On the numerical computation of controls for the 1-D heat equation

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$\omega \subset (0,1)$, $a \in C^1([0,1],\mathbb{R}^+_*)$, $y_0 \in L^2(0,1)$, $Q_T = (0,1)\times(0,T)$, $q_T = \omega \times (0,T)$

$$\begin{cases} Ly \equiv y_t - (a(x)y_x)_x = v1_\omega, & (x,t) \in Q_T \\ y(x,t) = 0, & (x,t) \in \{0,1\}\times(0,T) \\ y(x,0) = y_0(x), & x \in (0,1). \end{cases} \tag{1}$$

$\forall y_0 \in L^2(0,1)$, $T > 0$ and $v \in L^2(q_T)$, $y \in C^0([0,T];L^2(0,1)) \cap L^2(0,T;H^1_0(0,1))$.

We introduce the linear manifold

$$\mathcal{C}(y_0,T) = \{\, (y,v) : v \in L^2(q_T),\ \ y \text{ solves } (\mathbf{??}) \text{ and satisfies } y(T,\cdot) = 0 \,\}.$$

non empty (see FURSIKOV-IMANUVILOV'96, ROBBIANO-LEBEAU'95).

The goal is to compute numerically some elements of $\mathcal{C}(y_0,T)$, i.e. compute some controls for the heat equation

Outline

1- Ill-posedness for the control of minimal L^2-norm (the "HUM control")

2- Change of norm : framework of Fursikov-Imanuvilov'96
(with ENRIQUE FERNANDEZ-CARA)

3- Transmutation method : from wave to heat
(with ENRIQUE ZUAZUA)

4- Without dual variable via a variational approach
(with PABLO PEDREGAL)

5- Conclusions / Additional references

PART I

Control of minimal $L^2(0,1)$-norm assuming that $a(x) = a_0 > 0$

$$(P) \qquad \inf_{(y,v)\in\mathcal{C}(y_0,T)} J(v,y) = \frac{1}{2}\|v\|^2_{L^2(q_T)}$$

$L^2(0,1)$-norm of the HUM control with respect to time

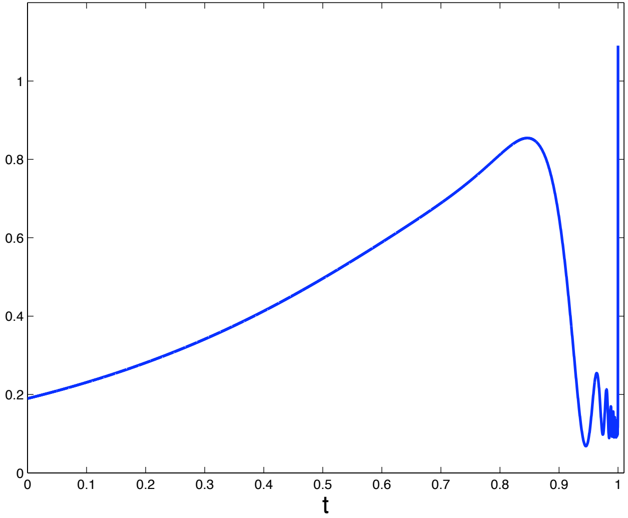

Figure: $y_0(x) = \sin(\pi x)$ - $T = 1$ - $\omega = (0.2, 0.8)$ - $t \rightarrow \|v(\cdot, t)\|_{L^2(0,1)}$ in $[0, T]$

L^2-norm of the HUM control with respect to time: Zoom near T

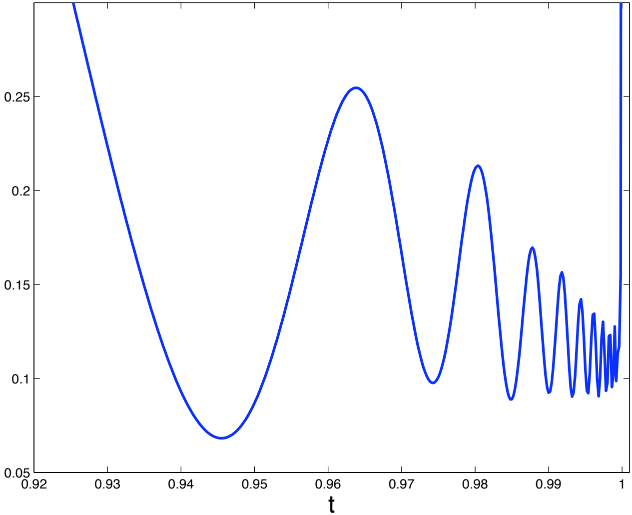

Figure: $y_0(x) = \sin(\pi x)$ - $T = 1$ - $\omega = (0.2, 0.8)$ - $t \to \|v(\cdot, t)\|_{L^2(0,1)}$ in $[0.92T, T]$

Minimal L^2 norm control

Since it is difficult to construct pairs $(v, y) \in \mathcal{C}(y_0, T)$ (*a fortiori* minimizing sequences for J !), it is by now standard to consider the corresponding dual :

$$\inf_{(y,v)\in\mathcal{C}(y_0,T)} J(y,v) = -\inf_{\phi_T\in H} J^\star(\phi_T),\ J^\star(\phi_T) = \frac{1}{2}\int_{q_T}\phi^2 dxdt + \int_\Omega \phi(0,\cdot)y_0 dx$$

where ϕ solves the backward system

$$\begin{cases} L^\star\phi \equiv -\phi' - (a(x)\phi_x)_x = 0 & Q_T = (0,T)\times\Omega, \\ \phi = 0 \quad \Sigma_T = (0,T)\times\partial\Omega, & \phi(T,\cdot) = \phi_T \quad \Omega. \end{cases}$$

The Hilbert space H is defined as the completion of $\mathcal{D}(0,1)$ with respect to the norm

$$\|\phi_T\|_H = \left(\int_{q_T}\phi^2(t,x)dxdt\right)^{1/2}.$$

From the observability inequality

$$C(T,\omega)\|\phi(0,\cdot)\|^2_{L^2(\Omega)} \leq \|\phi_T\|^2_H \quad \forall \phi_T \in L^2(\Omega),$$

$J^\star$ is coercive on H. The HUM control is given by $v = \phi\mathcal{X}_\omega$ on Q_T.

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- The completed space H is huge:

$$H^{-s} \subset H \quad \forall s > 0!$$

(H may also contain elements which are not distribution !!) and the minimizer is singular [Micu-Zuazua preprint 2010] [1]

-Due to the strong regularization effect of the heat operator, the constraint

$$y(\cdot, T) = 0, \qquad (0, 1)$$

can be viewed as an equality in a "very small" space; accordingly, the dual variable ϕ_T which is nothing but the Lagrange multiplier for the constraint may belong to a "large" dual space, much larger than L^2.

-Ill-posedness here is therefore related to the hugeness of H, poorly approximated numerically.

-This phenomenon is unavoidable (unless $\omega = (0, 1)$!) and is independent of the choice of the norm !

[1] S. Micu, E. Zuazua, *Regularity issues for a null-controllability of the linear 1-d heat equation*, Preprint 2010.

Regularization

For any $\epsilon > 0$, consider $J_\epsilon(y,v) = J(y,v) + \frac{\epsilon^{-1}}{2}\|y(T)\|^2_{H^{-s}(0,1)}$ and

$$\inf_{\phi_{T,\epsilon}\in L^2(0,1)} J^\star_\epsilon(\phi_{T,\epsilon}), \quad J^\star_\epsilon(\phi_{T,\epsilon}) = \frac{1}{2}\int_{q_T}\phi^2 dxdt + \int_\Omega \phi(0,\cdot)y_0 dx + \frac{\epsilon}{2}\|\phi_{T,\epsilon}\|^2_{H^s(0,1)}$$

and minimize in L^2 the quadratic and strictly convex function $J^\star_\epsilon$ by a conjugate gradient algorithm as initially proposed in Carthel-Glowinski-Lions'94 [2].

$$\phi_T(x) = \sum_{k\geq 1} a_k \sin(k\pi x) \Longleftrightarrow y_T(x) = \sum_{p\geq 1} b_p \sin(p\pi x), \quad x \in \Omega$$

and taking $y_0 = 0$ (for simplicity), we obtain the relation

$$b_p = \sum_{k\geq 1}\Big(c_{p,k}(\omega)g_{p,k}(T) + \epsilon(k\pi)^{2s}\delta_{p,k}\Big)a_{k,\epsilon}, \quad s = 0, 1.$$

$$c_{p,k}(\omega) = 2\int_\omega \sin(k\pi x)\sin(p\pi x)dx, \quad g_{p,k}(T) = \frac{1 - e^{-c(\lambda_p+\lambda_k)T}}{\lambda_k + \lambda_p}, \quad \lambda_k = (k\pi)^2$$

[2] Carthel-Glowinski-Lions, *On exact and approximate boundary controllabilities for the heat equation: a numerical approach*, JOTA (1994)

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$$T = 1, \quad y_T(x) = e^{-a_0 \pi^2 T} \sin(\pi x), \quad a_0 = 1/10, \quad \omega = (0.2, 0.8)$$

ϵ	10^{-1}	10^{-3}	10^{-5}	10^{-7}	10^{-9}
$\Vert\phi^N_{T,\epsilon}\Vert_{L^2(\Omega)}$	5.47×10^{-1}	2.52×10^{0}	1.42×10^{1}	9.20×10^{1}	6.66×10^{2}
$\Vert v^N_\epsilon\Vert_{L^2((0,T)\times\omega)}$	2.23×10^{-1}	3.85×10^{-1}	4.28×10^{-1}	4.43×10^{-1}	4.49×10^{-1}
$cond(\mathbf{\Lambda_{N,\epsilon}})$	5.44×10^{0}	5.87×10^{2}	7.46×10^{4}	7.45×10^{6}	7.18×10^{8}

Table: $N = 80$ - $\Vert \boldsymbol{v^N - v^N_\epsilon}\Vert_{\boldsymbol{L^2((0,T)\times\omega)}} \approx \boldsymbol{O(\epsilon^{0.295})}$.

