About Least-squares type approach to address direct and controllability problems

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joint work with PABLO PEDREGAL (Ciudad Real, Spain)

Introduction (the linear heat eq. to fix ideas)

$\omega \subset \Omega \subset \mathbb{R}^N$, $N \geq 1$, $a \in C^1(\overline{\Omega}, \mathbb{R}^+_*)$, $d \in L^\infty(Q_T)$, $T > 0$, $Q_T = \Omega \times (0,T)$,
$q_T = \omega \times (0,T)$, $\Gamma_T := \partial\Omega \times (0,T)$

$$\begin{cases} Ly \equiv y_t - \nabla \cdot (a(x)\nabla y) + dy = v 1_\omega, & \text{in} \quad Q_T \\ y = 0, & \text{on} \quad \Gamma_T \\ y(\cdot, 0) = y_0, & \text{in} \quad \Omega. \end{cases} \tag{1}$$

$$(y_0 \in L^2(\Omega), v \in L^2(q_T)) \Longrightarrow y \in C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega)).$$

Null controllability - $\forall T > 0, \omega \subset \Omega, \quad \exists v \in L^2(q_T)$ s.t. $y(\cdot, T) = 0$
(FURSIKOV-IMANUVILOV'96, ROBBIANO-LEBEAU'95, etc)

Control of minimal L^2- norm-

$$\begin{cases} \min J(y,v) := \|v\|^2_{L^2(q_T)} \quad \text{over} \quad \mathcal{C}(y_0, T) \\ \mathcal{C}(y_0, T) = \{ (y, v) : v \in L^2(q_T), \quad y \text{ solves (1) and satisfies } y(T, \cdot) = 0 \} \end{cases} \tag{2}$$

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Minimal L^2 norm control using duality [Glowinski-Lions 94']

$$\inf_{(y,v)\in\mathcal{C}(y_0,T)} J(y,v) = -\inf_{\phi_T\in H} J^\star(\phi_T), \; J^\star(\phi_T) := \frac{1}{2}\int_{q_T}\phi^2 dxdt + \int_\Omega \phi(0,\cdot)y_0 dx$$

where ϕ solves the backward system

$$\begin{cases} L^\star\phi \equiv -\phi_t - \nabla\cdot(a(x)\nabla\phi) + d\phi = 0 \qquad Q_T = (0,T)\times\Omega, \\ \phi = 0 \quad \Sigma_T = (0,T)\times\partial\Omega, \qquad \phi(T,\cdot) = \phi_T \quad \Omega. \end{cases}$$

H-completion of $\mathcal{D}(\Omega)$ with respect to the norm

$$\|\phi_T\|_H = \left(\int_{q_T}\phi^2(t,x)dxdt\right)^{1/2}.$$

From the observability inequality

$$C(T,\omega)\|\phi(0,\cdot)\|^2_{L^2(\Omega)} \leq \|\phi_T\|^2_H \quad \forall\phi_T\in L^2(\Omega),$$

$J^\star$ is coercive on H. The control is given by $v = \phi\mathcal{X}_\omega$ on Q_T.

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$N = 1$ - $L^2(0,1)$-norm of the HUM control with respect to time

Hugeness of H: $H^{-s} \subset H$ for any $s \geq 0 \Longrightarrow$ Ill-posedness

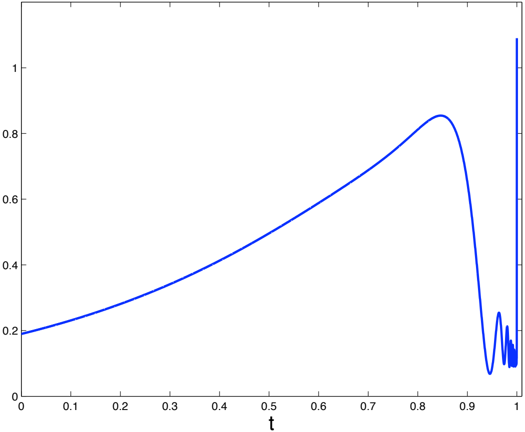

Figure: $y_0(x) = \sin(\pi x)$ - $T = 1$ - $\omega = (0.2, 0.8)$ - $t \to \|v(\cdot,t)\|_{L^2(0,1)}$ in $[0, T]$

