenvectors of A is a Hilbert basis of $\mathcal{H}$, if we write $u_0 = \sum_{n\in I} u_0^{(n)} \phi_{0,n}$, then $(\phi_{0,n} = (y_1^{(n)}, z_1^{(n)}, \eta_1^{(n)}))$

$$u_1(t) = \sum_{n\in I} u_0^{(n)} e^{i\mu_n t} \phi_{0,n} \Rightarrow P\eta_1(t) = \sum_{n\in I} u_0^{(n)} e^{i\mu_n t} P\eta_1^{(n)}.$$

$$\text{(Ingham's inequality)} \Rightarrow \int_0^T \|P\eta_1(t)\|^2 dt \geq c \sum_{n\in I} |u_0^{(n)}|^2 \|P\eta_1^{(n)}\|^2.$$

As for $|n|$ large enough

$$P\eta_1^{(n)} = (-1)^n \frac{1}{i^{m-1}} \left(\frac{P(B_0^m C)}{k_n^{m+1} \pi^{m+1}} + o\left(\frac{1}{k_n^{m+1}}\right)\right), \text{ with } k_n \sim \mu_n,$$

(A_2) and (9) $\Rightarrow$

$$\int_0^T \|P\eta_1\|^2 dt \gtrsim \sum_{n\in I} |u_0^{(n)}|^2 |\mu_n|^{-2(m+1)} \sim \|u_0\|^2_{D(A^{-(m+1)})}.$$

The polynomial stability

Theorem

Let u be a solution of problem (3) with initial datum $u_0 \in D(\mathcal{A})$. *Let the assumptions* $(A_1), (A_2), (A_3)$ *be satisfied. Assume moreover (9) and the existence of m as defined by equation (8), then we obtain the following polynomial energy decay:*

$$E(t) \leq \frac{M}{(1+t)^{\frac{1}{m+1}}} \|u_0\|^2_{D(\mathcal{A})},$$

for some $M > 0$.

Polynomial stability

Proof. Set $E_1(0) = \frac{1}{2}\left(\|u_0\|^2_{\mathcal{H}} + \|Au_0\|^2_{\mathcal{H}}\right)$. Prop. 3 $\Rightarrow$

$$E(T) - E(0) = \int_0^T (R\eta, \eta)dt \leq -K\|u_0\|^2_{D(A^{-(m+1)})}.$$

Using $\|u_0\|^2_{D(A^{-(m+1)})} \geq \frac{\|u_0\|^{2(m+2)}_{\mathcal{H}}}{\|u_0\|^{2(m+1)}_{D(A)}} \geq \frac{E(0)^{m+2}}{E_1(0)^{m+1}}$, we get

$$E((k+1)T) \leq E(kT) - K\frac{E((k+1)T)^{m+2}}{E_1(kT)^{m+1}},$$

dividing by $E_1(0)$,

$$\varepsilon_{k+1} \leq \varepsilon_k - C\varepsilon_{k+1}^{m+2}, \forall k \geq 0,$$

with $\varepsilon_k = \frac{E(kT)}{E_1(0)}$. Hence (due to [Jaffard, Tucsnak and Zuazua, 98])

$$\varepsilon_k \leq \frac{M}{(1+k)^{\frac{1}{m+1}}}, \forall k \geq 0.$$

Spectral analysis of $\mathcal{A}$

To obtain optimality, we need
1. the asymptotic behavior of the eigenvalues of $\mathcal{A}$ near the imaginary axis,
2. the Riesz basis property of the set of generalized eigenvectors of $\mathcal{A}$.

- Rouché's Thm $\Rightarrow$ The number of eigenvalues of $\mathcal{A}$ counted with multiplicities is equal to that of A in the square $C_n = [-n\pi, n\pi] \times [-n\pi, n\pi]$, for n large enough.
- All the eigenvalues of $\mathcal{A}$ have finite algebraic multiplicities. Moreover, the eigenvalues with large enough moduli are algebraically simple.