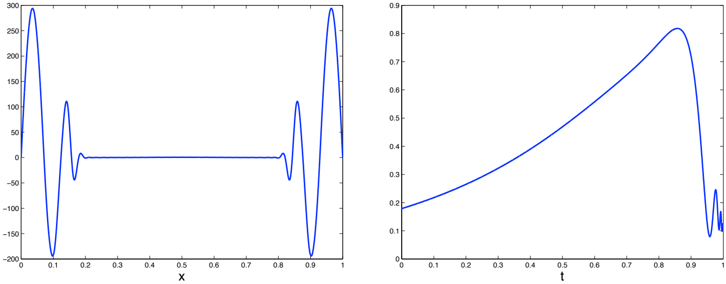

Figure: L^2 regularization for $\epsilon = 10^{-7}$ and $N = 80$ -**Left**: Adjoint solution $\phi_{T,\epsilon}$ - **Right**: L^2- norm of the control vs. t.

$$\epsilon = 10^{-14},\ T = 1, \quad y_T(x) = e^{-c\pi^2 T}\sin(\pi x), \quad c = 0.1$$

	$N = 10$	$N = 20$	$N = 40$	$N = 80$
$\|\phi_T^N\|_{L^2(\Omega)}$	4.27	3.22×10^1	1.68×10^3	5.38×10^6
$\|\phi^N \mathcal{X}_\omega\|_{L^2(Q_T)}$	4.194×10^{-1}	4.410×10^{-1}	4.526×10^{-1}	4.586×10^{-1}

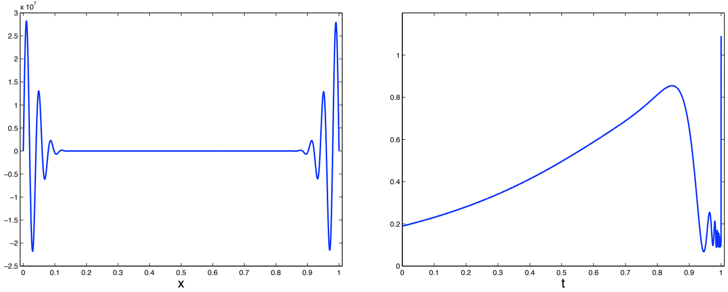

Figure: $T = 1$ - $\omega = (0.2, 0.8)$ - ϕ_T^N for $N = 80$ on Ω (**Left**) and on ω (**Right**).

$$T = 1, \quad y_0(x) = \sin(\pi x), \quad a(x) = a_0 = 1/10, \quad \omega = (0.2, 0.8)$$

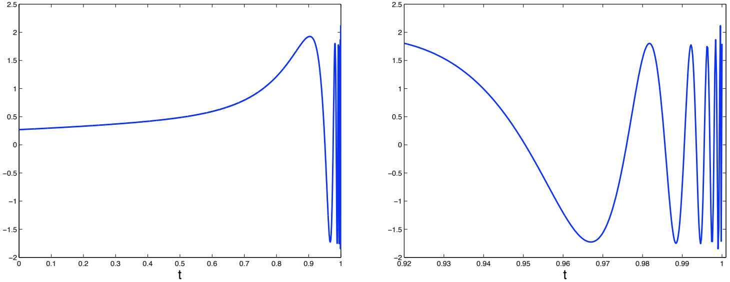

Figure: $T = 1$ - $\omega = (0.2, 0.8)$ - $\phi^N(\cdot, 0.8)$ for $N = 80$ on $[0, T]$ (**Left**) and on $[0.92T, T]$ (**Right**).

Minimization of $J_h^\star$ in $L^2(0,1)$ using a conjugate gradient method

$$T=1, \quad y_0(x)=\sin(\pi x), \quad a(x)=a_0=1/10, \quad \omega=(0.2,0.8)$$

h	$1/20$	$1/40$	$1/80$	$1/160$
$\sharp$ Iteration	36	218	574	1588
$\Vert v_h\Vert_{L^2((0,T)\times\omega)}$	4.05×10^{-1}	4.322×10^{-1}	4.426×10^{-1}	4.492×10^{-1}
$\Vert y_h(T,\cdot)-y_{Th}\Vert_{L^2(\Omega)}$	2.11×10^{-9}	1.58×10^{-9}	2.65×10^{-9}	2.35×10^{-9}
$\frac{\Vert\phi_h(0,x)\Vert^2_{L^2(\Omega)}}{\Vert\phi_h\mathcal{X}_\omega\Vert^2_{L^2(Q_T)}}$	4.072×10^{-1}	4.329×10^{-1}	4.429×10^{-1}	4.439×10^{-1}

Table: Semi-discrete scheme $\omega=(0.2,0.8)$ - $\Omega=(0,1)$ - $T=1$.

$\Longrightarrow$ The conditioning number of the problem blows up exponentially w.r.t. $1/h$.

Minimization of $J^\star$ by a conjugate gradient method

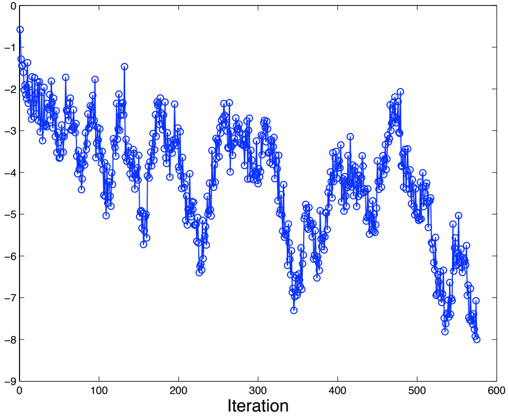

Figure: Semi-discrete scheme - $h = 1/80$ - Evolution of the residu w.r.t. the iteration of the GC algorithm

Lack of uniform observability vs. ill-posedness

$$C_{1h}\|\phi_h(0)\|^2_{L^2(\Omega)} \leq \int_0^T\int_\omega \phi_h^2(t,x)dxdt \leq C_{2h}\|\phi_h(0)\|^2_{L^2(\Omega)}, \quad \forall \phi_{Th} \in L^2(\Omega)$$

$$cond(\Lambda_h) \leq C_{1h}^{-1} C_{2h} h^{-2}$$

$$C_{2h} \to \infty \quad h \to 0$$

(more in [AM-Zuazua, Inverse Problems 2010]).

Other regularization / perturbation are considered in [AM-Zuazua'10]
1- Replace the heat equation by the hyperbolic equation

$$y_{\epsilon,t} - cy_{\epsilon,xx} + \boldsymbol{\epsilon y_{\epsilon,tt}} = v_\epsilon 1_\omega, \quad \text{in} \quad Q_T,$$

2- Singular (non uniformly controllable w.r.t. ε) perturbation

$$y_{\epsilon,t} - cy_{\epsilon,xx} - \boldsymbol{\epsilon y_{\epsilon,txx}} = v_\epsilon 1_\omega \quad \text{in} \quad Q_T.$$

$\Longrightarrow$ The main open issue is to characterize deeper the space H !!

$$(P_\infty) \quad \inf_{(y,v)\in\mathcal{C}(y_0,T)} J(v,y) = \|v\|_{L^\infty(q_T)}$$

$\Longrightarrow$ Bang-Bang control (piecewise constant in q_T) [Fabre-Puel-Zuazua,95] [3]

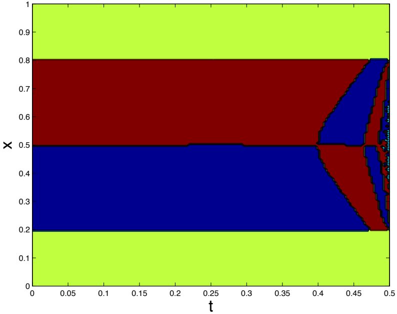

Figure: $y_0(x) = \sin(2\pi x)$ - $a_0 = 1/10$ - $s' = 1.$ - $\omega = (0.2, 0.8)$ - Iso-values of the control function $v_h \in Q_T$.

[3] C. Fabre, J.-P. Puel and E. Zuazua, *Approximate controllability of the semilinear heat equation*, Proc. Roy. Soc. Edinburgh Sect. A (1995).

$\Longrightarrow$ Set $v = [\lambda \mathcal{X}_\mathcal{O} + (-\lambda)(1 - \mathcal{X}_\mathcal{O})]1_\omega$

$\Longrightarrow$ Reformulate (P_∞) as follows :

$$(T_\infty) \begin{cases} \text{Minimize } \lambda^2 \\ \text{Subject to } \ (\lambda, \mathcal{X}_\mathcal{O}) \in \mathcal{D}(y_0, T) \end{cases}$$

$$\mathcal{D}(y_0, T) = \{(\lambda, \mathcal{X}_\mathcal{O}) \in \mathbb{R}^+ \times L^\infty(Q_T, \{0,1\}) \ \ y = y(\lambda, \mathcal{X}_\mathcal{O}) \text{ solves } (\textbf{??}) \text{ and } \|y(T)\|_{L^2(\Omega)} = 0\}$$

with

$$\begin{cases} y_t - (a(x)y_x)_x = [\lambda \mathcal{X}_\mathcal{O} + (-\lambda)(1 - \mathcal{X}_\mathcal{O})]1_\omega, & (x,t) \in Q_T \\ y(x,t) = 0, & (x,t) \in \{0,1\} \times (0,T) \\ y(x,0) = y_0(x), & x \in (0,1). \end{cases} \tag{2}$$

$\Longrightarrow$ Relaxation of the (time dependent) optimal design problem (T_∞) and capture of the oscillation near T via time-dependent density and (Young) measure [4].