Remedies via Carleman approach and convergence results in [Fernandez-Cara, Münch, 2011-2014]

Least-squares approach

We define the non-empty set

$$\mathcal{A} = \Big\{(u,f);\ u \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega));\ u' \in L^2(0,T,H^{-1}(\Omega)),$$
$$u(\cdot,0) = u_0, u(\cdot,T) = 0, f \in L^2(q_T)\Big\}$$

and find $(u,f) \in \mathcal{A}$ solution of the heat eq. !

For any $(u,f) \in \mathcal{A}$, we define the "corrector" $v = v(u,f) \in H^1(Q_T)$ solution of the Q_T-elliptic problem

$$\begin{cases} -v_{tt} - \nabla\cdot(a(x)\nabla v) + \Big(u_t - \nabla\cdot(a(x)\nabla u) + du - f\,1_\omega\Big) = 0, & (x,t) \in Q_T, \\ v_t = 0, & x \in \Omega, t \in \{0,T\} \\ v = 0, & x \in \Sigma_T. \end{cases} \quad (3)$$

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Least-squares approach (2)

Theorem
u is a controlled solution of the heat eq. by the control function $f\,1_\omega \in L^2(q_T)$ if and only if (u, f) is a solution of the extremal problem

$$\inf_{(u,f)\in\mathcal{A}} E(u,f) := \frac{1}{2}\iint_{Q_T} (|v_t|^2 + a(x)|\nabla v|^2)dx\,dt. \tag{4}$$

Proof.
$\Longleftarrow$ From the null controllability of the heat eq., the extremal problem is well-posed in the sense that the infimum, equal to zero, is reached by any controlled solution of the heat eq. (the minimizer is not unique).

$\Longrightarrow$ Conversely, we check that any minimizer of E is a solution of the (controlled) heat eq.:
We define the vector space

$$\mathcal{A}_0 = \Big\{(u,f);\ u \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega));\ u' \in L^2(0,T,H^{-1}(\Omega)),$$
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Least-squares approach (2)

The first variation of E at (u, f) in the admissible direction $(U, F) \in \mathcal{A}_0$ defined by

$$< E'(u,f),(U,F) >= \lim_{\eta \to 0} \frac{E((u,f)+\eta(U,F)) - E(u,f)}{\eta} \tag{5}$$

exists and is given by

$$< E'(u,f),(U,F) >= \iint_{Q_T} (v_t V_t + a(x)\nabla v \cdot \nabla V) dx\, dt, \tag{6}$$

where the corrector $V \in H^1(Q_T)$ associated to (U, F) is the solution of

$$\begin{cases} U_t - V_{tt} - \nabla \cdot (a(x)(\nabla U + \nabla V)) - F\, 1_\omega = 0, & (x,t) \in Q_T, \\ V_t(x,0) = V_t(x,T) = 0, & x \in \Omega, \\ V(0,t) = V(1,t) = 0, & t \in (0,T). \end{cases} \tag{7}$$

Using that

$$-\int_0^T < U_t, v >_{H^{-1}(\Omega),H^1(\Omega)} \, dt = \iint_{Q_T} U v_t \, dx\, dt - \int_0^1 [Uv]_0^T \, dx = \iint_{Q_T} U v_t \, dx\, dt,$$

we get that

$$< E'(u,f),(U,F) >= \iint_{Q_T} (U v_t - a(x)\nabla U \cdot \nabla v + F v\, 1_\omega)\, dx\, dt, \quad \forall (U,F) \in \mathcal{A}_0$$

Least-squares approach (2)

Therefore, if (u, f) minimizes E, the equality $< E'(u, f), (U, F) >= 0$ for all $(U, F) \in A_0$ implies that the corrector $v = v(u, f)$ satisfies

$$\begin{cases} -v_t - \nabla \cdot (a(x)\nabla v) + dv = 0, & (x,t) \in Q_T, \\ v = 0, & (x,t) \in q_T \end{cases}$$

in addition to the boundary conditions: $v = 0$ on Σ_T and $v_t = 0$ on $\Omega \times \{0, T\}$.
Unique continuation property implies that $v = 0$ in Q_T and so $E(u, f) = 0$ and so $(u, f) \in \mathcal{A}$ solves the heat eq.

