Controllability of the linear 1D wave equation with inner moving forces

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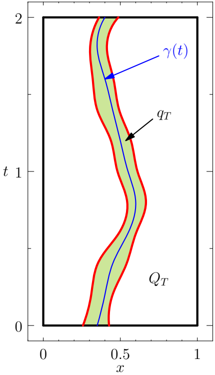
joint work with CARLOS CASTRO (Madrid) and NICOLAE CÎNDEA (Clermont-Ferrand)

$Q_T = (0,1) \times (0,T)$, $q_T \subset Q_T$, $\boldsymbol{V} := H_0^1(0,1) \times L^2(0,1)$, $a, b \in C([0,T],]0,1[)$

$$\begin{cases} y_{tt} - y_{xx} = v 1_{q_T}, & (x,t) \in Q_T \\ y = 0, & (x,t) \in \partial\Omega \times (0,T) \\ (y(\cdot,0), y_t(\cdot,0)) = (y_0, y_1) \in \boldsymbol{V}, & x \in (0,1). \end{cases}$$



Dependent domains q_T included in Q_T.

$$q_T = \Big\{ (x,t) \in Q_T;\ a(t) < x < b(t),\ t \in (0,T) \Big\}$$

Goals of the works -

- For some $T > 0$ and q_T, prove the existence of uniform null $L^2(q_T)$-controls.
- Approximate numerically the control of minimal $L^2(q_T)$-norm.

Problem statement

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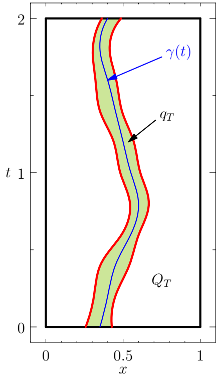

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Combination of two works

This contribution is a combination of two recent works :

- C. Castro : Exact controllability of the 1D wave equation from a moving interior point, COCV - 2013

$$\begin{cases} y_{tt} - y_{xx} = v(x,t)\,1_{x=\gamma(t)}, & (x,t) \in Q_T, \\ \gamma \in C^1([0,T],(0,1)), & 0 < |\gamma'(t)| < 1, t \in (0,T). \end{cases}$$

  Existence of $H^{-1}(\cup_{t\in(0,T)}\gamma(t) \times (0,T))$ null controls for $(y_0, y_1) \in L^2(0,1) \times H^{-1}(0,1)$, $T > 2$

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  Robust numerical approximation of the control of minimal $L^2(\omega \times (0,T))$-norm using a space-time formulation, well-adapted to our non cylindrical case.

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Set $\boldsymbol{H} = L^2(0,1) \times H^{-1}(0,1)$. Let $T > 0$.
q_T-non-cylindrical domain, $L\varphi = \varphi_{tt} - \varphi_{xx}$. Let $q_T \subset (0,1) \times (0,T)$ be an open set.

$$\Phi = \left\{ \varphi \in C([0,T]; L^2(0,1)) \cap C^1([0,T]; H^{-1}(0,1)), \text{ such that } L\varphi \in L^2(0,T; H^{-1}(0,1)) \right\}.$$

Theorem (Castro, Cîndea, Münch)

Assume that $q_T \subset (0,1) \times (0,T)$ is a finite union of connected open sets and satisfies the following hypotheses:

Any characteristic line starting at a point $x \in (0,1)$ at time $t = 0$ and following the optical geometric laws when reflecting at the boundary Σ_T must meet q_T.

Then, there exists $C > 0$ such that the following estimate holds :

$$\|\varphi(\cdot,0), \varphi_t(\cdot,0)\|_{\boldsymbol{H}}^2 \leq C\left(\|\varphi\|_{L^2(q_T)}^2 + \|L\varphi\|_{L^2(0,T;H^{-1}(0,1))}^2 \right), \quad \forall \varphi \in \Phi. \qquad (1)$$

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Step 1: Let φ be a smooth solution of the wave eq. φ can be extended in a unique way to a function, still denoted by φ, in $(x,t) \in (0,1) \times \mathbf{R}$ satisfying $L\varphi = 0$ and the boundary conditions $\varphi(0,t) = \varphi(1,t) = 0$ for all $t \in \mathbf{R}$.

For such extension the following holds: For each $t \in \mathbf{R}$, $x \in (0,1)$ and $\delta > 0$

$$\int_{t-\delta}^{t+\delta} |\varphi_x(0,s)|^2\, ds \leq \frac{1}{\delta} \iint_{\mathcal{U}^{\delta}_{(x,t+x)}} \left(|\varphi_x(y,s)|^2 + |\varphi_t(y,s)|^2\right) dy\, ds,$$

$$\int_{t-\delta}^{t+\delta} |\varphi_x(0,s)|^2\, ds \leq \frac{1}{\delta} \iint_{\mathcal{U}^{\delta}_{(x,t-x)}} \left(|\varphi_x(y,s)|^2 + |\varphi_t(y,s)|^2\right) dy\, ds,$$

where $\mathcal{U}^{\delta}_{(x,t)}$ is a neighborhood of (x,t) of the form

$$\mathcal{U}^{\delta}_{(x,t)} = \{(y,s) \text{ such that } |x-y| + |t-s| < \delta\}$$

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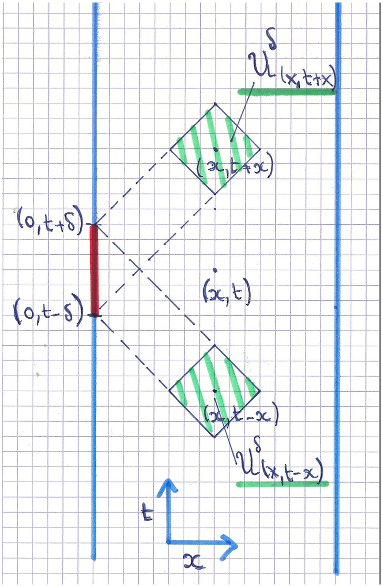

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Step 1: Wave equation is symmetric with respect to the time and space variables. The D'Alembert formulae can be used changing the time and space role, i.e.

$$\frac{1}{2}\int_{t-x}^{t+x} \varphi_x(0,s)\,ds = \varphi(x,t), \quad (x,t) \in (0,1)\times \mathbf{R}, \tag{2}$$

where we have taken into account the boundary condition $\varphi(0,t) = 0$. Consider now $x = x_0 - t$ in (2) and differentiate with respect to time. Then,

$$-\varphi_x(0, 2t - x_0) = -\varphi_x(x_0 - t, t) + \varphi_t(x_0 - t, t),$$

that written in the original variables (x,t) gives, $\varphi_x(0, t-x) = \varphi_x(x,t) - \varphi_t(x,t)$, or equivalently

$$\varphi_x(0,t) = \varphi_x(x, t+x) - \varphi_t(x, t+x), \quad t \in \mathbf{R}, \quad x \in (0,1). \tag{3}$$

Integrating the square of (3) in $(y,s) \in \mathcal{U}^{\delta}_{(x,t+x)}$ with the parametrization

$$y = x + \frac{u-v}{\sqrt{2}}, \qquad s = t + \frac{u+v}{\sqrt{2}}, \qquad |u|, |v| < \frac{\delta}{\sqrt{2}},$$

we obtain

$$\int_{-\delta/\sqrt{2}}^{\delta/\sqrt{2}} \int_{-\delta/\sqrt{2}}^{\delta/\sqrt{2}} |\varphi_x(0, t + 2v/\sqrt{2})|^2\,du dv = \iint_{\mathcal{U}^{\delta}_{x,t+x}} |\varphi_x(y,s) - \varphi_t(y,s)|^2\,dyds.$$

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Proof: Step 2

Set $W = \{\varphi : \varphi \in \Phi \text{ such that } L\varphi = 0\} \subset \Phi$. We show that for some constant $C > 0$,

$$\|\varphi(\cdot,0),\varphi_t(\cdot,0))\|_{\boldsymbol{V}}^2 \leq C\left(\|\varphi_t\|_{L^2(q_T)}^2 + \|\varphi_x\|_{L^2(q_T)}^2\right), \quad (4)$$

for any $\varphi \in W$ and initial data in $\boldsymbol{V}$.