[4] F. Periago, AM, *Approximation of bang-bang controls for the heat equation: dual method versus optimal design approach:* (2010) Preprint

PART II

Change of the norm : framework of Fursikov-Imanuvilov'96 [5]

$$\begin{cases} \text{Minimize } \; J(y,v) = \dfrac{1}{2}\displaystyle\iint_{Q_T} \rho^2 |y|^2 \, dx\, dt + \dfrac{1}{2}\displaystyle\iint_{q_T} \rho_0^2 |v|^2 \, dx\, dt \\ \text{Subject to } \; (y,v) \in \mathcal{C}(y_0, T). \end{cases} \tag{3}$$

where ρ, ρ_0 are non-negative continuous weights functions such that $\rho, \rho_0 \in L^\infty(Q_{T-\delta}) \quad \forall \delta > 0$.

[5] A.V. Fursikov and O. Yu. Imanuvilov, *Controllability of Evolution Equations,* Lecture Notes Series, number 34. Seoul National University, Korea, (1996) 1–163.

Primal (direct) approach

Following Fursikov-Imanuvilov'96, we assume Carleman type weights :

$$\left\{\begin{array}{l} \rho(x,t) = \exp\left(\dfrac{\beta(x)}{T-t}\right), \quad \rho_0(x,t) = (T-t)^{3/2}\rho(x,t), \quad \beta(x) = K_1\left(e^{K_2} - e^{\beta_0(x)}\right) \\ \text{where the } K_i \text{ are sufficiently large positive constants (depending on } T, a_0 \text{ and } \|a\|_{C^1}) \\ \text{and } \beta_0 \in C^\infty([0,1]),\ \beta_0 > 0 \text{ in } (0,1),\ \beta_0(0) = \beta_0(1) = 0,\ |\beta_0'| > 0 \text{ outside } \omega. \end{array}\right. \tag{4}$$

We introduce

$$P_0 = \{\, q \in C^2(\overline{Q}_T) : q = 0 \ \text{ on } \ \Sigma_T \,\}.$$

In this linear space, the bilinear form

$$(p,q)_P := \iint_{Q_T} \rho^{-2} L^\star p\, L^\star q\, dx\, dt + \iint_{q_T} \rho_0^{-2} p\, q\, dx\, dt$$

with $L^\star p = -p_t - (a(x)p_x)_x$, is a scalar product (unique continuation property).

Let P be the completion of P_0 for this scalar product.

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Let P be the completion of P_0 for this scalar product.

Lemma (Fursikov-Imanuvilov'96)

*Let ρ and ρ_0 be given by (**??**). Let us also set*

$$\rho_1(x,t) = (T-t)^{1/2}\rho(x,t), \quad \rho_2(x,t) = (T-t)^{-1/2}\rho(x,t). \qquad (5)$$

Then there exists $C>0$, only depending on ω, T, a_0 and $\|a\|_{C^1}$, such that

$$\left\{ \begin{aligned} &\iint_{Q_T} \left[\rho_2^{-2}\left(|q_t|^2+|q_{xx}|^2\right)+\rho_1^{-2}|q_x|^2+\rho_0^{-2}|q|^2\right] dx\,dt \\ &\qquad \le C\left(\iint_{Q_T}\rho^{-2}|L^\star q|^2\,dx\,dt+\iint_{q_T}\rho_0^{-2}|q|^2\,dx\,dt\right),\ \forall q\in P. \end{aligned}\right. \qquad (6)$$

Lemma (Fursikov-Imanuvilov'96, Fernández-Cara-Guerrero'06)

Under the same assumptions, for any $\delta>0$, one has

$$P \hookrightarrow C^0([0,T-\delta];H_0^1(0,1)),$$

where the embedding is continuous. In particular, there exists $C>0$, only depending on ω, T, a_0 and $\|a\|_{C^1}$, such that

$$\|q(\cdot\,,0)\|^2_{H_0^1(0,1)} \le C\left(\iint_{Q_T}\rho^{-2}|L^\star q|^2\,dx\,dt+\iint_{q_T}\rho_0^{-2}|q|^2\,dx\,dt\right) \qquad (7)$$

for all $q\in P$.

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for all $q \in P$.

Primal (direct) approach

Proposition

*Let ρ and ρ_0 be given by (**??**). Let (y, v) be the corresponding optimal pair for J. Then there exists $p \in P$ such that*

$$y = \rho^{-2} L^* p \equiv \rho^{-2}(-p_t - (a(x)p_x)_x), \quad v = -\rho_0^{-2} p|_{q_T}. \tag{8}$$

The function p is the unique solution in P of

$$\iint_{Q_T} \rho^{-2} L^* p \, L^* q \, dx\, dt + \iint_{q_T} \rho_0^{-2} p\, q \, dx\, dt = \int_0^1 y_0(x)\, q(x,0)\, dx, \quad \forall q \in P \tag{9}$$

Remark

p solves, at least in $\mathcal{D}'$, the following differential problem, that is second order in time and fourth order in space:

$$\begin{cases} L(\rho^{-2}L^*p) + \rho_0^{-2} p\, 1_\omega = 0, & (x,t) \in (0,1) \times (0,T) \\ p(x,t) = 0, \ \ (-\rho^{-2}L^*p)(x,t) = 0 & (x,t) \in \{0,1\} \times (0,T) \\ (-\rho^{-2}L^*p)(x,0) = y_0(x), \ \ (-\rho^{-2}L^*p)(x,T) = 0, & x \in (0,1). \end{cases} \tag{10}$$

*The "boundary" conditions at $t = 0$ and $t = T$ appear in (**??**) as Neumann conditions.*

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*The “boundary” conditions at $t = 0$ and $t = T$ appear in (**??**) as Neumann conditions.*

Conformal discretization

For large integers N_x and N_t, we set $\Delta x = 1/N_x$, $\Delta t = T/N_t$ and $h = (\Delta x, \Delta t)$. Let us introduce the associated uniform quadrangulations $\mathcal{Q}_h$, with

$$Q_T = \bigcup_{K \in \mathcal{Q}_h} K.$$

The following (conformal) finite element approximations of the space P are introduced:

$$P_h = \{ q_h \in P : q_h|_K \in (\mathbb{P}_{3,x} \otimes \mathbb{P}_{1,t})(K) \ \ \forall K \in \mathcal{Q}_h \}. \quad (11)$$

Here, $\mathbb{P}_{\ell,\xi}$ denotes the space of polynomial functions of order ℓ in the variable ξ. Notice that

$$P_h = \{ q_h \in C^{1,0}_{x,t}(\overline{Q}_T) : q_h|_K \in (\mathbb{P}_{3,x} \otimes \mathbb{P}_{1,t})(K) \ \ \forall K \in \mathcal{Q}_h, \ \ q_h|_{\Sigma_T} \equiv 0 \},$$

where $C^{1,0}_{x,t}(\overline{Q}_T)$ is the space of the functions in $C^0(\overline{Q}_T)$ that are continuously differentiable with respect to x in $\overline{Q}_T$.

The variational equality (**??**) is approximated as follows:

$$\begin{cases} \displaystyle\iint_{Q_T} \rho^{-2} L^* p_h \, L^* q_h \, dx\, dt + \iint_{q_T} \rho_0^{-2} p_h \, q_h \, dx\, dt = \int_0^1 y_0(x)\, q_h(x,0)\, dx \\ \forall q_h \in P_h; \quad p_h \in P_h. \end{cases} \quad (12)$$

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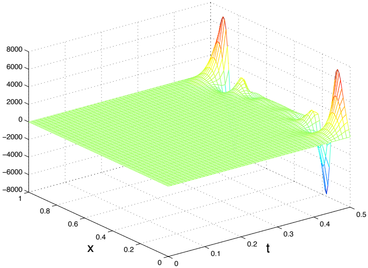
Experiment with $\omega = (0.2, 0.8)$

$\Delta x = \Delta t$	$1/20$	$1/40$	$1/80$	$1/160$	$1/320$
conditioning	1.33×10^{14}	1.76×10^{22}	7.86×10^{32}	2.17×10^{44}	2.30×10^{54}
$\|p_h(\cdot,T)\|_{L^2(0,1)}$	2.85×10^{1}	2.04×10^{2}	1.59×10^{3}	4.70×10^{4}	6.12×10^{6}
$\|y_h(\cdot,T)\|_{L^2(0,1)}$	4.37×10^{-2}	2.18×10^{-2}	1.09×10^{-2}	5.44×10^{-3}	2.71×10^{-3}
$\|v_h\|_{L^2(q_T)}$	1.228	1.251	1.269	1.281	1.288

Table: $T = 1/2$, $y_0(x) \equiv \sin(\pi x)$, $a(x) \equiv 10^{-1}$. $\|y_h(\cdot,T)\|_{L^2(0,1)} = \mathcal{O}(h)$.



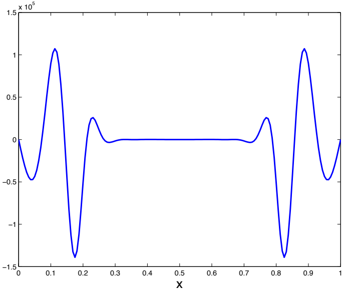


Figure: $\omega = (0.2, 0.8)$. The adjoint state p_h and its restriction to $(0,1) \times \{T\}$.