Remark The proposition educes the search of ONE control f distributed in ω to the minimization of the functional E over $\mathcal{A}$.

Remark Least squares terminology :

$$E(u,f) := \frac{1}{2}\|u_t - \nabla \cdot (a(x)\nabla u) + d\,u - f 1_\omega\|^2_{H^{-1}(Q_T)}$$

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Least-squares approach (3): convergence of (some) minimizing sequences

For any $s_{\mathcal{A}} := (u_{\mathcal{A}}, f_{\mathcal{A}}) \in \mathcal{A}$, we consider the equivalent problem :

$$\min_{(u,f)\in\mathcal{A}_0} E_{s_{\mathcal{A}}}(u,f), \qquad E_{s_{\mathcal{A}}}(u,f) := E(s_{\mathcal{A}} + (u,f)). \tag{8}$$

$(\mathcal{A}_0, \|\cdot\|_{\mathcal{A}_0})$ is a Hilbert space with and introduce

$$\|u,f\|^2_{\mathcal{A}_0} := \iint_{Q_T} (|u|^2 + |\nabla u|^2)\, dx\, dt + \int_0^T \|u_t(\cdot,t)\|^2_{H^{-1}(\Omega)} dt + \iint_{Q_T} |f|^2\, dx\, dt \tag{9}$$

The boundedness of $E_{s_{\mathcal{A}}}$ implies only the boundedness of the corrector v for the $H^1(Q_T)$-norm.

It turns out that minimizing sequences for $E_{s_{\mathcal{A}}}$ which belong to a precise subset of $\mathcal{A}_0$ remain bounded uniformly.

Actually, this property is mainly due to the fact the functional $E_{s_{\mathcal{A}}}$ is invariant in the subset of $\mathcal{A}_0$ which satisfies the state equations.

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We note

- $\mathbf{T}$ which maps a triplet $(u,f) \subset \mathcal{A}$ into the corresponding vector $v \in H^1(Q_T)$.
- $A = \mathrm{Ker}\,\mathbf{T} \cap \mathcal{A}_0$ composed of the elements (u,f) satisfying the heat eq. and such that u vanishes on the boundary ∂Q_T.
- $A^\perp = (\mathrm{Ker}\,\mathbf{T} \cap \mathcal{A}_0)^\perp$ the orthogonal complement of A in $\mathcal{A}_0$
- $P_{A^\perp} : \mathcal{A}_0 \to A^\perp$ the (orthogonal) projection on $A^\perp$.

We define the minimizing sequence $(u^k, f^k)_{k\geq 0} \in A^\perp$ as follows:

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where $(\overline{u}^k, \overline{f}^k) \in \mathcal{A}_0$ is defined as the unique solution of the formulation

$$\langle (\overline{u}^k, \overline{f}^k), (U,F) \rangle_{\mathcal{A}_0} = \langle E'_{s_\mathcal{A}}(u^k, f^k), (U,F) \rangle, \quad \forall (U,F) \in \mathcal{A}_0. \tag{11}$$

Proposition

For any $s_\mathcal{A} \in \mathcal{A}$ and any $\{u^0, f^0\} \in A^\perp$, the sequence $s_\mathcal{A} + \{(u^k, f^k)\}_{k\geq 0} \in \mathcal{A}$ converges strongly to a solution of the extremal problem for E.