We may assume that φ is smooth since the general case can be obtained by a usual density argument. We also assume that φ is extended to $(x,t) \in (0,1) \times \mathbf{R}$ by assuming that it satisfies the wave equation and boundary conditions in this region. This extension is unique and 2-periodic in time. The region q_T is also extended to $\tilde{q}_T$ to take advantage of the time periodicity of the solution φ. We define,

$$\tilde{q}_T = \bigcup_{k\in \boldsymbol{Z}} \{(x,t) \quad \text{such that } (x,t+2k) \in q_T\}\,.$$

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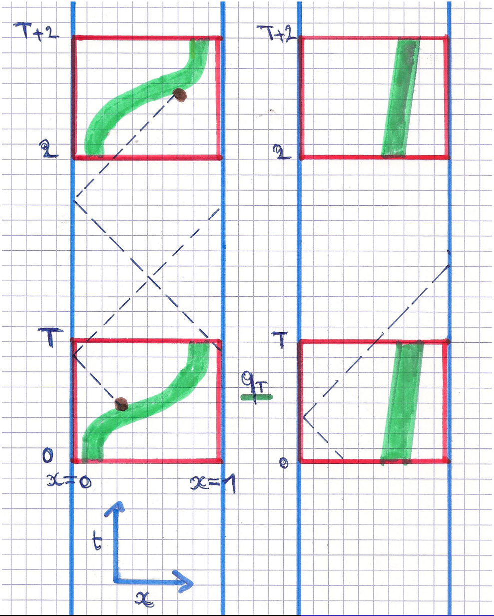
Proof : Step 2 (Picture)



Proof: Step 2

$$\tilde{q}_T = \bigcup_{k \in \mathbf{Z}} \{(x,t) \quad \text{such that } (x, t+2k) \in q_T\}.$$

The key point now is to observe that the hypotheses on q_T, namely the fact that

"any characteristic line starting at point $(x, 0)$ and following the optical geometric laws when reflecting at the boundary must meet q_T",

$\Longleftrightarrow$

"for any point $(0, t)$ with $t \in [0, 2]$ there exists one characteristic line (either $(x, t+x)$ with $x \in (0, 1)$ or $(x, t-x)$) that meets $\tilde{q}_T$".

Proof: Step 2

Thus, given $t \in [0,2]$ we can apply STEP 1 with $(x, t+x) \in \tilde{q}_T$ or with $(x, t-x) \in \tilde{q}_T$ with δ sufficiently small so that $\mathcal{U}^{\delta}_{(x,t+x)} \subset \tilde{q}_T$ or $\mathcal{U}^{\delta}_{(x,t-x)} \subset \tilde{q}_T$. In particular we see that for any $t \in [0,2]$ there exists $\delta_t > 0$ and $C_t > 0$ such that

$$\int_{t-\delta_t}^{t+\delta_t} |\varphi_x(0,s)|^2 \, ds \leq C_t \iint_{\tilde{q}_T} \left(|\varphi_t|^2 + |\varphi_x|^2\right) dxdt.$$

By compacity, there exists a finite number of times $t_1, ..., t_n$ such that $\cup_{i=1,\ldots,n}(t_i - \delta_{t_i}, t_i + \delta_{t_i})$ covers the whole interval $[0,2]$ and therefore, by adding the corresponding inequalities, there exists $C > 0$ such that

$$\int_{0}^{2} |\varphi_x(0,s)|^2 \, ds \leq C \iint_{\tilde{q}_T} \left(|\varphi_t|^2 + |\varphi_x|^2\right) dxdt. \tag{5}$$

The fact that we can replace $\tilde{q}_T$ by q_T is due to the 2-periodicity of φ in time. Finally, the result is a consequence of the well-known boundary observability inequality

$$\|\varphi(\cdot,0), \varphi_t(\cdot,0))\|_V^2 \leq C \int_0^2 |\varphi_x(0,s)|^2 \, ds.$$

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$$\|\varphi(\cdot,0), \varphi_t(\cdot,0))\|_V^2 \le C \int_0^2 |\varphi_x(0,s)|^2 \, ds.$$

Step 3. We show that we can substitute φ_x by φ in the right hand side of (4), i.e.

$$\|\varphi(\cdot,0),\varphi_t(\cdot,0))\|^2_{\boldsymbol{V}} \leq C\left(\|\varphi_t\|^2_{L^2(q_T)} + \|\varphi\|^2_{L^2(q_T)}\right), \quad (6)$$

for any $\varphi \in W$ and initial data in $\boldsymbol{V}$.
In fact, this requires to extend slightly the observation zone q_T. Instead, we observe that if q_T satisfies the hypotheses in Proposition 2 then there exists a smaller open subset $\tilde{q}_T \subset q_T$ that still satisfies the same hypotheses and such that the closure of $\tilde{q}_T$ is included in q_T. Thus, (4) must hold as well for $\tilde{q}_T$. Let us introduce now a function $\eta \geq 0$ which satisfies the following hypotheses:

$$\eta \in C^1((0,1)\times(0,T)), \quad supp(\eta) \subset q_T, \quad \|\eta_t\|_{L^\infty} + \|\eta_x^2/\eta\|_{L^\infty} \leq C_1 \text{ in } q_T$$

$$\eta > \eta_0 > 0 \text{ in } \tilde{q}_T, \text{ with } \eta_0 > 0 \text{ constant.}$$

As q_T is a finite union of connected open sets, the function η can be easily obtained by convolution of the characteristic function of $\tilde{q}_T$ with a positive mollifier.

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for any $\varphi \in W$ and initial data in $\boldsymbol{V}$.
In fact, this requires to extend slightly the observation zone q_T. Instead, we observe that if q_T satisfies the hypotheses in Proposition 2 then there exists a smaller open subset $\tilde{q}_T \subset q_T$ that still satisfies the same hypotheses and such that the closure of $\tilde{q}_T$ is included in q_T. Thus, (4) must hold as well for $\tilde{q}_T$. Let us introduce now a function $\eta \geq 0$ which satisfies the following hypotheses:

$$\eta \in C^1((0,1)\times(0,T)), \quad supp(\eta) \subset q_T, \quad \|\eta_t\|_{L^\infty} + \|\eta_x^2/\eta\|_{L^\infty} \leq C_1 \text{ in } q_T$$

$$\eta > \eta_0 > 0 \text{ in } \tilde{q}_T, \text{ with } \eta_0 > 0 \text{ constant.}$$

As q_T is a finite union of connected open sets, the function η can be easily obtained by convolution of the characteristic function of $\tilde{q}_T$ with a positive mollifier.

Proof: Step 3

Multiplying the equation of φ by $\eta\varphi$ and integrating by parts we easily obtain

$$\begin{aligned}\iint_{q_T} \eta|\varphi_x|^2\,dx\,dt &= \iint_{q_T} \eta|\varphi_t|^2\,dx\,dt + \iint_{q_T} (\eta_t\varphi\varphi_t - \eta_x\varphi\varphi_x)\,dx\,dt \\ &\leq \iint_{q_T} \eta|\varphi_t|^2\,dxdt + \frac{\|\eta_t\|_{L^\infty(q_T)}}{2} \iint_{q_T} (|\varphi|^2 + |\varphi_t|)\,dx\,dt \\ &\quad + \frac{1}{2}\iint_{q_T} (\frac{\eta_x^2}{\eta}\varphi^2 + \eta\varphi_x^2)\,dx\,dt.\end{aligned}$$

Therefore,

$$\iint_{q_T} \eta|\varphi_x|^2\,dx\,dt \leq C\iint_{q_T} (|\varphi_t|^2 + |\varphi|^2)\,dx\,dt,$$

for some constant $C > 0$, and we obtain

$$\|\varphi_x\|^2_{L^2(\tilde{q}_T)} \leq C_2^{-1}\iint_{q_T} \eta|\varphi_x|^2\,dx\,dt \leq C_2^{-1}C\iint_{q_T} (|\varphi_t|^2 + |\varphi|^2)\,dx\,dt.$$

This combined with (4) for $\tilde{q}_T$ provides (6).

Step 4. Here we prove that we can remove the second term in the right hand side of (6), i.e.

$$\|\varphi(\cdot,0),\varphi_t(\cdot,0))\|^2_{\boldsymbol{V}} \leq C\|\varphi_t\|^2_{L^2(q_T)}, \tag{7}$$

for any $\varphi \in W$ and initial data in $\boldsymbol{V}$.

We substitute the inequality (energy estimate)

$$\|\varphi\|^2_{L^2(q_T)} \leq T\|\varphi(\cdot,0),\varphi_t(\cdot,0))\|^2_{\boldsymbol{H}}.$$

in (6)

$$\|\varphi(\cdot,0),\varphi_t(\cdot,0))\|^2_{\boldsymbol{V}} \leq C\Big(\|\varphi_t\|^2_{L^2(q_T)} + \|\varphi(\cdot,0),\varphi_t(\cdot,0))\|^2_{\boldsymbol{H}}\Big).$$

Inequality (7) is finally obtained by contradiction. Assume that it is not true. Then, there exists a sequence $(\varphi^k(\cdot,0),\varphi^k_t(\cdot,0)))_{k>0} \in \boldsymbol{V}$ such that

$$\|\varphi^k(\cdot,0),\varphi^k_t(\cdot,0))\|^2_{\boldsymbol{V}} = 1, \quad \forall k>0, \qquad \|\varphi^k_t\|^2_{L^2(q_T)} \to 0, \text{ as } k \to \infty.$$

There exists a subsequence such that $(\varphi^k(\cdot,0),\varphi^k_t(\cdot,0)) \to (\varphi^\star(\cdot,0),\varphi^\star_t(\cdot,0))$ weakly in $\boldsymbol{V}$ and strongly in $\boldsymbol{H}$. Passing to the limit in the equation we see that the solution associated to $(\varphi^\star(\cdot,0),\varphi^\star_t(\cdot,0))$, $\varphi^\star$ must vanish at q_T and therefore, by (6), $\varphi^\star = 0$.