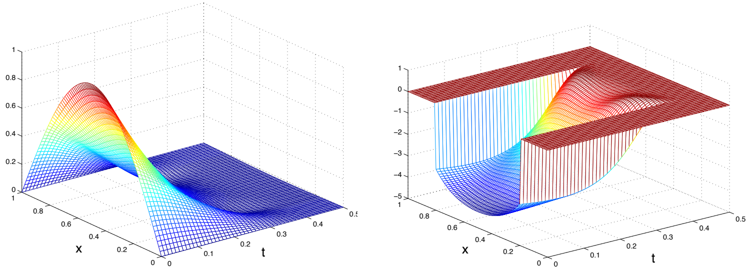
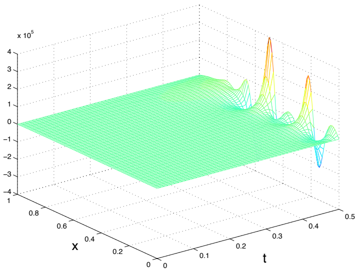


Figure: $\omega = (0.2, 0.8)$. The state y_h (**Left**) and the control v_h (**Right**).

$\Delta x = \Delta t$	$1/20$	$1/40$	$1/80$	$1/160$	$1/320$
conditioning	3.06×10^{14}	5.24×10^{22}	2.13×10^{33}	5.11×10^{44}	4.03×10^{54}
$\lVert p_h(\cdot, T)\rVert_{L^2(0,1)}$	1.37×10^{3}	5.51×10^{3}	5.12×10^{4}	2.16×10^{6}	3.90×10^{6}
$\lVert y_h(\cdot, T)\rVert_{L^2(0,1)}$	1.55×10^{-1}	9.46×10^{-2}	6.12×10^{-2}	3.91×10^{-2}	2.41×10^{-2}
$\lVert v_h\rVert_{L^2(q_T)}$	5.813	8.203	10.68	13.20	15.81

Table: $T = 1/2$, $y_0(x) \equiv \sin(\pi x)$, $a(x) \equiv 10^{-1}$. $\lVert y_h(\cdot, T)\rVert_{L^2(0,1)} = \mathcal{O}(h^{0.66})$.



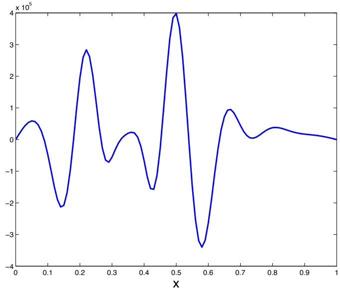


Figure: $\omega = (0.3, 0.4)$. The adjoint state p_h in Q_T (**Left**) and its restriction to $(0, 1) \times \{T\}$ (**Right**).

Experiments with $\omega = (0.3, 0.4)$

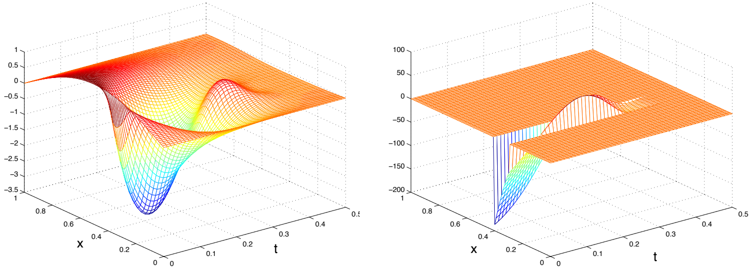


Figure: $\omega = (0.3, 0.4)$. The state y_h (**Left**) and the control v_h (**Right**).

Avoiding C^1 finite element in space : Mixed formulation

We keep the variable $y = \rho^{-2}L^\star p$ explicit, introduce $z = \rho^{-2}L^\star q$ and we transform the formulation : find $p \in P$ s.t.

$$\iint_{Q_T} \rho^{-2} L^* p \, L^* q \, dx\, dt + \iint_{q_T} \rho_0^{-2} p\, q \, dx\, dt = \int_0^1 y_0(x)\, q(x,0)\, dx, \quad \forall q \in P$$

into : find $(p, y) \in P \times Z$ s.t.

$$\begin{cases} \iint_{Q_T} \rho^2 y\, z \, dx\, dt + \iint_{q_T} \rho_0^{-2} p\, q\, dx\, dt = \int_0^1 y_0(x)\, q(x,0)\, dx \\ \forall (z,q) \text{ with } L^\star q - \rho^2 z = 0 \text{ and } q \in P; \quad (y, v) \text{ with } L^\star p - \rho^2 y = 0 \text{ and } p \in P. \end{cases}$$

and then into : find $(p, y, \lambda) \in P \times Z \times Z$ s.t.

$$\begin{cases} \iint_{Q_T} \rho^2 y\, z\, dx\, dt + \iint_{q_T} \rho_0^{-2} p\, q\, dx\, dt + \iint_{Q_T} \lambda (L^\star q - \rho^2 z)\, dx\, dt = \int_0^1 y_0(x)\, q(x,0)\, dx \\ \iint_{Q_T} \mu (L^\star p - \rho^2 y)\, dx\, dt = 0 \end{cases}$$

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Avoiding C^1 finite element in space : Mixed formulation

Let us introduce the space

$$Z = L^2(\rho^2; Q_T) = \{ z \in L^1_{\text{loc}}(Q_T) : \iint_{Q_T} \rho^2 |z|^2 \, dx\, dt < +\infty \},$$

the bilinear forms

$$a(\,(y,p),(z,q)\,) = \iint_{Q_T} \rho^2 y\, z\, dx\, dt + \iint_{q_T} \rho_0^{-2} p\, q\, dx\, dt \quad \forall (y,p),(z,q) \in Z \times P$$

and

$$b(\,(z,q),\mu\,) = \iint_{Q_T} (L^* q - \rho^2 z)\, \mu\, dx\, dt \quad \forall (z,q) \in Z \times P, \quad \forall \mu \in Z$$

and the linear form

$$\langle \ell, (z,q) \rangle = \int_0^1 y_0(x)\, q(x,0)\, dx \quad \forall (z,q) \in Z \times P.$$

Then $a(\cdot\,,\cdot)$, $b(\cdot\,,\cdot)$ and ℓ are well-defined and continuous and the announced mixed formulation is the following:

$$\begin{cases} a(\,(y,p),(z,q)\,) + b(\,(z,q),\lambda\,) &= \langle \ell, (z,q) \rangle \quad \forall (z,q) \in Z \times P \\ b(\,(y,p),\mu\,) &= 0 \quad \forall \mu \in Z \\ (y,p) \in Z \times P, \quad \lambda \in Z \end{cases} \tag{13}$$

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Theorem

*There exists a unique solution (y, p, λ) to (**??**). Moreover, y is, together with $v = \rho_0^{-2} p|_{q_T}$, the unique solution to (**??**).*

PROOF: Let us introduce the space

$$V = \{ (z,q) \in Z \times P : b(\,(z,q),\mu\,) = 0 \;\; \forall \mu \in Z \}.$$

- $a(\cdot\,,\cdot)$ is coercive on V, that is:

$$a(\,(z,q),(z,q)\,) \geq \kappa \|(z,q)\|^2_{Z\times P} \quad \forall (z,q) \in V, \quad \kappa > 0. \tag{14}$$

- $b(\cdot\,,\cdot)$ satisfies the usual "inf-sup" condition with respect to $Z \times P$ and Z, i.e.

$$\beta := \inf_{\mu \in Z} \sup_{(z,q)\in Z\times P} \frac{b(\,(z,q),\mu\,)}{\|(z,q)\|_{Z\times P}\|\mu\|_Z} > 0. \tag{15}$$

Theorem

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$$V = \{ (z, q) \in Z \times P : b((z, q), \mu) = 0 \ \ \forall \mu \in Z \}.$$

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For any $h = (\Delta x, \Delta t)$ as before, let us consider again the associated uniform quadrangulation $\mathcal{Q}_h$. We now introduce the following finite dimensional spaces:

$$Z_h = \{ z_h \in C^0(\overline{Q}_T) : z_h|_K \in (\mathbb{P}_{1,x} \otimes \mathbb{P}_{1,t})(K) \;\; \forall K \in \mathcal{Q}_h \,, \;\; z_h \in Z \},$$

$$Q_h = \{ q_h \in C^0(\overline{Q}_T) : q_h|_K \in (\mathbb{P}_{1,x} \otimes \mathbb{P}_{1,t})(K) \;\; \forall K \in \mathcal{Q}_h \,, \;\; q_h|_{\Sigma_T} \equiv 0 \}.$$

We have $Z_h \subset Z$ but, contrarily, $Q_h \not\subset P$. Let us introduce the bilinear form

$$\begin{cases} b_h(\,(z_h, q_h), \mu_h\,) = \displaystyle\iint_{Q_T} (-(q_h)_t\, \mu_h + a(x)(q_h)_x(\mu_h)_x - \rho^2 z_h \mu_h)\, dx\, dt \\ \forall (z_h, q_h) \in Z_h \times Q_h \,, \; \forall \mu_h \in Z_h. \end{cases}$$