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Least-squares approach (4): convergence of the minimizing sequences

This proposition is the consequence of the following abstract result :

Lemma

Suppose $\mathbf{T}: X \mapsto Y$ *is a linear, continuous operator between Hilbert spaces, and* $H \subset X$*, a closed subspace,* $u_0 \in X$*. Put*

$$E : u_0 + H \mapsto \mathbb{R}^+, \quad E(u) = \frac{1}{2}\|\mathbf{T}u\|^2, \quad A = Ker\mathbf{T} \cap H.$$

1. $E : u_0 + A^\perp \to \mathbb{R}$ *is quadratic, non-negative, and strictly convex, where* $A^\perp$ *is the orthogonal complement of* A *in* H.
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Least-squares approach (4): convergence of the minimizing sequences

This proposition is the consequence of the following abstract result :

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Least-squares approach (4): convergence of the minimizing sequences

PROOF OF THE LEMMA - 1-) Suppose there are $u_i \in A^\perp$, $i = 1, 2$, such that

$$E\left(u_0 + \frac{1}{2}u_1 + \frac{1}{2}u_2\right) = \frac{1}{2}E(u_0 + u_1) + \frac{1}{2}E(u_0 + u_2).$$

Due to the strict convexity of the norm in a Hilbert space, we deduce that this equality can only occur if $\mathbf{T}u_1 = \mathbf{T}u_2$. So therefore $u_1 - u_2 \in A \cap A^\perp = \{0\}$, and $u_1 = u_2$.

2-) Note that for arbitrary $U \in A$, $\mathbf{T}U = 0$, and so

$$E(u_0 + u + U) = \frac{1}{2}\|\mathbf{T}u_0 + \mathbf{T}u + \mathbf{T}U\|^2 = \frac{1}{2}\|\mathbf{T}u_0 + \mathbf{T}u\|^2 = E(u_0 + u).$$

Therefore the derivative $E'(u_0 + u)$, the steepest descent direction for E at $u_0 + u$, has to be orthogonal to all such $U \in A$.

Least-squares approach (4): convergence of the minimizing sequences

PROOF OF THE LEMMA 1- 3-) Finally, assume $E(u_0 + \overline{u}) = 0$. It is clear that this minimizer is unique in $u_0 + A^{\perp}$ (recall the strict convexity in (i)). This, in particular, implies that for arbitrary $u \in A^{\perp}$,

$$\langle E'(u_0 + u), \overline{u} - u \rangle \leq 0, \tag{12}$$

because this inner product is the derivative of the section $t \mapsto E(u_0 + t\overline{u} + (1-t)u)$ at $t = 0$, and this section must be a positive parabola with the minimum point at $t = 1$. If we consider the gradient flow

$$u'(t) = -E'(u_0 + u(t)), \quad t \in [0, +\infty),$$

then, because of (12),

$$\frac{d}{dt}\left(\frac{1}{2}\|u(t) - \overline{u}\|^2\right) = \langle u(t) - \overline{u}, u'(t) \rangle = \langle u(t) - \overline{u}, -E'(u_0 + u(t)) \rangle \leq 0.$$

This implies that sequences produced through a steepest descent method will be minimizing for E, uniformly bounded in X (because $\|u(t) - \overline{u}\|$ is a non-increasing function of t), and due to the strict convexity of E restricted to $u_0 + A^{\perp}$, they will have to converge towards the unique minimizer $u_0 + \overline{u}$.

Least-squares approach (4): convergence of the minimizing sequences

PROOF OF THE PROPOSITION-

The result is obtained by applying the previous lemma 1 with :

- $B = \{\mathbf{y} \in \mathbf{L}^2(0, T, H_0^1(\Omega)) : \mathbf{y}_t \in \mathbf{L}^2(0, T; H^{-1}(\Omega))\}$,
- X is taken to be $B \times L^2(q_T)$
- H is taken to be $\mathcal{A}_0$, $u_0 = s_{\mathcal{A}} \in \mathcal{A} \subset X$.
- The operator $\mathbf{T}$ maps $(u, f) \in \mathcal{A} \subset X$ into $v \in Y := H^1(Q_T)$