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Step 5. We now write (7) with respect to the weaker norm. In particular, we obtain

$$\|\varphi(\cdot,0),\varphi_t(\cdot,0))\|^2_{\boldsymbol{H}} \leq C\|\varphi\|^2_{L^2(q_T)}, \tag{8}$$

for any $\varphi \in \Phi$ with $L\varphi = 0$.

Let $\eta \in \Phi$ be defined by $\eta(x,t) = \eta(x,0) + \int_0^t \varphi(x,s)\,ds$, for all $(x,t) \in Q_T$ such that

$$(\eta(\cdot,0),\eta_t(\cdot,0)) = (\Delta^{-1}\varphi_t(\cdot,0),\varphi(\cdot,0)) \in \boldsymbol{V}$$

where Δ designates the Dirichlet Laplacian in $(0,1)$. Then $L\eta = 0$ in Q_T.

Then, inequality (7) on η and the fact that Δ is an isomorphism from $H^1_0(0,1)$ to $L^2(0,1)$, provide

$$\begin{aligned}\|(\varphi(\cdot,0),\varphi_t(\cdot,0),)\|^2_{\boldsymbol{H}} &= \|(\Delta^{-1}\varphi_t(\cdot,0),\varphi(\cdot,0))\|^2_{\boldsymbol{V}}\\ &= \|(\eta(\cdot,0),\eta_t(\cdot,0))\|^2_{\boldsymbol{V}}\\ &\leq C\|\eta_t\|^2_{L^2(q_T)} = C\|\varphi\|^2_{L^2(q_T)}.\end{aligned}$$

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Then, inequality (7) on η and the fact that Δ is an isomorphism from $H_0^1(0,1)$ to $L^2(0,1)$, provide

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Proof

Step 6. Here we finally obtain (10). Given $\varphi \in \Phi$ we can decompose it as $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1, \varphi_2 \in \Phi$ solve

$$\begin{cases} L\varphi_1 = L\varphi, \\ \varphi_1(\cdot,0) = (\varphi_1)_t(\cdot,0) = 0 \end{cases} \qquad \begin{cases} L\varphi_2 = 0, \\ \varphi_2(\cdot,0) = \varphi(\cdot,0), \quad (\varphi_2)_t(\cdot,0) = \varphi_t(\cdot,0). \end{cases}$$

From Duhamel's principle, we can write

$$\varphi_1(\cdot,t) = \int_0^t \psi(\cdot,t-s,s)ds$$

where $\psi(x,t,s)$ solves, for each value of the parameter $s \in (0,t)$,

$$\begin{cases} L\psi(\cdot,\cdot,s) = 0, \\ \psi(\cdot,0,s) = 0, \quad \psi_t(\cdot,0,s) = L\varphi(\cdot,s). \end{cases}$$

Therefore,

$$\begin{aligned} \|\varphi_1\|^2_{L^2(q_T)} &\leq \int_0^T \|\psi(\cdot,\cdot,s)\|^2_{L^2(q_T)} ds \leq C \int_0^T \|\psi(\cdot,0,s),\psi_t(\cdot,0,s))\|^2_{\boldsymbol{H}} ds \\ &\leq C\|L\varphi\|^2_{L^2(0,T;H^{-1}(0,1))} \end{aligned} \tag{9}$$

Combining (9) and estimate (8) for φ_2 we obtain

$$\begin{aligned} \|\varphi(\cdot,0),\varphi_t(\cdot,0))\|^2_{\boldsymbol{H}} &= \|\varphi_2(\cdot,0),(\varphi_2)_t(\cdot,0))\|^2_{\boldsymbol{H}} \leq C\|\varphi_2\|^2_{L^2(q_T)} \\ &\leq C\left(\|\varphi\|^2_{L^2(q_T)} + \|\varphi_1\|^2_{L^2(q_T)}\right) \leq C\left(\|\varphi\|^2_{L^2(q_T)} + \|L\varphi\|^2_{L^2(0,T;H^{-1})}\right). \end{aligned}$$

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$$\|\varphi_1\|^2_{L^2(q_T)} \leq \int_0^T \|\psi(\cdot,\cdot,s)\|^2_{L^2(q_T)} ds \leq C \int_0^T \|\psi(\cdot,0,s), \psi_t(\cdot,0,s))\|^2_{\boldsymbol{H}} ds$$
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Therefore,

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Set $\boldsymbol{H} = L^2(0,1) \times H^{-1}(0,1)$. Let $T > 0$.

Theorem (Castro, Cîndea, Münch)

Assume $q_T \subset (0,1) \times (0,T)$ is a finite union of connected open sets and satisfies the following hypotheses:

"Any characteristic line starting at the point $x \in (0,1)$ at time $t = 0$ and following the optical geometric laws when reflecting at the boundaries $x = 0, 1$ must meet q_T".

Then, there exists $C > 0$ such that the following estimate holds :

$$\|\varphi(\cdot,0), \varphi_t(\cdot,0))\|^2_{\boldsymbol{H}} \leq C\left(\|\varphi\|^2_{L^2(q_T)} + \|L\varphi\|^2_{L^2(0,T;H^{-1}(0,1))}\right), \quad \forall \varphi \in \Phi. \qquad (10)$$

1. The hypotheses on q_T stated in the Theorem are optimal in the following sense: If there exists a subinterval $\omega_0 \subset (0,1)$ for which all characteristics starting in ω_0 and following the geometrical optics conditions when getting to the boundary $x = 0, 1$, do not meet q_T, then the inequality fails to hold. This is easily seen by considering particular solutions of the wave equation which initial data supported in ω_0.

2. The proof of inequality (10) above does not provide an estimate on the dependence of the constant with respect to q_T.

3. In the cylindrical situation, i.e. $q_T = (a,b) \times (0,T)$, a generalized Carleman inequality, valid for the wave equation with variable coefficients, have been obtained in Cindea, Fernandez-Cara and Munch (2013) (see also Yao'2011). The extension of Proposition 2 to the wave equation with variable coefficients is still open and *a priori* can not be obtained by the method used in this section.

Corollary

Under the hypotheses on q_T, the space Φ is a Hilbert space with the scalar product,

$$(\varphi, \overline{\varphi})_\Phi = \iint_{q_T} \varphi(x,t)\overline{\varphi}(x,t)\,dx\,dt + \eta \int_0^T < L\varphi, L\overline{\varphi} >_{H^{-1}(0,1),H^{-1}(0,1)}\,dt, \qquad (11)$$

for any fixed $\eta > 0$.

PROOF: The seminorm associated to this inner product $\|\cdot\|_\Phi$ is a norm from (10). We check that Φ is closed with respect to this norm.
Let us consider a convergence sequence $\{\varphi_k\}_{k\geq 1} \subset \Phi$ such that $\varphi_k \to \varphi$ in the norm $\|\cdot\|_\Phi$.
From (10), there exist $(\varphi_0, \varphi_1) \in \boldsymbol{H}$ and $f \in L^2(0,T;H^{-1}(0,1))$ such that $(\varphi_k(\cdot,0), \varphi_{k,t}(\cdot,0)) \to (\varphi_0,\varphi_1)$ in $\boldsymbol{H}$ and $L\varphi_k \to f$ in $L^2(0,T;H^{-1}(0,1))$. Therefore, φ_k can be considered as a sequence of solutions of the wave equation with convergent initial data and second hand term $L\varphi_k \to f$.
By the continuous dependence of the solutions of the wave equation on the data, $\varphi_k \to \varphi$ in $C([0,T];L^2(0,1)) \cap C^1([0,T];H^{-1}(0,1))$, where φ is the solution of the wave equation with initial data $(\varphi_0,\varphi_1) \in \boldsymbol{H}$ and second hand term $L\varphi = f \in L^2(0,T;H^{-1}(0,1))$. Therefore $\varphi \in \Phi$.