Then the mixed finite element approximation of (**??**) is the following:

$$\begin{cases} a(\,(y_h, p_h), (z_h, q_h)\,) + b_h(\,(z_h, q_h), \lambda_h\,) & = \langle \ell, (z_h, q_h) \rangle & \forall (z_h, q_h) \in Z_h \times Q_h \\ b(\,(y_h, p_h), \mu_h\,) & = 0 & \forall \mu_h \in Z_h \\ (y_h, p_h) \in Z_h \times Q_h \,, \quad \lambda_h \in Z_h. \end{cases} \tag{16}$$

$$a(\,(y,p),(z,q)\,) = \iint_{Q_T} \rho^2 y\, z\, dx\, dt + \iint_{q_T} \rho_0^{-2} p\, q\, dx\, dt \quad \forall (y,p),(z,q) \in Z \times P$$

$$\rho(x,t) \rightarrow \rho_\eta(x,t) = \exp\left(\frac{\beta(x)}{T-t+\eta}\right), \quad \rho_{0,\eta}(x,t) = (T-t+\eta)^{3/2}\rho_\eta(x,t). \qquad (17)$$

$\Delta x = \Delta t$	$1/40$	$1/80$	$1/160$	$1/320$
conditioning	1.68×10^{119}	6.55×10^{127}	3.76×10^{117}	1.47×10^{116}
$\lVert p_h(\cdot\,,T)\rVert_{L^2(0,1)}$	1.91×10^{43}	2.37×10^{43}	2.14×10^{43}	8.59×10^{43}
$\lVert y_h(\cdot\,,T)\rVert_{L^2(0,1)}$	1.84×10^{-12}	7.64×10^{-13}	2.77×10^{-13}	1.36×10^{-11}
$\lVert v_h\rVert_{L^2(q_T)}$	1.272	1.275	1.282	1.289

Table: Mixed approach imposing $y_h(\cdot\,,0) = y_{0h}$, $\omega = (0.2, 0.8)$, $\eta = 10^{-2}$, $y_0(x) \equiv \sin(\pi x)$, $a(x) \equiv 10^{-1}$.

Mixed formulation :A nonconstant C^1 diffusion

$$y_0(x) = \sin(\pi x),\; T = 1/2,\;\; a \in C^1([0,1]),\;\; a(x) = 1 \text{ in } (0, 0.45),\;\; a(x) = 1/15 \text{ in } (0.55, 1) \qquad (18)$$

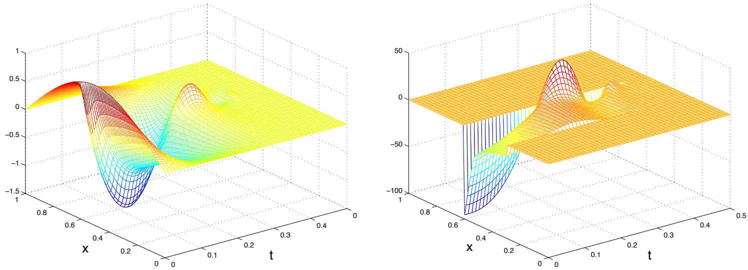


Figure: $\omega = (0.3, 0.6)$, nonconstant C^1 coefficient a; first case. The state y_h (**Left**) and the control v_h (**Right**).

$\Delta x = \Delta t$	$1/40$	$1/80$	$1/160$	$1/320$
conditioning	1.05×10^{37}	2.02×10^{36}	7.80×10^{35}	2.49×10^{35}
$\Vert p_h(\cdot, T)\Vert_{L^2(0,1)}$	2.15×10^{8}	3.71×10^{8}	7.56×10^{8}	2.31×10^{9}
$\Vert y_h(\cdot, T)\Vert_{L^2(0,1)}$	1.97×10^{-9}	2.93×10^{-9}	6.62×10^{-9}	1.21×10^{-8}
$\Vert v_h\Vert_{L^2(q_T)}$	5.721	6.159	6.4721	6.550

$$y_0(x) = \sin(\pi x),\ T = 1/2,\ a \in C^1([0,1]),\ a(x) = 1/15 \text{ in } (0, 0.45),\ a(x) = 1 \text{ in } (0.55, 1) \qquad (19)$$

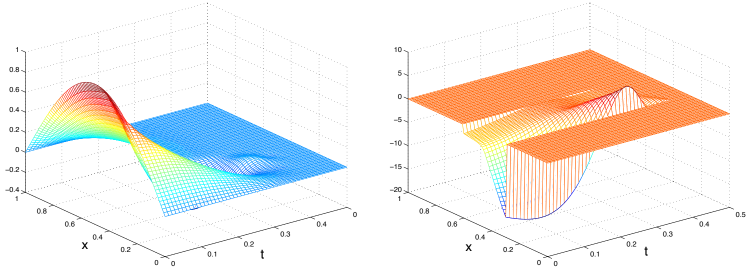


Figure: $\omega = (0.3, 0.6)$, nonconstant C^1 coefficient a; second case. The state y_h (**Left**) and the control v_h (**Right**).

$\Delta x = \Delta t$	$1/40$	$1/80$	$1/160$	$1/320$
conditioning	2.82×10^{37}	1.73×10^{36}	7.07×10^{35}	8.31×10^{35}
$\Vert p_h(\cdot, T)\Vert_{L^2(0,1)}$	3.95×10^{7}	4.28×10^{7}	8.09×10^{7}	2.51×10^{8}
$\Vert y_h(\cdot, T)\Vert_{L^2(0,1)}$	4.92×10^{-10}	3.26×10^{-10}	7.68×10^{-10}	2.94×10^{-9}
$\Vert v_h\Vert_{L^2(q_T)}$	1.704	1.796	1.872	1.890

Table: Mixed approach imposing $y_h(\cdot, 0) = y_{0h}$, $\omega = (0.3, 0.6)$, $\eta = 3 \times 10^{-2}$, $y_0(x) \equiv \sin(\pi x)$, nonconstant C^1 coefficient a. second case: the coefficient is “small” in ω.

[Benabdallah-Dermenjian-Le Rousseau, 2007] [6]

$$y_0(x) = \sin(\pi x),\ T = 1/2, \quad , a = a_1\, 1_{D_1} + a_2\, 1_{D_2} \quad D_1 = (0, 0.5), D_2 = (0.5, 1), \quad (a_1, a_2) = (1, 1/15).$$

$$\omega = (0.1, 0.4)$$

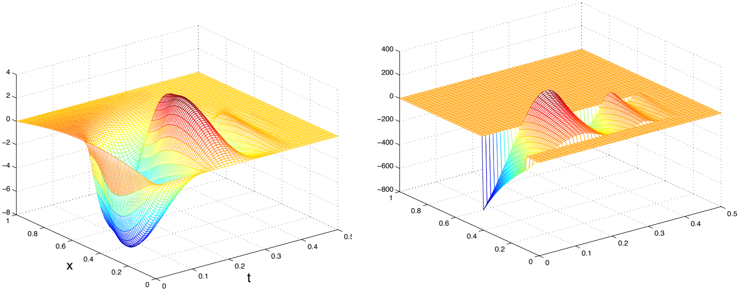


Figure: Piecewise constant diffusion a: The optimal pairs (y_h, v_h) for $\omega = (0.1, 0.4)$. $\|v_h\|_{L^2(q_T)} = 46.56$.

[6] *Carleman estimates for the 1D heat equation with a discontinuous coefficient and applications to controllability and an inverse problems*,JMAA (2007).

$$y_0(x) = \sin(\pi x),\ T = 1/2, \quad , a = a_1\, 1_{D_1} + a_2\, 1_{D_2} \quad D_1 = (0, 0.5), D_2 = (0.5, 1), \quad (a_1, a_2) = (1, 1/15).$$

$$\omega = (0.6, 0.9)$$

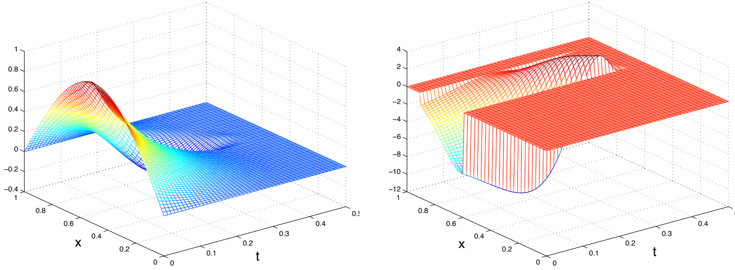


Figure: Piecewise constant diffusion a: The optimal pairs (y_h, v_h) for $\omega = (0.6, 0.9)$. $\|v_h\|_{L^2(q_T)} = 1.35$.

7

[7] E. Fernandez-Cara, AM *Numerical exact controllability of the 1D heat equation: primal algorithms*, preprint 2009.

Mixed formulation : non cylindrical situation

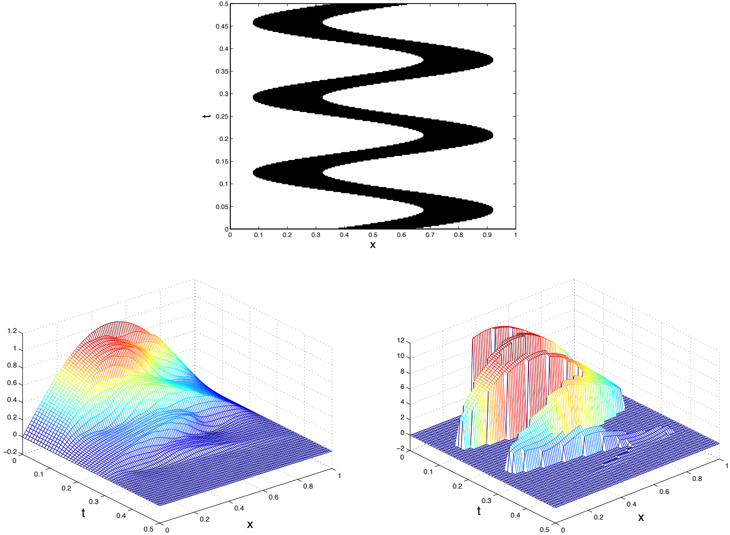


Figure: $\Delta x = \Delta t = 10^{-2}$ - Null controllability with non cylindrical control domains G_T (**Top**), the computed states y_h (**Left**) and the control v_h (**Right**).