Remark

Direct problem

$$\mathcal{A} = \left\{ u;\ u \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)); u' \in L^2(0,T,H^{-1}(\Omega)), u(\cdot,0) = u_0 \right\}$$

$< E'(u), U >= 0$ for all $U \in \mathcal{A}_0$ implies that the corrector v solves

$$\begin{cases} -v_t - \nabla \cdot (a(x)\nabla v) + dv = 0, & (x,t) \in Q_T, \\ v(\cdot, T) = 0, & x \in \Omega \end{cases}$$

Boundary controllability

$\Sigma_T \subset \Gamma_T := \partial\Omega \times (0,T)$.

$$\mathcal{A} = \left\{ u;\ u \in H^1(Q_T), u = 0 \text{ on } \Gamma_T \setminus \Sigma_T,\ u(\cdot,0) = u_0, u(\cdot,T) = 0 \right\}$$

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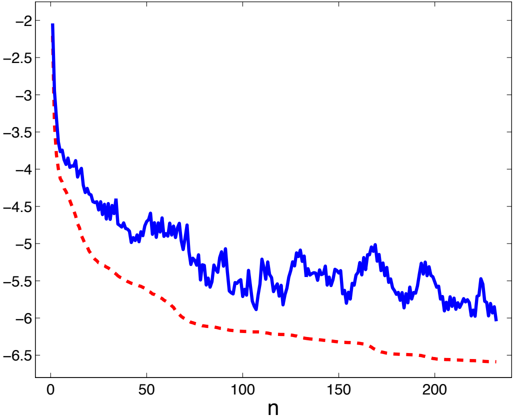
A numerical application in 1D (inner controllability)

$N = 1$, $\Omega = (0, 1)$, $\omega = (0.2, 0.5)$, $u_0(x) = sin(\pi x)$, $a(x) = a_0 = 0.25$, $T = 1/2$, $d := 0$

Starting point of the algorithm: $(u, f) = (u_0(x)(1 - t/T)^2, 0) \in \mathcal{A}$



$u_0(x) = \sin(\pi x)$ - Control acting on $\omega = (0.2, 0.5)$ - $\varepsilon = 10^{-6}$ - $\log_{10}(E_h(u_h^n)$ (**dashed line**) and $\log_{10}(\|g_h^n\|_{\mathcal{A}})$ (**full line**) vs. the iteration n of the CG algorithm.

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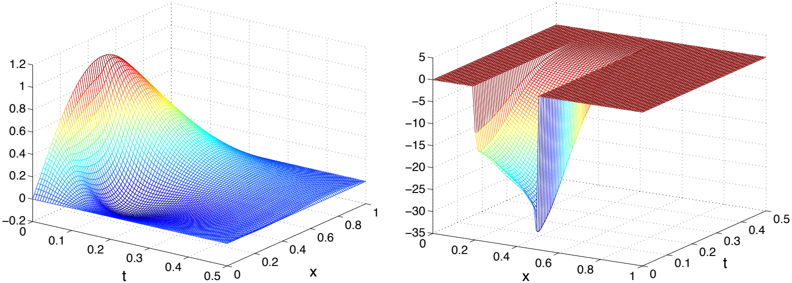
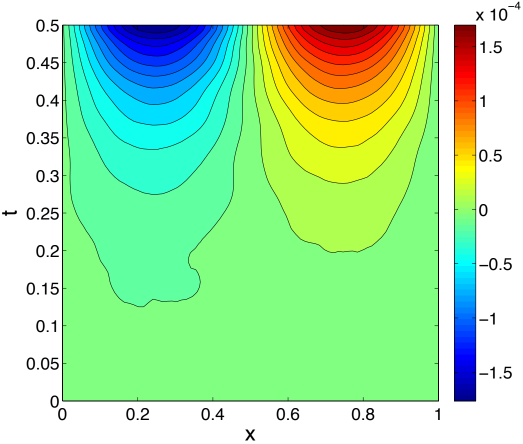