Strong convergent (numerical approximation)"

[CINDEA, FERNANDEZCARA, MUNCH, COCV13],

$$\text{Minimize } \; J(y,v) := \frac{1}{2}\iint_{Q_T} |y|^2\,dx\,dt + \frac{1}{2}\iint_{q_T} |v|^2\,dx\,dt.$$

The optimal pair (y, v) is

$$y = L\varphi \quad \text{in} \quad Q_T, \qquad v = \varphi 1_\omega \quad \text{in} \quad Q_T$$

where $\varphi \in \Phi$ solves the variational problem

$$\iint_{Q_T} L\varphi\, L\overline{\varphi}\,dx\,dt + \iint_{q_T} \varphi\,\overline{\varphi}\,dx\,dt, = (y_0, \overline{\varphi}_t(\cdot,0))_{H^1,H^{-1}} - (y_1, \overline{\varphi}(\cdot,0))_{L^2} \quad \forall \overline{\varphi} \in \Phi.$$

Let $\varphi_h \in \Phi_h \subset \Phi$ solves the variational problem

$$\iint_{Q_T} L\varphi_h\, L\overline{\varphi_h}\,dx\,dt + \iint_{q_T} \varphi_h\,\overline{\varphi_h}\,dx\,dt, = (y_0, \overline{\varphi_h}_t(\cdot,0))_{H^1,H^{-1}} - (y_1, \overline{\varphi_h}(\cdot,0))_{L^2} \quad \forall \overline{\varphi_h} \in \Phi_h.$$

$\varphi_h \to \varphi$ in Φ as $h \to 0 \Longrightarrow y_h := L\varphi_h \to y$ in $L^2(Q_T)$ and $v_h := \varphi_h 1_{q_T} \to v$ in $L^2(q_T)$

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$\varphi_h \to \varphi$ in Φ as $h \to 0 \Longrightarrow y_h := L\varphi_h \to y$ in $L^2(Q_T)$ and $v_h := \varphi_h 1_{q_T} \to v$ in $L^2(q_T)$

[CINDEA, MUNCH, CALCOLO14],

$$\min_{(\varphi_0,\varphi_1)\in \boldsymbol{H}} J^\star(\varphi_0,\varphi_1)=\frac{1}{2}\iint_{q_T}|\varphi|^2\,dx\,dt+<\varphi_1,y_0>_{H^{-1}(0,1),H_0^1(0,1)}-\int_0^1\varphi_0\,y_1\,dx.$$

where $L\varphi=0$ in Q_T; $\varphi=0$ on Σ_T, $(\varphi,\varphi_t)(\cdot,0)=(\varphi_0,\varphi_1)$ and

$$<\varphi_1,y_0>_{H^{-1}(0,1),H_0^1(0,1)}=\int_0^1\partial_x((-\Delta)^{-1}\varphi_1)(x)\,\partial_x y_0(x)\,dx$$

where $-\Delta$ is the Dirichlet Laplacian in $(0,1)$.
Since the variable φ is completely and uniquely determined by (φ_0,φ_1), the idea of the reformulation is to keep φ as variable and consider the following extremal problem:

$$\begin{aligned}&\min_{\varphi\in W}\hat{J}^\star(\varphi)=\frac{1}{2}\iint_{q_T}|\varphi|^2\,dx\,dt+<\varphi_t(\cdot,0),y_0>_{H^{-1}(0,1),H_0^1(0,1)}-\int_0^1\varphi(\cdot,0)\,y_1\,dx,\\&W=\left\{\varphi:\varphi\in L^2(q_T),\ \varphi=0\text{ on }\Sigma_T,\ L\varphi=0\in L^2(0,T;H^{-1}(0,1))\right\}.\end{aligned}\tag{12}$$

From (10), the property $\varphi\in W$ implies that $(\varphi(\cdot,0),\varphi_t(\cdot,0))\in\boldsymbol{H}$, so that the functional $\hat{J}^\star$ is well-defined over W.

[CINDEA, MUNCH, CALCOLO14],

$$\min_{(\varphi_0,\varphi_1)\in \boldsymbol{H}} J^\star(\varphi_0,\varphi_1) = \frac{1}{2}\iint_{q_T} |\varphi|^2\,dx\,dt + <\varphi_1, y_0>_{H^{-1}(0,1),H_0^1(0,1)} - \int_0^1 \varphi_0\, y_1\, dx.$$

where $L\varphi = 0$ in Q_T; $\varphi = 0$ on Σ_T, $(\varphi,\varphi_t)(\cdot,0) = (\varphi_0,\varphi_1)$ and

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From (10), the property $\varphi\in W$ implies that $(\varphi(\cdot,0),\varphi_t(\cdot,0))\in \boldsymbol{H}$, so that the functional $\hat{J}^\star$ is well-defined over W.

Control of minimal L^2-norm: a mixed formulation

The main variable is now φ submitted to the constraint equality $L\varphi = 0$ as an $L^2(0,T;H^{-1}(0,1))$ function. This constraint is addressed introducing a Lagrangian multiplier $\lambda \in L^2(0,T;H_0^1(\Omega))$:

We consider the following problem : find $(\varphi,\lambda) \in \Phi \times L^2(0,T;H_0^1(0,1))$ solution of

$$\begin{cases} a_r(\varphi,\overline{\varphi}) + b(\overline{\varphi},\lambda) = l(\overline{\varphi}), & \forall \overline{\varphi} \in \Phi \\ b(\varphi,\overline{\lambda}) = 0, & \forall \overline{\lambda} \in L^2(0,T;H_0^1(0,1)), \end{cases} \tag{13}$$

where ($r \geq 0$ - augmentation parameter)

$$a_r : \Phi \times \Phi \to \mathbb{R}, \quad a_r(\varphi,\overline{\varphi}) = \iint_{q_T} \varphi\,\overline{\varphi}\,dx\,dt + r\int_0^T < L\varphi, L\overline{\varphi} >_{H^{-1},H^{-1}} dt$$

$$b : \Phi \times L^2(0,T;H_0^1(0,1)) \to \mathbb{R}, \quad b(\varphi,\lambda) = \int_0^T < L\varphi, \lambda >_{H^{-1}(0,1),H_0^1(0,1)} dt$$

$$= \iint_{Q_T} \partial_x(-\Delta^{-1}(L\varphi)) \cdot \partial_x\lambda\,dx\,dt$$

$$l : \Phi \to \mathbb{R}, \quad l(\varphi) = -< \varphi_t(\cdot,0), y_0 >_{H^{-1}(0,1),H_0^1(0,1)} + \int_0^1 \varphi(\cdot,0)\,y_1\,dx.$$

Well-posedness of the mixed formulation

Theorem

1. *The mixed formulation* (13) *is well-posed.*
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The well-posedness of the mixed formulation is a consequence of two properties [Fortin-Brezzi'91] :

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Inf-Sup condition

For any $\lambda_0 \in L^2(H_0^1)$, we define the (unique) element φ_0 such that

$$L\varphi_0 = -\Delta\lambda_0 \quad Q_T, \qquad \varphi_0(\cdot,0) = \varphi_{0,t}(\cdot,0) = 0 \quad \Omega, \qquad \varphi_0 = 0 \quad \Sigma_T$$

From the direct inequality,

$$\|\varphi_0\|_{L^2(Q_T)} \leq C_{\Omega,T} \| -\Delta\lambda_0\|_{L^2(0,T;H^{-1}(0,1))} \leq C_{\Omega,T} \|\lambda_0\|_{L^2(0,T;H_0^1(0,1))}$$

we get that $\varphi_0 \in \Phi$. In particular, $b(\varphi_0,\lambda_0) = \|\lambda_0\|^2_{L^2(0,T;H_0^1(0,1))}$ and

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Combining the above two inequalities, we obtain

$$\sup_{\varphi_0\in\Phi} \frac{b(\varphi_0,\lambda_0)}{\|\varphi_0\|_\Phi \|\lambda_0\|_{L^2(0,T;H_0^1(0,1))}} \geq \frac{1}{\sqrt{C^2_{\Omega,T} + \eta}}$$

and, hence, (14) holds with $\delta = \left(C^2_{\Omega,T} + \eta\right)^{-\frac{1}{2}}$.

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Lemma

Let A_r be the linear operator from $L^2(H_0^1)$ into $L^2(H_0^1)$ defined by

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For any $r > 0$, the operator A_r is a strongly elliptic, symmetric isomorphism from $L^2(H_0^1)$ into $L^2(H_0^1)$.