Use duality to minimize J:

$$\begin{cases} \text{Minimize } \ J(y,v) = \dfrac{1}{2}\displaystyle\iint_{Q_T} \rho^2 |y|^2 \, dx \, dt + \dfrac{1}{2}\iint_{q_T} \rho_0^2 |v|^2 \, dx \, dt \\ \text{Subject to } \ (y,v) \in \mathcal{C}(y_0, T). \end{cases} \tag{20}$$

by

$$\begin{cases} \text{Minimize } \ J_{R,\varepsilon}(y,v) = \dfrac{1}{2}\displaystyle\iint_{Q_T} \rho_R^2 |y|^2 \, dx \, dt + \dfrac{1}{2}\iint_{q_T} \rho_0^2 |v|^2 \, dx \, dt + \dfrac{1}{2\varepsilon}\|y(\cdot\,,T)\|_{L^2}^2 \\ \text{Subject to } \ (y,v) \in \mathcal{A}(y_0, T) \end{cases} \tag{21}$$

where $\rho_R = min(\rho, R)$ and

$$\mathcal{A}(y_0, T) = \{\, (y,v) : v \in L^2(q_T), \ \ y \text{ solves } (\textbf{??}) \,\}.$$

8

[8] E. Fernández-Cara, AM, *Numerical exact controllability of the 1D heat equation: dual algorithm*, Preprint 2010.

Conjugate functions $J^{\star}_{R,\varepsilon}$ of $J_{R,\varepsilon}$

$$\left\{\begin{array}{l} \text{Minimize } \; J^{*}_{R,\varepsilon}(\mu,\varphi_T) = \dfrac{1}{2}\left(\displaystyle\iint_{Q_T} \rho_R^{-2}|\mu|^2\,dx\,dt + \iint_{q_T} \rho_0^{-2}|\varphi|^2\,dx\,dt\right) \\ \qquad\qquad + \displaystyle\int_0^1 \varphi(x,0)\,y_0(x)\,dx + \frac{\varepsilon}{2}\|\varphi_T\|^2_{L^2} \\ \text{Subject to } \; (\mu,\varphi_T) \in L^2(Q_T)\times L^2(0,1). \end{array}\right. \tag{22}$$

where $\varphi = M^{*}\mu + B^{*}\varphi_T$, i.e. φ is the solution to

$$\left\{\begin{array}{ll} L^{\star}\varphi = -\varphi_t - (a(x)\varphi_x)_x = \mu, & (x,t)\in(0,1)\times(0,T) \\ \varphi(x,t) = 0, & (x,t)\in\{0,1\}\times(0,T) \\ \varphi(x,T) = \varphi_T(x), & x\in(0,1). \end{array}\right. \tag{23}$$

Proposition

*The unconstrained extremal problems (**??**) is the dual problems to (**??**) in the sense of the Fenchel-Rockafellar theory. Furthermore, (**??**) and (**??**) are stable and possess unique solutions. Finally, if we denote by $(y_{R,\varepsilon}, v_{R,\varepsilon})$ the unique solution to (**??**), we denote by $(\mu_{R,\varepsilon}, \varphi_{T,R,\varepsilon})$ the unique solution to (**??**) and we set $\varphi_{R,\varepsilon} = M^{*}\mu_{R,\varepsilon} + B^{*}\varphi_{T,R,\varepsilon}$, then the following relations hold:*

$$v_{R,\varepsilon} = \rho_0^{-2}\varphi_{R,\varepsilon}|_{q_T}\,, \quad y_{R,\varepsilon} = -\rho_R^{-2}\mu_{R,\varepsilon}\,, \quad y_{R,\varepsilon}(\cdot\,,T) = -\varepsilon\varphi_{T,R,\varepsilon}\,. \tag{24}$$

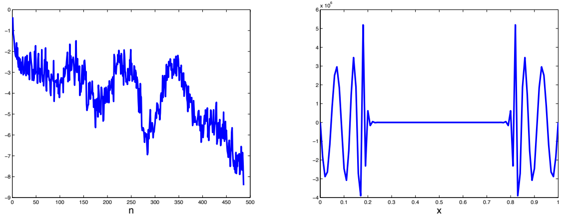
Moreover,

$$v_{R,\varepsilon} \to v \;\; \textit{strongly in } L^2(q_T) \;\; \textit{and} \;\; y_{R,\varepsilon} \to y \;\; \textit{strongly in } L^2(Q_T) \tag{25}$$

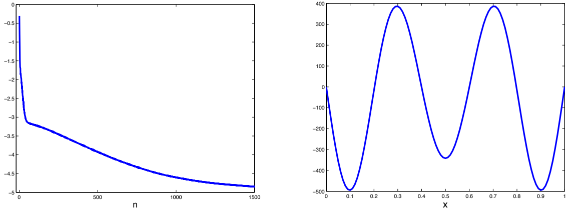
as $\varepsilon \to 0^+$, $R \to \infty$ where (y, v) minimizes J.

With and Without weights: $\omega = (0.2, 0.8)$ - $y_0(x) = \sin(\pi x)$

Evolution of the residue (in $\log_{10}$-scale) and $\varphi_{T,h}$ on $(0,1)$ for $(\rho, \rho_0) \equiv (0, 1)$



Evolution of the residue (in $\log_{10}$-scale) and $\varphi_{T,R,\varepsilon,h}$ on $(0,1)$ for Carleman type weights with $R = 10^{10}$ and $\varepsilon = 10^{-10}$.



$\Longrightarrow$ Very low variation of the cost around the minimizer with respect to the high frequencies of $\varphi_{T,R,\varepsilon}$.

$$\begin{cases} y_t - (a(x)y_x)_x + f(y) = v1_\omega, & (x,t) \in (0,1) \times (0,T) \\ y(x,t) = 0, & (x,t) \in \{0,1\} \times (0,T) \\ y(x,0) = y_0(x), & x \in (0,1). \end{cases}$$

We assume that

$$|f'(s)| \leq C(1 + |s|^p), \quad \text{a.e., with } \; p \leq 5. \tag{26}$$

so that the system posseses a local (in time) solution. [9]

Theorem (Fernandez-Cara and Zuazua'00)

*Let $T > 0$. Assume that $f(0) = 0$ and $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz-continuous and satisfies (**??**) and*

$$\frac{f(s)}{|s| \log^{3/2}(1 + |s|)} \to 0 \quad \textit{as} \quad |s| \to \infty. \tag{27}$$

*Then (**??**) is null controllable at time T; for any $y_0 \in L^2(0,1)$, there exists a control $v \in L^\infty(q_T)$ such that $y(T) = 0$.*

---

[9] Fernández-Cara, Zuazua, *Null and approximate controllability for weakling blowing up semilinear heat equation*, Ann. Inst. Poincaré (2000).

We consider the Newton method for $F(y, v) = (y_t - (a(x)y_x)_x + f(y) - v\,1_\omega, y(T))$.

Assuming that $(y^n, v^n) \in \mathcal{C}(y_0, T)$ is known, solve (y^{n+1}, v^{n+1}) over $\mathcal{C}(y_0, T)$ the unique solution of the linear extremal problem :

$$\text{Minimize } J(y^{n+1}, v^{n+1}) = \frac{1}{2}\iint_{Q_T} \rho^2 |y^{n+1}|^2\,dx\,dt + \frac{1}{2}\iint_{q_T} \rho_0^2 |v^{n+1}|^2\,dx\,dt$$

where $v^{n+1} \in L^2(q_T)$ is a null control for y^{n+1} solution of the

$$\begin{cases} y_t^{n+1} - (a(x)y_x^{n+1})_x + f'(y^n) \cdot y^{n+1} = v^{n+1}\,1_\omega + G(y^n), & (x,t) \in (0,1) \times (0,T) \\ y^{n+1}(x,t) = 0, & (x,t) \in \{0,1\} \times (0,T) \\ y^{n+1}(\cdot, 0) = y_0, & x \in (0,1). \end{cases}$$

with $G(y) = f'(y) \cdot y - f(y)$.

We take

$$f(y) = K\,y \log^{\alpha}(1 + |y|), \alpha > 0 \Longrightarrow G(y) = K\alpha |y|^2 \frac{\log^{\alpha-1}(1+|y|)}{1+|y|}.$$

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$$\begin{cases} y_t^{n+1} - (a(x)y_x^{n+1})_x + f'(y^n)\cdot y^{n+1} = v^{n+1}\,1_\omega + G(y^n), & (x,t) \in (0,1)\times(0,T) \\ y^{n+1}(x,t) = 0, & (x,t) \in \{0,1\}\times(0,T) \\ y^{n+1}(\cdot,0) = y_0, & x \in (0,1). \end{cases}$$

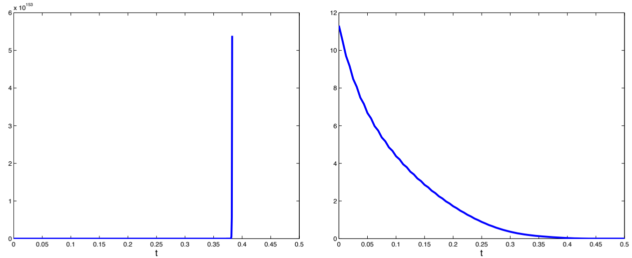
with $G(y) = f'(y)\cdot y - f(y)$.