$(u, f) \in \mathcal{A}$ along Q_T at convergence

A numerical application in 1D (inner controllability)



Isovalues along Q_T of the corresponding corrector v: $\|v\|_{H^1(Q_T)} \approx 10^{-4}$

A numerical application in 1D (boundary controllability)

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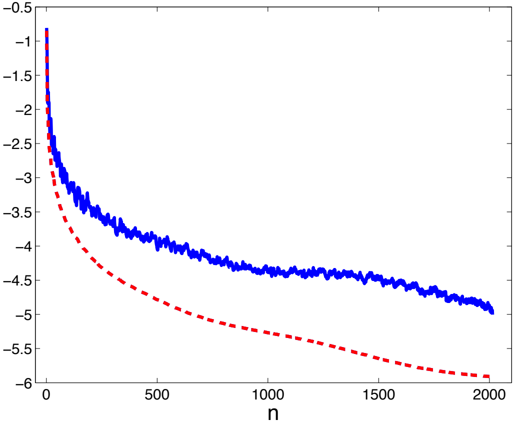

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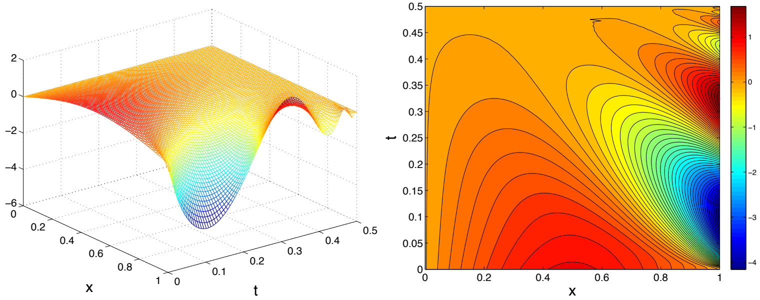

Left: $u \in \mathcal{A}$ along Q_T at convergence; Right: Iso-values of u

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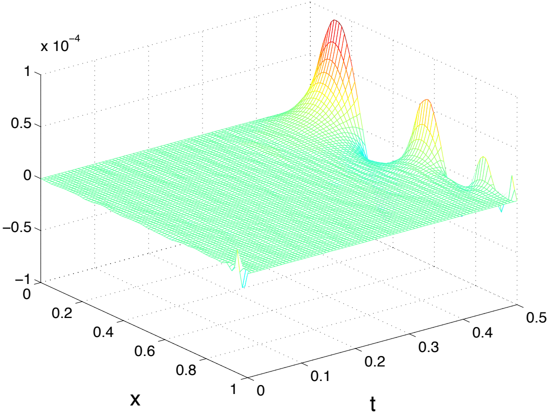

Corresponding corrector v: $\|v\|_{H^1(Q_T)} \approx 10^{-4}$

Direct and control problem for Stokes

$$\begin{cases} \mathbf{y}_t - \nu\Delta\mathbf{y} + \nabla\pi = \mathbf{f}\, 1_\omega, \quad \nabla\cdot\mathbf{y} = 0 \quad \text{in } Q_T \\ \mathbf{y} = \mathbf{0} \quad \text{on } \Sigma_T, \quad \mathbf{y}(\cdot,0) = \mathbf{y}_0 \quad \text{in } \Omega \end{cases} \tag{13}$$

$$\begin{aligned} \mathbf{H} &= \{\boldsymbol{\varphi} \in \mathbf{L}^2(\Omega) : \nabla\cdot\boldsymbol{\varphi} = 0 \text{ in } \Omega,\ \boldsymbol{\varphi}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}, \\ \mathbf{V} &= \{\boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega) : \nabla\cdot\boldsymbol{\varphi} = 0 \text{ in } \Omega\}, \\ U &= \left\{\psi \in L^2(\Omega) : \int_\Omega \psi(\mathbf{x})\, d\mathbf{x} = 0\right\}. \end{aligned} \tag{14}$$

Then, for any $\mathbf{y}_0 \in \mathbf{H}$, $T > 0$, and $\mathbf{f} \in \mathbf{L}^2(q_T)$, there exists exactly one solution $(\mathbf{y}, \pi)$ of (13) with the following regularity :

$$\mathbf{y} \in C^0([0,T];\mathbf{H}) \cap L^2(0,T;\mathbf{V}), \ \pi \in L^2(0,T;U)$$

Theorem

For any $\mathbf{y}_0 \in \mathbf{H}$, the linear system (13) is null-controllable at any time $T > 0$.