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$$\sup_{\lambda \in L^2(H_0^1)} \inf_{\varphi \in \Phi} \mathcal{L}_r(\varphi,\lambda) = -\inf_{\lambda \in L^2(0,T,H_0^1(0,1))} J^{\star\star}(\lambda) \quad + \mathcal{L}_r(\varphi_0, 0)$$

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Let then Φ_h and M_h be two finite dimensional spaces parametrized by the variable h such that

$$\Phi_h \subset \Phi, \quad M_h \subset L^2(0,T;H_0^1(0,1)), \qquad \forall h > 0.$$

Then, we can introduce the following approximated problems : find $(\varphi_h, \lambda_h) \in \Phi_h \times M_h$ solution of

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The well-posedness is again a consequence of two properties : the coercivity of the bilinear form a_r on the subset $\mathcal{N}_h(b) = \{\varphi_h \in \Phi_h; b(\varphi_h, \lambda_h) = 0 \quad \forall \lambda_h \in M_h\}$. From the relation

$$a_r(\varphi, \varphi) \geq \frac{r}{\eta} \|\varphi\|_\Phi^2, \quad \forall \varphi \in \Phi$$

the form a_r is coercive on the full space Φ, and so *a fortiori* on $\mathcal{N}_h(b) \subset \Phi_h \subset \Phi$. The second property is a discrete inf-sup condition : there exists $\delta_h > 0$ such that

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For any fixed h, the spaces M_h and Φ_h are of finite dimension so that the infimum and supremum in (16) are reached: moreover, from the property of the bilinear form a_r, δ_h is strictly positive. Consequently, for any fixed $h > 0$, there exists a unique couple (φ_h, λ_h) solution of (15).

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The space Φ_h must be chosen such that $L\varphi_h \in L^2(0,T,H^{-1}(0,1))$ for any $\varphi_h \in \Phi_h$. This is guaranteed for instance as soon as φ_h possesses second-order derivatives in $L^2_{loc}(Q_T)$. A conformal approximation based on standard triangulation of Q_T is obtained with spaces of functions continuously differentiable with respect to both x and t.

We introduce a triangulation $\mathcal{T}_h$ such that $\overline{Q_T} = \cup_{K\in\mathcal{T}_h} K$ and we assume that $\{\mathcal{T}_h\}_{h>0}$ is a regular family. We note $h := \max\{\mathrm{diam}(K), K \in \mathcal{T}_h\}$.

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Discretization

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[Bramble, Gunzburger]
Remark that if there exist two constants $C_0 > 0$ and $\alpha > 0$ such that

$$\|\psi_h\|^2_{L^2(Q_T)} \geq C_0 h^\alpha \|\psi_h\|^2_{L^2(0,T;H^1_0(0,1))}, \qquad \forall \psi_h \in \Phi_h \tag{17}$$

then a similar inequality it holds for weaker norms. More precisely, we have

$$\|\varphi_h\|^2_{L^2(0,T;H^{-1}(0,1))} \geq C_0 h^\alpha \|\varphi_h\|^2_{L^2(Q_T)}, \qquad \forall \varphi_h \in \Phi_h. \tag{18}$$

Indeed, to obtain (18) it suffices to take $\psi_h(\cdot,t) = (-\Delta)^{\frac{1}{2}}\varphi_h(\cdot,t)$ in (17). That gives

$$\int_0^T \left\|(-\Delta)^{-\frac{1}{2}}\varphi_h(\cdot,t)\right\|^2_{L^2(0,1)} dt \geq C_0 h^\alpha \int_0^T \left\|(-\Delta)^{-\frac{1}{2}}\varphi_{h,x}(\cdot,t)\right\|^2_{L^2(0,1)} dt.$$

Since $-\Delta$ is a self-adjoint positive operator and $\varphi_h \in \Phi_h \subset H^1_0(Q_T)$ we can integrate by parts in both hand-sides of the above inequality and hence we deduce estimate (18).

C_0 and α does not depend on T.

Change of the norm $\|\cdot\|_{L^2(H^{-1})}$ over the discrete space Φ_h

[Bramble, Gunzburger]
Remark that if there exist two constants $C_0 > 0$ and $\alpha > 0$ such that

$$\|\psi_h\|^2_{L^2(Q_T)} \geq C_0 h^\alpha \|\psi_h\|^2_{L^2(0,T;H^1_0(0,1))}, \qquad \forall \psi_h \in \Phi_h \tag{17}$$

then a similar inequality it holds for weaker norms. More precisely, we have

$$\|\varphi_h\|^2_{L^2(0,T;H^{-1}(0,1))} \geq C_0 h^\alpha \|\varphi_h\|^2_{L^2(Q_T)}, \qquad \forall \varphi_h \in \Phi_h. \tag{18}$$

Indeed, to obtain (18) it suffices to take $\psi_h(\cdot,t) = (-\Delta)^{\frac{1}{2}}\varphi_h(\cdot,t)$ in (17). That gives

$$\int_0^T \left\|(-\Delta)^{-\frac{1}{2}}\varphi_h(\cdot,t)\right\|^2_{L^2(0,1)} dt \geq C_0 h^\alpha \int_0^T \left\|(-\Delta)^{-\frac{1}{2}}\varphi_{h,x}(\cdot,t)\right\|^2_{L^2(0,1)} dt.$$

Since $-\Delta$ is a self-adjoint positive operator and $\varphi_h \in \Phi_h \subset H^1_0(Q_T)$ we can integrate by parts in both hand-sides of the above inequality and hence we deduce estimate (18).

C_0 and α does not depend on T.

Change of the norm $\|\cdot\|_{L^2(H^{-1})}$ over the discrete space Φ_h

We consider, for any fixed $h > 0$, the following equivalent definitions of the form $a_{r,h}$ and b_h over the finite dimensional spaces $\Phi_h \times \Phi_h$ and $\Phi_h \times M_h$ respectively :

$$a_{r,h} : \Phi_h \times \Phi_h \to \mathbb{R}, \qquad a_{r,h}(\varphi_h, \overline{\varphi_h}) = a(\varphi_h, \overline{\varphi_h}) + rC_0 h^\alpha \iint_{Q_T} L\varphi_h L\overline{\varphi_h} dxdt$$

$$b_h : \Phi_h \times M_h \to \mathbb{R}, \qquad b_h(\varphi_h, \lambda_h) = C_0 h^\alpha \iint_{Q_T} L\varphi_h \lambda_h dxdt.$$

Let $n_h = \dim \Phi_h$, $m_h = \dim M_h$ and let the real matrices $A_{r,h} \in \mathbb{R}^{n_h,n_h}$ defined by

$$a_{r,h}(\varphi_h, \overline{\varphi_h}) = \langle A_{r,h}\{\varphi_h\}, \{\overline{\varphi_h}\}\rangle_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}}, \quad \forall \varphi_h, \overline{\varphi_h} \in \Phi_h,$$

where $\{\varphi_h\} \in \mathbb{R}^{n_h,1}$ denotes the vector associated to φ_h and $\langle\cdot,\cdot\rangle_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}}$ the usual scalar product over $\mathbb{R}^{n_h}$. The problem reads: find $\{\varphi_h\} \in \mathbb{R}^{n_h,1}$ and $\{\lambda_h\} \in \mathbb{R}^{m_h,1}$ such that

$$\begin{pmatrix} A_{r,h} & B_h^T \\ B_h & 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h,n_h+m_h}} \begin{pmatrix} \{\varphi_h\} \\ \{\lambda_h\} \end{pmatrix}_{\mathbb{R}^{n_h+m_h,1}} = \begin{pmatrix} L_h \\ 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h,1}}.$$

The matrix of order $m_h + n_h$ is symmetric but not positive definite. We use exact integration methods and the LU decomposition method.

From φ_h, an approximation v_h of the control v is given by $v_h = -\varphi_h 1_{q_T} \in L^2(Q_T)$.

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In order to approximate the values of the constants C_0, α appearing in (17)-(18) we consider the following problem :

$$\text{find } \alpha > 0 \text{ and } C_0 > 0 \text{ such that } \sup_{\varphi_h \in \Phi_h} \frac{\|\varphi_h\|^2_{L^2(0,T;H^1_0(0,1))}}{\|\varphi_h\|^2_{L^2(Q_T)}} \leq \frac{1}{C_0 h^\alpha}, \qquad \forall h > 0.$$

Since $\dim \Phi_h < \infty$, the supremum is, for any fixed $h > 0$, the solution of the following eigenvalue problem :

$$\forall h > 0, \quad \gamma_h = \sup\Big\{\gamma : K_h\{\psi_h\} = \gamma \bar{J}_h\{\psi_h\}, \quad \forall \{\psi_h\} \in \mathbb{R}^{m_h} \setminus \{0\}\Big\}$$

We determine C_0 and α such that $C_0 h^\alpha = \gamma_h^{-1}$. We obtain

$$C_0 \approx 1.48 \times 10^{-2}, \quad \alpha \approx 2.1993.$$

We check that the constant γ_h (and so C_0 and α) does not depend on T nor on the controllability domain.