We take

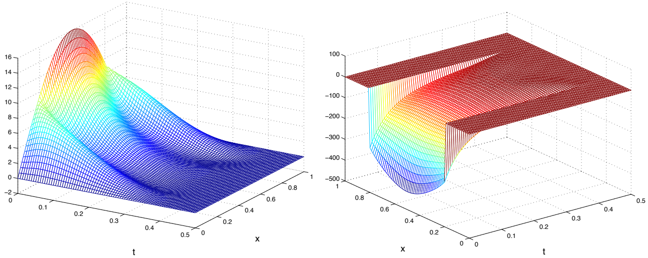
$$f(y) = K\,y\log^\alpha(1+|y|), \alpha > 0 \Longrightarrow G(y) = K\alpha|y|^2\frac{\log^{\alpha-1}(1+|y|)}{1+|y|}.$$

semi-linear situation

$$\omega = (0.2, 0.8),\ \ T = 1/2,\ \ a(x) = 1/2,\ \ f(s) = -5s\log^{\frac{3}{2}}(1+|s|),\ \ y_0(x) = 16\sin(\pi x)$$



$\|y_h(\cdot, t)\|_{L^2(0,1)}$- norm vs. $t \in (0, T)$ of the uncontrolled and controlled solution.



PART III

Computation of control using the transmutation method

with Enrique Zuazua :
AM-EZ, Inverse Problems (2010) [10]

$a(x) = a_0 > 0$

[10] AM-EZ,*Numerical approximation of null controls for the heat equation: ill posedeness and remedies*, (2010).

[11] Let $L > 0$ and $y_0 \in H_0^1(\Omega)$. IF $f \in L^2([0,L] \times \omega)$ is a null-control for w, solution of the wave equation

$$\begin{cases} w_{ss} - w_{xx} = f\, 1_\omega & (s,x) \in (0,L) \times \Omega, \\ w = 0 & (0,L) \times \partial\Omega, \\ (w(0), w_s(0)) = (y_0, 0) \Longrightarrow (w(L), w_s(L)) = (0,0) \end{cases}$$

AND if $H \in C^0([0,T], \mathcal{M}(]-L,L[)$ is a fundamental controlled solution for the heat equation

$$\begin{cases} \partial_t H - \partial_s^2 H = 0 \quad \text{in} \quad \mathcal{D}'(]0,T[\times]-L,L[), \\ H(t=0) = \delta, \quad H(t=T) = 0 \end{cases}$$

THEN the fonction

$$v(t,x) = 2\int_0^L H(t,s) f(s,x) ds\, 1_\omega(x), \quad (0,T) \times \Omega$$

is a null control in $L^2(q_T)$ for $y(t,x) = 2\int_0^L H(t,s) w(s,x) ds$ solution of the heat equation

$$\begin{cases} y_t - y_{xx} = v\, 1_\omega & (0,T) \times \Omega, \\ y = 0 & (0,T) \times \partial\Omega, \\ y(0) = y_0 \end{cases}$$

[11] L. Miller, The control transmutation method and the cost of fast controls, SICON 2006

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[11] L. Miller, The control transmutation method and the cost of fast controls, SICON 2006

Computation of the fundamental solution for the heat equation

Jones [12], Rouchon [13]. Let $\delta \in (0, T)$. For $t \in (0, \delta)$, H is taken as the Gaussian :

$$H(t,s) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{s^2}{4t}}, \quad (t,s) \in (0,\delta) \times \mathbb{R}.$$

so that it remains to join $H(\delta, s)$ to 0 at time T. For any $a > 0$ and any $\alpha \geq 1$, we consider the *bump* function

$$h(n) = \exp\left(-\frac{a}{((n-\delta)(T-n))^{\alpha}}\right), \quad n \in (\delta, T)$$

and then the function

$$p(t) = \frac{1}{\sqrt{4\pi t}} \begin{cases} 1 & t \in (0,\delta) \\ \frac{\int_t^T h(n)dn}{\int_\delta^T h(n)dn} & t \in (\delta, T) \end{cases}$$

so that $p(T) = 0$. $h \in C_c^{\infty}([\delta, T])$ and $p \in C^{\infty}([0, T])$. h and p are both Gevrey functions of order $1 + 1/\alpha \in (1, 2]$ so that the serie

$$H(t,s) = \sum_{k \geq 0} p^{(k)}(t) \frac{s^{2k}}{(2k)!} \tag{28}$$

is convergent. (**??**) defines a solution of the heat equation and satisfies $H(T, s) = 0$ for all $s \in \mathbb{R}$ and $\lim_{t\to 0^+} H(t,s) = \delta_{s=0}$.

[12] B. Jones, A fundamental solution for the heat equation which is supported in a strip, J. Math. Anal. Appl. 1977

[13] B. Laroche, P. Martin and P. Rouchon, Motion planning for the heat equation, Int. Journal of Robust and Nonlinear Control, (2000)

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[13] B. Laroche, P. Martin and P. Rouchon, Motion planning for the heat equation, Int. Journal of Robust and Nonlinear Control, (2000)

Fundamental solution for the heat equation: example

$a_0 = 1$ by the change of variable $(\tilde{x}, \tilde{t}) = (a_0 t, x)$

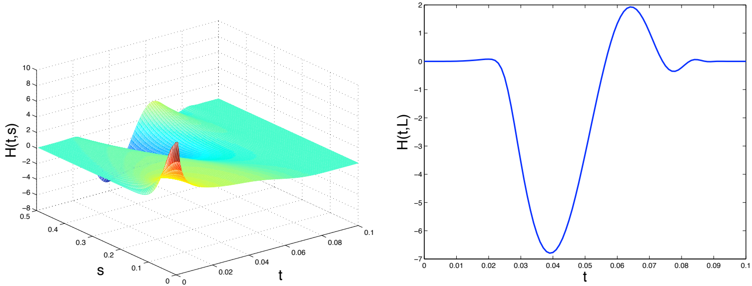

Figure: $L = 0.5$ - $T = 0.1$ - $(a, \alpha, \delta) = (10^{-2}, 1, T/5)$ - **Left**: fundamental solution H on $(0, T) \times (0, L)$ - **Right**: $H(t, L)$ vs. $t \in (0, T)$.

Fundamental solution for the heat equation: example

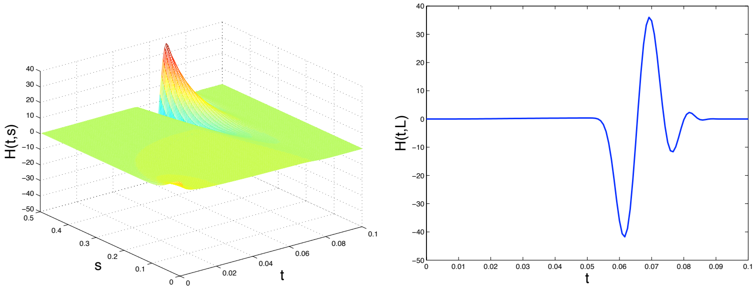

Figure: $L = 0.5$ - $T = 0.1$ - $(a, \alpha, \delta) = (10^{-2}, 1, T/2)$ - **Left**: fundamental solution H on $(0, T) \times (0, L)$ - **Right**: $H(t, L)$ vs. $t \in (0, T)$.

Control by the transmutation method

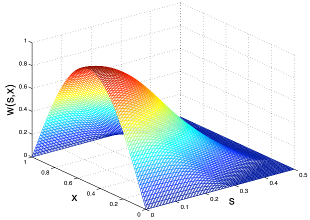

Figure: $y_0(x) = \sin(\pi x), L = 0.5$ - Controlled wave solution w (**Left**) and corresponding HUM control f (**Right**) on $(0, L) \times \Omega$.

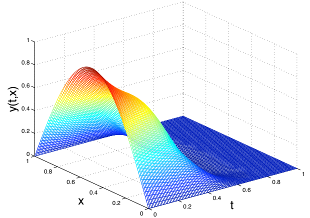

Figure: $y_0(x) = \sin(\pi x), T = 1, a_0 = 1/10, (\delta, \alpha) = (T/5, 1)$ - Controlled heat solution y (**Left**) and corresponding transmutted control v (**Right**) on $(0, T) \times \Omega$.