Least-squares for the controllability of Stokes

$$\mathcal{A} = \Big\{ (\mathbf{y}, \pi, \mathbf{f});\ \ \mathbf{y} \in \mathbf{L}^2(0,T,\mathbf{H}_0^1(\Omega)), \mathbf{y}_t \in \mathbf{L}^2(0,T;\mathbf{H}^{-1}(\Omega)),$$
$$\mathbf{y}(\cdot,0) = \mathbf{y}_0,\ \ \mathbf{y}(\cdot,T) = \mathbf{0},\ \ \pi \in L^2(0,T;U),\ \ \mathbf{f} \in \mathbf{L}^2(q_T) \Big\}. \quad (15)$$

Then, we define the functional $E : \mathcal{A} \to \mathbb{R}^+$ by

$$E(\mathbf{y}, \pi, \mathbf{f}) = \frac{1}{2} \iint_{Q_T} (|\mathbf{v}_t|^2 + |\nabla \mathbf{v}|^2 + |\nabla \cdot \mathbf{y}|^2)\, d\mathbf{x}\, dt \quad (16)$$

where the corrector $\mathbf{v}$ is the unique solution in $\mathbf{H}^1(Q_T)$ of the (elliptic) boundary value problem

$$\begin{cases} -\mathbf{v}_{tt} - \Delta \mathbf{v} + (\mathbf{y}_t - \nu \Delta \mathbf{y} + \nabla \pi - \mathbf{f}\, 1_\omega) = 0, & \text{in} \quad Q_T, \\ \mathbf{v} = 0 \quad \text{on} \quad \Sigma_T, \quad \mathbf{v}_t = 0 \quad \text{on} \quad \Omega \times \{0, T\}. \end{cases} \quad (17)$$

Least-squares for the controllability of Stokes

[Pedregal, Münch 2014], [Münch 2015]

Proposition

$(\mathbf{y}, \pi)$ is a controlled solution of the Stokes system (13) by the control function $\mathbf{f}\,1_\omega \in \mathbf{L}^2(q_T)$ if and only if $(\mathbf{y}, \pi, \mathbf{f})$ is a solution of the extremal problem :

$$\inf_{(\mathbf{y},\pi,\mathbf{f})\in\mathcal{A}} E(\mathbf{y}, \pi, \mathbf{f}). \tag{18}$$

Proof- $\Longrightarrow$
$\langle E'(\mathbf{y}, \pi, \mathbf{f}), (\mathbf{Y}, \Pi, \mathbf{F})\rangle = 0\ \forall(\mathbf{Y}, \Pi, \mathbf{F}) \in \mathcal{A}_0$ implies that the corrector $\mathbf{v} = \mathbf{v}(\mathbf{y}, \pi, \mathbf{f})$ solution of (30) satisfies the conditions

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The unique continuation property for the Stokes system implies that $\mathbf{v} = 0$ in Q_T and that $\nabla\cdot\mathbf{y}$ is a constant in Q_T. Eventually, from

$$\langle E'(\mathbf{y}, \pi, \mathbf{f}), (\mathbf{Y}, \Pi, \mathbf{F})\rangle = (\nabla\cdot\mathbf{y})\iint_{Q_T} \nabla\cdot\mathbf{Y}\,dx\,dt = 0, \quad \forall(\mathbf{Y}, \Pi, \mathbf{F}) \in \mathcal{A}_0$$

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Convergence of the minimizing sequence