Proposition (Prop 4)

Let λ be an eigenvalue of $\mathcal{A}$ with $|\lambda|$ large enough. Then λ satisfies the following expansion for some k large enough,

$$\lambda = i\left(k\pi + \frac{\pi}{2} + \frac{\|C\|^2}{k\pi} - \frac{\|C\|^2}{2k^2\pi} + \frac{(B_0C, C)}{ik^2\pi^2}\right) + \frac{(RC, C)}{k^2\pi^2} + o\left(\frac{1}{k^2}\right).$$

Riesz basis property

Proposition (Prop 5)

Assume that (9) holds and

$$\sigma(\mathcal{A}) \cap \sigma(B) = \phi, \tag{12}$$

then the system of generalized eigenvectors of $\mathcal{A}$ forms a Riesz basis of $\mathcal{H}$.

The proof is based on Bari's thm.

Optimality of the energy decay rate

Define

$$\omega(u_0) = \sup\{\alpha \in \mathbb{R}_+ : E(t) = \frac{1}{2}\|u(t)\|^2 \lesssim \frac{1}{t^\alpha}, \forall t > 1\}.$$

Prop. 4 and 5 + [Littman and Markus, 88] $\Rightarrow$

Proposition

Under the assumptions of Prop. 5, if $\Re(\lambda_k) \sim -\frac{1}{k^\delta}$*, with* $\delta \geq 2(m+1)$*, then*

$$\inf_{u_0 \in D(\mathcal{A})} \omega(u_0) = \frac{1}{m+1}.$$

Corollary

- *If* $m = 0$*, we obtain optimal polynomial energy decay given by* $E(t) \leq \frac{c}{t}\|\mathcal{A}u_0\|^2_{\mathcal{H}}$.
- *If* $m = 1$*,* $(RC, C) = 0$*, and* $\Im(B^2C, C) = 0$ *then the polynomial energy decay rate is optimal.*

An example: the acoustic bc

Consider

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) & = & 0, & 0 < x < 1, t > 0, \\ y(0,t) & = & 0, & t > 0, \\ y_x(1,t) + \delta_t(t) & = & 0, & t > 0, \\ \delta_{tt}(t) + b_1\delta_t(t) + b_0\delta(t) - y_t(1,t) & = & 0, & t > 0, \end{cases}$$

$$n = 2, \quad \eta = \begin{pmatrix} \delta \\ \delta_t \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -b_0 & -b_1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Example

$$\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right)_{\mathbb{C}^2} = b_0 x \bar{x}_1 + y \bar{y}_1, \text{ or } M = \begin{pmatrix} b_0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$B^* = \begin{pmatrix} 0 & -1 \\ b_0 & -b_1 \end{pmatrix}, B_0 = \begin{pmatrix} 0 & 1 \\ -b_0 & 0 \end{pmatrix}, R = \begin{pmatrix} 0 & 0 \\ 0 & -b_1 \end{pmatrix}.$$

Example

- If $\mu^2 = b_0 \neq k^2\pi^2$ for all $k \in \mathbb{Z}$, then

$$C \notin \ker(i\mu I - B_0)^{\perp}.$$

Indeed, computing

$$\left(\begin{pmatrix} 1 \\ i\mu \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = i\mu \neq 0,$$

thus $\sigma(A) \cap \sigma(B_0) = \phi$.

- If $b_0 = k^2\pi^2$ for some $k \in \mathbb{N}^*$, then $\sigma(B_0) = \{\pm ik\pi\}$. Computing the associated eigenvectors we get

$$\eta_{\pm k\pi} = \begin{pmatrix} 1 \\ \pm ik\pi \end{pmatrix} \text{ and } P(\eta_{\pm k\pi}) = \begin{pmatrix} 0 \\ \pm ik\pi \end{pmatrix} \neq 0.$$

Example

- Since $PC \neq 0$, we can choose $m = 0$, and the system satisfies the following optimal polynomial decay

$$E(t) \leq \frac{C}{1+t} \|u_0\|^2_{D(\mathcal{A})}.$$

- Suppose that for the same B we choose the system with $C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then we obtain $\frac{1}{2}$ as an optimal polynomial decay rate.

For Further Reading

J.T. Beale.
Spectral Properties of an acoustic boundary condition.
Indiana Univ. Math.J., Vol. 25 (1976), pp. 895-917.

J.E. Munoz Rivera ,Y. Qin.
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Thank you for your attention.