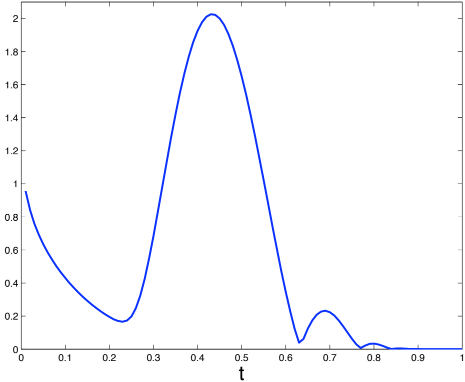

Figure: $L^2(\omega)$ norm of the control v vs $t \in [0, T]$ for $(y_0(x), T, a_0) = (\sin(\pi x), 1, 1/10)$

$\|v\|_{L^2(q_T)} \leq 2\|f\|_{L^2((0,L)\times\omega)}\|H\|_{L^2((0,T)\times(0,L)}$
$\|H\|_{L^2((0,T)\times(0,L)}$ is reduced if δ is small (reduce the time period where the dissipation is governed by the gaussian), and $\alpha_1 > 1$ (allows to take δ small) and $\alpha_2 < 1$ (increase the magnitude of the control near T).

$$h(s) = \exp\left(-\frac{a}{(s-\delta)^{\alpha_1}(T-s)^{\alpha_2}}\right)$$

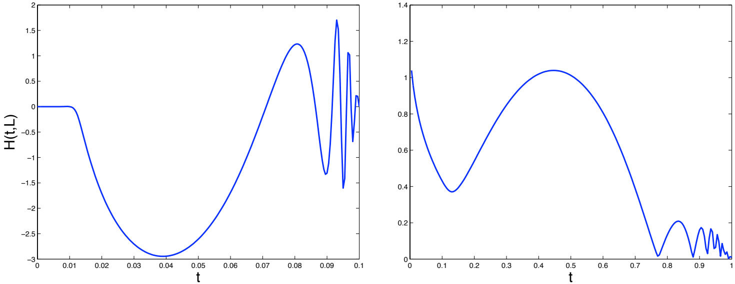

Figure: $(y_0(x), a_0) = (\sin(\pi x), 1/10)$ - Heat fundamental solution $H(t, L)$ vs. $t \in [0, \tilde{T}]$ (**Left**) and $L^2(\Omega)$-norm of corresponding control v (**Right**).
$\alpha_1 = 1.1, \alpha_2 = 0.7$ $\|g\|_{L^2(Q_T)} \approx 5.67 \times 10^{-1}$

-The transmuted control $v_h = (v)_{h>0}$ ensures that $\|y_h(T,\cdot)\|_{L^2(\Omega)} \approx 10^{-5}$

-Once a solution H in the one dimensional is constructed, we can take

$$H_n(t, x_1, x_2, \cdots, x_n) = H(t, x_1) \times H(t, x_2) \times \cdots \times H(t, x_n)$$

as a fundamental control solution for $(t, x) \in (0, T) \times [-L, L]^n$. Consequently, the transmutation provides also a control in any dimension, provided some geometric condition on the support ω.

-The transmutation method provides uniformly bounded discrete control $\{v_h\}$ discretization of

$$v(t,x) = 2\sum_{k\geq 0} p^{(k)}(t) \int_0^L \frac{s^{2k}}{(2k)!} f(s,x) ds \, 1_\omega(x)$$

- The main difficulty is the robust evaluation of $p^{(k)}$.

PART IV

Numerical null controllability through a variational approach

with Pablo Pedregal, Preprint 2010.

Introduced in [Pedregal, (2010)] [14]

Assume that $y_0 \in H^{1/2}(0,1)$, $y_0(0)=0$.

1. Consider the following class of feasible functions that comply with initial, boundary and final conditions :

$$\mathcal{A}=\left\{y \in H^1(Q_T) : y(x,0)=y_0(x), y(x,T)=0, x\in(0,1), y(0,t)=0, t\in(0,T)\right\}$$

2. Find an element $y \in \mathcal{A}$ solution of the heat equation, that is,

$$\int_{Q_T} (y_t w + a(x) y_x w_x)\, dxdt = 0, \quad \forall w \in L^2(0,T;H^1_0(0,1)) \tag{29}$$

3. Define a control v as the trace of y on $\{1\}\times(0,T)$, that is

$$v(t)=y(1,t), \quad t\in(0,T)$$

[14] P. Pedregal, *A variational perspective on controllability,* Inverse Problems(2010)

Consider the problem

$$\inf_{y\in\mathcal{A}} E(y) = \frac{1}{2}\iint_{Q_T} \left(|u_t|^2 + a(x)|u_x|^2\right)\, dx\, dt, \tag{30}$$

where $u = u(y) \in H^1_{0,x}(Q_T) = \{u \in H^1(Q_T), u = 0 \text{ on } \{0,1\} \times (0,T)\}$ is the solution of the elliptic problem over Q_T:

$$\begin{cases} -u_{tt} - (a(x)u_x)_x = -(y_t - (a(x)y_x)_x), & (x,t) \in Q_T, \\ u_t(x,0) = u_t(x,T) = 0, & x \in (0,1), \\ u(0,t) = u(1,t) = 0, & t \in (0,T). \end{cases} \tag{31}$$

Theorem (Pedregal 10)

- $\inf_{y\in\mathcal{A}} E(y) = \min_{y\in\mathcal{A}} E(y) = m$
- *The minimizers y of E solve the heat equation (i.e. the corrector u identically vanishes on Q_T)*

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- Reminiscent of a least square approach as introduced by Glowinski'83.

- In practice, for any $\overline{y} \in \mathcal{A}$, for instance $\overline{y}(x,t) = y_0(x)(1-t/T)^2$, we consider

$$min_{z \in \mathcal{A}_0} E(\overline{y} + z)$$

over $z \in \mathcal{A}_0 = \{z \in H^1(Q_T) : z(x,0) = z(x,T) = 0, z(0,t) = 0\}$ by a conjugate gradient algorithm.

- The corrector u solution an H^1-elliptic problem is approximated by $C^0(Q_T)$-finite element.

$$X_h = \{\varphi_h \in C^0([0,1] \times [0,T]) : \varphi_h|_K \in (\mathbb{P}_{1,x} \otimes \mathbb{P}_{1,t})(K) \;\; \forall K \in \mathcal{Q}_h\}.$$

$$X_{0h} = \{\, \varphi_h \in X_h : \varphi_h(0,t) = \varphi_h(1,t) = 0 \;\; \forall t \in (0,T)\,\},$$
$$X_{yh} = \{\, \varphi_h \in X_h : \varphi_h(0,t) = 0 \;\; \forall t \in (0,T), \varphi_h(x,0) = y_0(x), \varphi_h(x,T) = 0 \;\; \forall x \in (0,1)\}.$$

$$\begin{cases} \text{Minimize} \quad E_h(y_h) = \dfrac{1}{2} \displaystyle\iint_{Q_T} (|u_{h,t}|^2 + a(x)|u_{h,x}|^2)\, dx\, dt, \\ \text{subject to} \quad y_h \in X_{yh}. \end{cases} \tag{32}$$

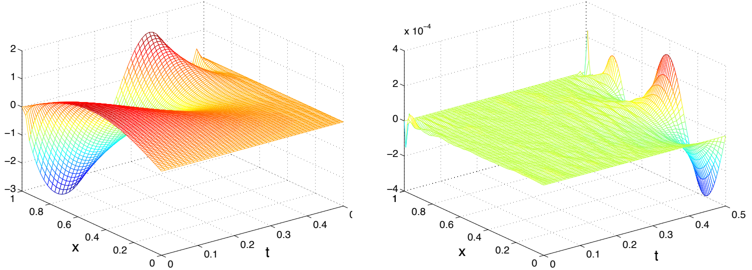

Figure: $y_0(x) = \sin(\pi x)$, $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - Solution in $y_h \in \mathcal{A}_h$ (**Left**) and corresponding corrector u_h (**Right**) along Q_T.

$\Delta x = \Delta t$	$1/25$	$1/50$	$1/100$	$1/200$
$\sharp$ CG iteration	846	2 132	2 014	2 834
$\Vert y_h\Vert_{H^1(Q_T)}$	6.024	6.658	5.920	6.021
$\Vert y_h\Vert_{L^2(\Sigma_T)}$	1.369	1.487	1.392	1.418
$E(y_h)$	4.88×10^{-6}	8.37×10^{-7}	1.22×10^{-6}	8.29×10^{-7}

Table: $y_0(x) = \sin(\pi x)$, $T = 1/2$, $a_0 = 1/4$ - $\varepsilon = 10^{-5}$ - Numerical results with respect to $h = (\Delta x, \Delta t)$.

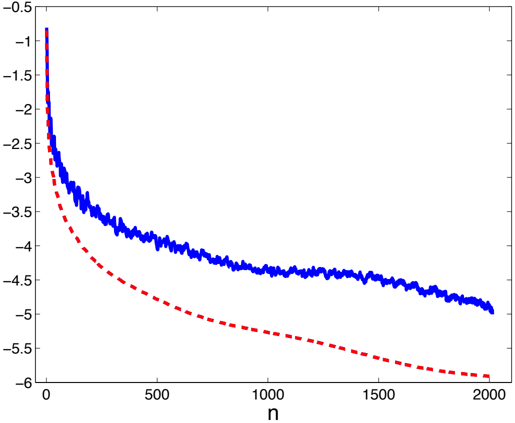

Figure: $y_0(x) = \sin(\pi x)$, $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - $\log_{10}(E_h(y_h^n))$ and $\log_{10}(\|g_h^n\|_{\mathcal{A}})$ vs. the iteration n of the conjugate gradient algorithm.

- The control y_h ensures that $\|\overline{y}_h(T,\cdot)\|_{L^2(\Omega)} \approx 10^{-3}$
- The distributed case is addressed in a similar way by considering the problem

$$E(u) = \frac{1}{2}\iint_{Q_T\setminus q_T} \left(|v_t|^2 + a(x)|v_x|^2\right)\, dx\, dt$$

so that v vanishes out of q_T.

- Main advantage : The approach does not introduce any dual variable and for instance allows to obtain fundamental solution for the heat eq.

- Main drawback: do not control the norm of the control

Final remarks

- NUMERICAL APPROXIMATIONS OF EXACT CONTROLS FOR THE HEAT IS SEVERALLY ILL-POSED, CONSEQUENCE OF THE REGULARIZATION PROPERTY.

- INTRODUCTION OF CARLEMAN TYPE WEIGHTS PROVIDES AN APPROPRIATE (ELLIPTIC) FRAMEWORK, VERY SUITABLE NUMERICALLY.

WORK IN PROGRESS : A POSTERIORI ESTIMATE FOR $\|p_h - p\|_P$ VS. h.

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THANK YOU FOR YOUR ATTENTION