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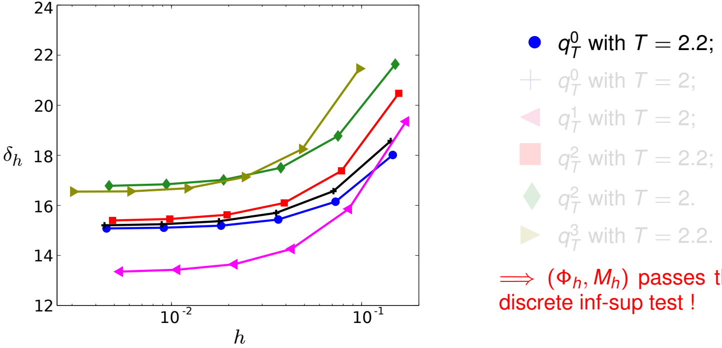
The discrete inf-sup test

In order to solve the mixed formulation (15), we first test numerically the discrete inf-sup condition (16). Taking $\eta = r > 0$ so that $a_{r,h}(\varphi, \overline{\varphi}) = (\varphi, \overline{\varphi})_\Phi$ for all $\varphi, \overline{\varphi} \in \Phi$, it is readily seen that the discrete inf-sup constant satisfies

$$\delta_h := \inf\left\{ \sqrt{\delta} : B_h A_{r,h}^{-1} B_h^T \{\lambda_h\} = \delta\, J_h\{\lambda_h\}, \quad \forall \{\lambda_h\} \in \mathbb{R}^{m_h} \setminus \{0\} \right\}.$$

The matrix $B_h A_{r,h}^{-1} B_h^T$ is symmetric, positive definite so that $\delta_h > 0$ for any $h > 0$.



$\Longrightarrow$ (Φ_h, M_h) passes the discrete inf-sup test !

Figure: δ_h vs. h for various control domains q_T, $T > 0$ and $r = 10^{-1}$.

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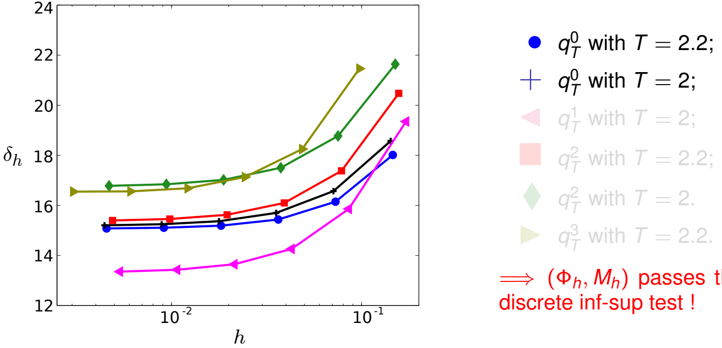

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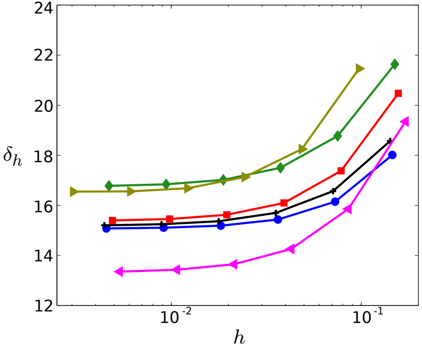

- q_T^0 with $T = 2.2$;
- q_T^0 with $T = 2$;
- q_T^1 with $T = 2$;
- q_T^2 with $T = 2.2$;
- q_T^2 with $T = 2$.
- q_T^3 with $T = 2.2$.

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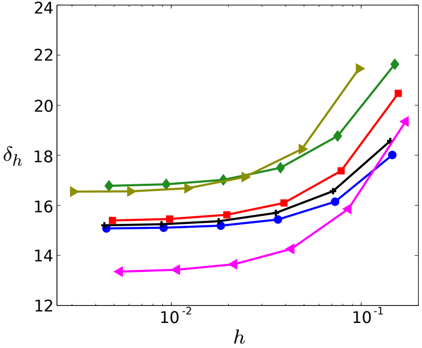

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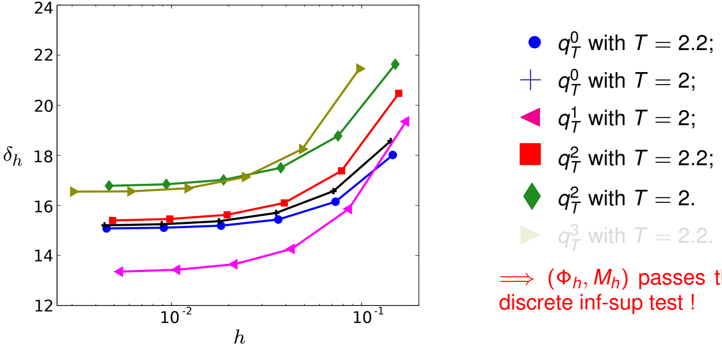

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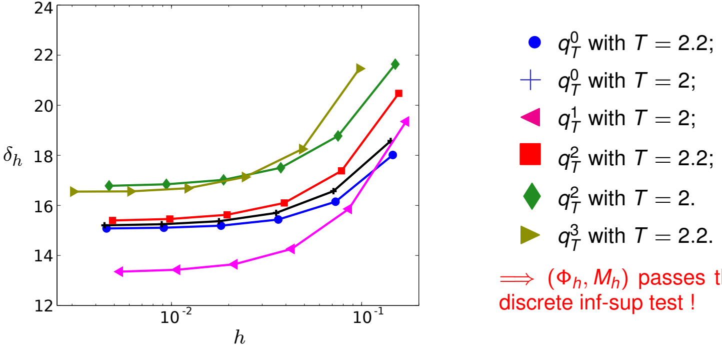

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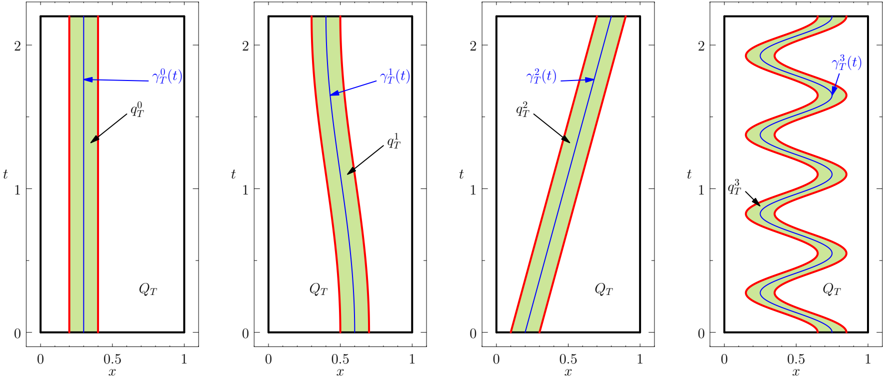


Figure: Time dependent domains q_T^i, $i \in \{0, 1, 2, 3\}$.

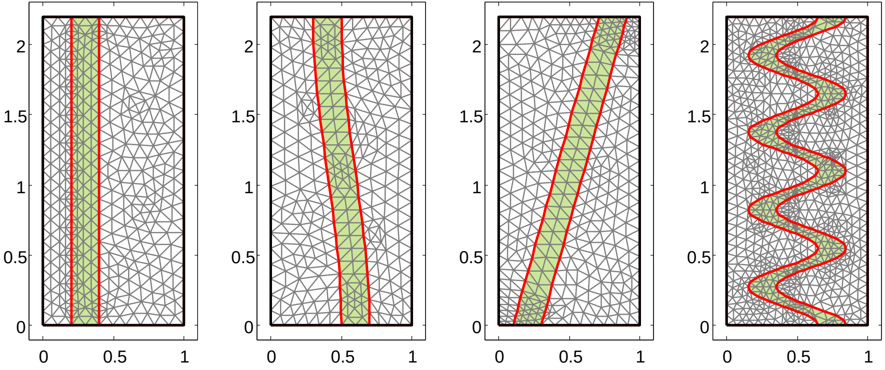


Figure: Meshes ♯1 associated with the domains $q^i_{T=2.2}$: $i = 0, 1, 2, 3$.

$$T = 2.; \quad y_0(x) = sin(\pi x); \quad y_1 = 0; \quad q_T = q_2^2$$

$\sharp$ Mesh	1	2	3	4	5
h	7.18×10^{-2}	3.59×10^{-2}	1.79×10^{-2}	8.97×10^{-3}	4.49×10^{-3}
$\Vert v_h \Vert_{L^2(q_T)}$	5.370	5.047	4.893	4.815	4.776
$\Vert L\varphi_h \Vert_{L^2(0,T;H^{-1}(0,1))}$	2.286	9.43×10^{-1}	3.76×10^{-1}	1.5×10^{-1}	6.15×10^{-2}
$\Vert v - v_h \Vert_{L^2(q_T)}$	2.45×10^{-1}	9.65×10^{-2}	4.32×10^{-2}	2.29×10^{-2}	1.10×10^{-2}
$\Vert y - \lambda_h \Vert_{L^2(Q_T)}$	5.63×10^{-3}	1.57×10^{-3}	4.04×10^{-4}	1.03×10^{-4}	2.61×10^{-5}
κ	2.46×10^{7}	2.67×10^{8}	2.96×10^{9}	3.03×10^{10}	3.08×10^{11}

Table: Norms vs. h for $r = 10^{-1}$.

$$r = 10^{-1} : \Vert v - v_h \Vert_{L^2(q_T)} \approx O(h^{1.3}), \Vert L\varphi_h \Vert_{L^2(0,T;H^{-1}(0,1))} \approx O(h^{1.3}), \Vert y - \lambda_h \Vert_{L^2(Q_T)} \approx O(h^{1.94})$$

$$r = 10^{3} : \Vert v - v_h \Vert_{L^2(q_T)} \approx O(h^{1.09}), \quad \Vert L\varphi_h \Vert_{L^2(Q_T)} \approx O(h^{1.04}), \quad \Vert y - \lambda_h \Vert_{L^2(Q_T)} \approx O(h^{2.01}).$$

$$T = 2.; \quad y_0(x) = sin(\pi x); \quad y_1 = 0; \quad q_T = q_2^2$$

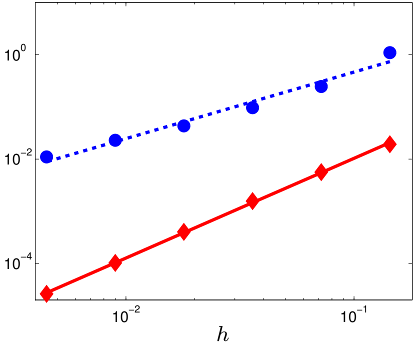


Figure: $r = 10^{-1}$; $q_T = q_{2.2}^2$; Norms $\|v - v_h\|_{L^2(q_T)}$ (●) and $\|y - \lambda_h\|_{L^2(Q_T)}$ (♦) vs. h.

$$T = 2.2; \quad y_0(x) = e^{-500(x-0.8)^2}; \quad y_1 = 0; \quad q_T = q^2_{2.2}$$

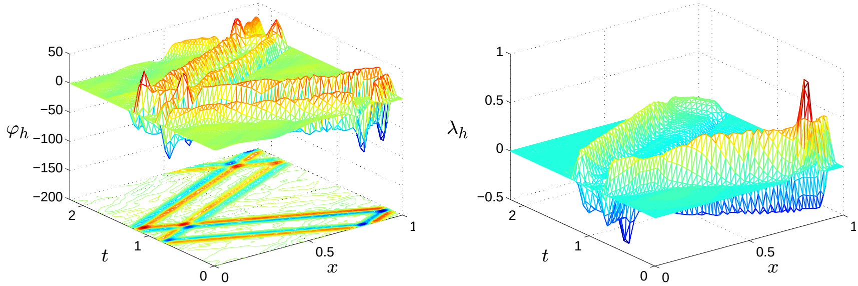


Figure: $r = 10^{-1}$; $q_T = q^2_{2.2}$: Functions φ_h (**Left**) and λ_h (**Right**) over Q_T.

$$\|v - v_h\|_{L^2(q_T)} \approx e^{5.85} h^{1.4}, \|L\varphi_h\|_{L^2(Q_T)} \approx e^{7.96} h^{1.31}, \|y - \lambda_h\|_{L^2(Q_T)} \approx e^{1.508} h^{1.62}$$

$$T = 2.2; \quad y_0(x) = \frac{x}{\theta} 1_{(0,\theta)}(x) + \frac{1-x}{1-\theta} 1_{(\theta,1)}(x), \quad y_1(x) = 0, \quad \theta \in (0,1) \quad q_T = q^2_{2.2}$$

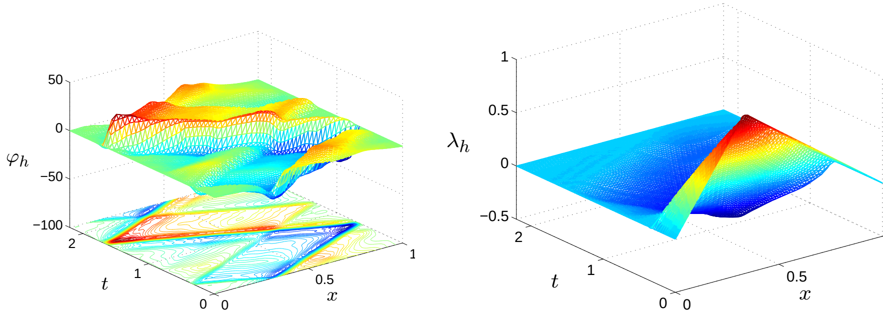


Figure: Example **EX3** with $\theta = 1/3$; $r = 10^{-1}$; $q_T = q^2_{2.2}$: Functions φ_h (**Left**) and λ_h (**Right**).

$$\|v - v_h\|_{L^2(q_T)} \approx e^{1.54} h^{0.47}, \quad \|L\varphi_h\|_{L^2(Q_T)} \approx e^{2.91} h^{0.54}, \quad \|y - \lambda_h\|_{L^2(Q_T)} \approx e^{-1.52} h^{1.29}.$$

$$T = 2.2; \quad y_0(x) = e^{-500(x-0.8)^2}; \quad y_1 = 0; \quad q_T = q^3_{2.2}$$

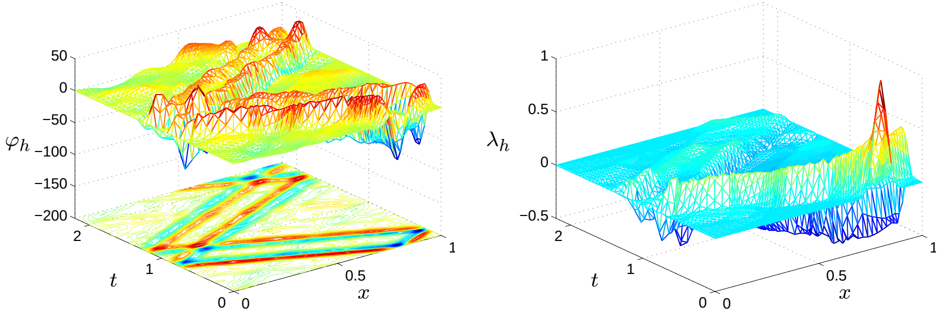


Figure: Example **EX2**: $q_T = q^3_{2.2}$ - Function φ_h (**Left**) and λ_h (**Right**) over Q_T.

$$T = 2.2; \quad y_0(x) = \frac{x}{\theta} 1_{(0,\theta)}(x) + \frac{1-x}{1-\theta} 1_{(\theta,1)}(x), \quad y_1(x) = 0, \quad \theta \in (0,1) \quad q_T = q^3_{2.2}$$

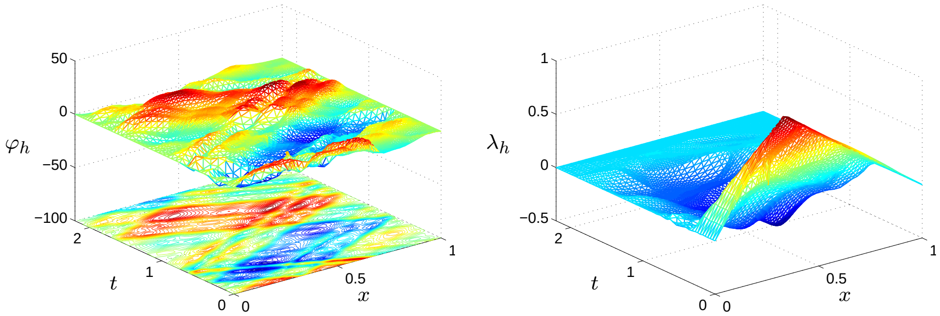


Figure: Example **EX3**, $\theta = 1/3$: $q_T = q^3_{2.2}$ - Function φ_h (**Left**) and λ_h (**Right**) over Q_T.

$$T = 2.2; \quad y_0(x) = \sin(\pi x), \quad y_1(x) = 0, \quad \theta \in (0,1) \quad q_T = q_2^2$$

δ_0	10^{-1}	$10^{-1}/2$	$10^{-1}/2^2$	$10^{-1}/2^3$	$10^{-1}/2^4$	$10^{-1}/2^5$	$10^{-1}/2^6$
$\sharp$ triangles	68 740	68 464	68 402	68 728	68 422	68 966	68 368
$\Vert v_h\Vert_{L^2(q_T)}$	4.8308	7.3308	11.5743	18.8056	29.7354	47.3157	123.9704
$\Vert v_h\Vert_{L^2(H^{-1})}$	0.0035	0.0042	0.0066	0.0107	0.0170	0.0270	0.0704

Table: Example **EX1**; $q_T = q_2^2$; Norms of the control v_h obtained for the **EX1** for control domains q_2^2 for different values of δ_0.

Non constant velocity

$$c(x) = \begin{cases} 1, & x \in [0, 0.45] \\ \in [1,5], \quad (c'(x) > 0), & x \in (0.45, 0.55) \\ 5, & x \in [0.55, 1]. \end{cases}$$

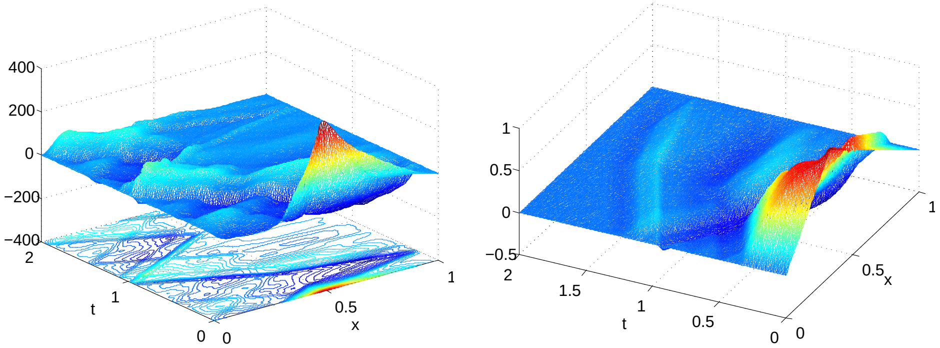


Figure: $r = 10^{-1}$:Example **EX3**, $\theta = 1/3$: $q_T = q_2^2$ for a non-constant velocity of propagation - Function φ_h (**Left**) and λ_h (**Right**) over Q_T.

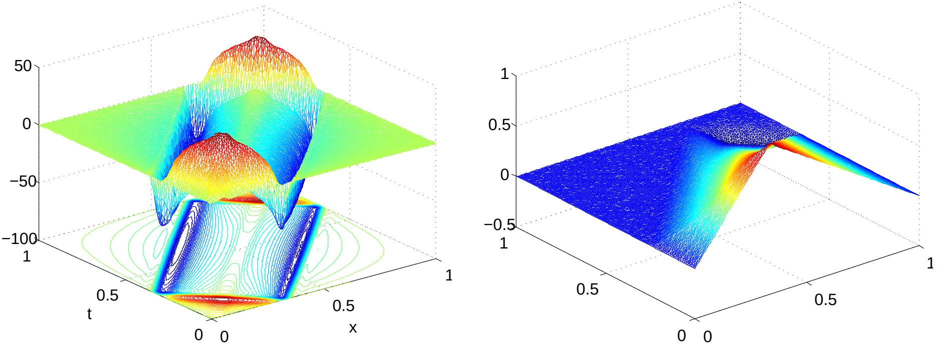


Figure: Example **EX3**, $\theta = 1/3$: $q_T = q_1^2$ - Function φ_h (**Left**) and λ_h (**Right**) over Q_T.

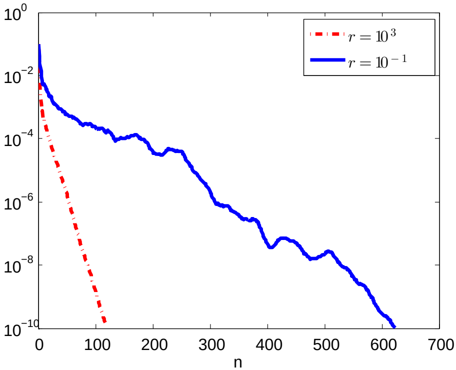


Figure: Example **EX3**. Evolution of the residue $\|g^n\|_{L^2(0,T;H_0^1(0,1))}/\|g^0\|_{L^2(0,T;H_0^1(0,1))}$ w.r.t. the iterate n.

$g^n = -\Delta^{-1}(L\varphi^n)$

♯ Mesh	1	2	3	4	5
h	7.18×10^{-2}	3.59×10^{-2}	1.79×10^{-2}	8.97×10^{-3}	4.49×10^{-3}
♯ iterate	87	105	119	140	166
$\|\lambda_h - y\|_{L^2(Q_T)}$	1.15×10^{-1}	5.2×10^{-2}	1.65×10^{-2}	6.03×10^{-3}	2.89×10^{-3}

CONTROLLABILITY HOLDS FOR ANY q_T SATISFYING THE GEOMETRIC OPTIC CONDITION

MIXED FORMULATION ALLOWS TO APPROXIMATE DIRECTLY L^2 MINIMAL CONTROL

THE MINIMISATION OF $J_r^{\star\star}(\lambda)$ IS VERY ROBUST AND FAST CONTRARY TO THE MINIMISATION OF $J^\star(\varphi_0, \varphi_1)$ (INVERSION OF SYMMETRIC DEFINITE POSITIVE AND VERY SPARSE MATRICE WITH DIRECT CHOLESKY SOLVERS)

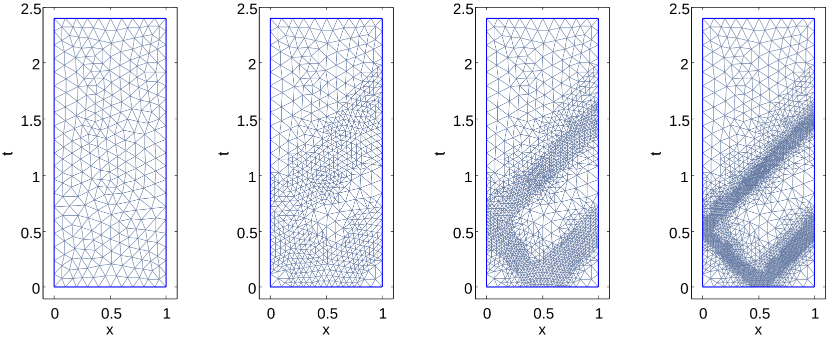
DIRECT APPROACH CAN BE USED FOR MANY OTHER CONTROLLABLE SYSTEMS FOR WHICH A GENERALIZED OBS. ESTIMATE IS AVAILABLE. IN PARTICULAR, HEAT, STOKES

THE PRICE TO PAY IS TO USED C^1 FINITE ELEMENTS (AT LEAST IN SPACE) UNLESS $L^\star\varphi = 0$ IS SEEN IN A WEAKER SPACE THAN $L^2(Q_T)$.

A NICE OPEN QUESTION IF THE DISCRETE INF-SUP PROPERTY !? A SIMPLE STRATEGY IS TO ADD THE LAGRANGIAN THE STABILIZED TERM

$$-\|L\lambda_h - \varphi_h 1_\omega\|^2_{L^2(Q_T)}, \quad -\|\lambda_h(\cdot,0) - y_0\|^2_{H^1_0}, \quad -\|(\lambda_h)_t(\cdot,0) - y_1\|^2_{L^2}$$

SPACE-TIME FINITE ELEMENT FORMULATION IS VERY WELL-ADAPTED TO MESH ADAPTATION AND TO NON-CYLINDRICAL SITUATION



Time-Space Refinement of the mesh according to the gradient of λ_h (from [Cîndea, Münch, 2014])

THIS WORK ALLOWS NOW TO CONSIDER THE OPTIMIZATION OF THE CONTROLS WITH RESPECT TO q_T:

$\forall (y_0, y_1) \in \boldsymbol{H}$, $T > 0$ and $L \in (0,1)$, the problem reads :

$$\inf_{q_T \in C_L} \|v_{q_T}\|_{L^2(q_T)}, \quad C_L = \{q_T : q_T \subset Q_T, |q_T| = L|Q_T| \text{ and such that } (10) \text{ holds}\}$$

where v_{q_T} denotes the control of minimal $L^2(q_T)$ norm for the wave eq. distributed over q_T.

THIS APPROACH MAY BE APPLIED FOR INVERSE PROBLEMS, OBSERVATION PROBLEMS, RECONSTRUCTION OF DATA, ...

Given the observation $z \in L^2(q_T)$, find y such that

$$Ly = 0 \quad \text{in} \quad Q_T, \quad y = z \quad \text{in} \quad q_T, \quad y = 0 \quad \text{on} \quad \Sigma_T$$

$$\text{Least Squares Problem} - \begin{cases} \inf_{y \in Y} \frac{1}{2} \iint_{q_T} (y - z)^2 \, dx \, dt \\ Y = \{y \in L^2(q_T), Ly = 0 \text{ in } L^2(Q_T), y = 0 \text{ on } \Sigma_T\} \end{cases}$$

through a mixed formulation ......

THANK YOU FOR YOUR ATTENTION

C. Castro, N. Cindea, A. Münch,
Controllability of the linear 1D wave equation with inner moving forces
Preprint 2014
Available at `http://hal.archives-ouvertes.fr/hal-00927076`