We then define the following minimizing sequence $(\mathbf{y}^k, \pi^k, \mathbf{f}^k)_{k\geq 0} \in A^\perp$ as follows:

$$\begin{cases} (\mathbf{y}^0, \pi^0, \mathbf{f}^0) \text{ given in } A^\perp, \\ (\mathbf{y}^{k+1}, \pi^{k+1}, \mathbf{f}^{k+1}) = (\mathbf{y}^k, \pi^k, \mathbf{f}^k) - \eta_k P_{A^\perp}(\overline{\mathbf{y}}^k, \overline{\pi}^k, \overline{\mathbf{f}}^k), \quad k \geq 0, \\ \langle(\overline{\mathbf{y}}^k, \overline{\pi}^k, \overline{\mathbf{f}}^k), (\mathbf{Y}, \Pi, \mathbf{F})\rangle_{\mathcal{A}_0} = \langle E'_{\mathbf{s_0}}(\mathbf{y}^k, \pi^k, \mathbf{f}^k), (\mathbf{Y}, \Pi, \mathbf{F})\rangle, \quad \forall(\mathbf{Y}, \Pi, \mathbf{F}) \in \mathcal{A}_0. \end{cases} \tag{20}$$

Proposition

For any $\mathbf{s}_\mathcal{A} \in \mathcal{A}$ and any $\{\mathbf{y}^0, \pi^0, \mathbf{f}^0\} \in A^\perp$, the sequence $\mathbf{s}_\mathcal{A} + \{(\mathbf{y}^\mathbf{k}, \pi^\mathbf{k}, \mathbf{f}^\mathbf{k})\}_{\mathbf{k}\geq\mathbf{0}} \in \mathcal{A}$ converges strongly to a solution of the extremal problem (29).

Proof- Applied the lemma with
$B = \{\mathbf{y} \in \mathbf{L}^2(0, T, \mathbf{H}_0^1(\Omega)) : \mathbf{y}_t \in \mathbf{L}^2(0, T; \mathbf{H}^{-1}(\Omega))\}$, X is taken to be
$B \times L^2(0, T; U) \times \mathbf{L}^2(q_T)$.
$H = \mathcal{A}_0$ and $u_0 = \boldsymbol{s}_\mathcal{A} \in \mathcal{A} \subset X$.
The operator $\mathbf{T}$ maps a triplet $(\mathbf{y}, \pi, \mathbf{f}) \in \mathcal{A} \subset X$ into
$(\mathbf{v}, \nabla \cdot \mathbf{y}) \in Y := \mathbf{H}^1(Q_T) \times L^2(Q_T)$.

Numerical application : controllability to trajectory

The Poiseuille flow $\overline{\mathbf{y}} = \left(-\frac{c}{2\nu}\, x_2(1-x_2), 0\right), \overline{\pi} = c\, x_1$ solves the stationary homogeneous Stokes eq.

$$-\nu\Delta\overline{\mathbf{y}} + \nabla\pi = \mathbf{0}, \quad \nabla\cdot\overline{\mathbf{y}} = 0 \quad \text{in } \; Q_T. \tag{21}$$

We introduce $(\mathbf{z}, \sigma) = (\mathbf{y} - \overline{\mathbf{y}}, \pi - \overline{\pi})$ where $(\mathbf{y}, \pi)$ solves the state equations of (13):

$$\mathbf{y}_t - \nu\Delta\mathbf{y} + \nabla\pi = \mathbf{f}\, 1_\omega, \quad \nabla\cdot\mathbf{y} = 0 \quad \text{in } \; Q_T, \quad \mathbf{y}(\cdot, 0) = \mathbf{y}_0 \quad \text{in } \; \Omega \tag{22}$$

so that $(\mathbf{z}, \sigma)$ solves

$$\mathbf{z}_t - \nu\Delta\mathbf{z} + \nabla\sigma = \mathbf{f}\, 1_\omega, \quad \nabla\cdot\mathbf{z} = 0 \quad \text{in } \; Q_T, \quad \mathbf{z}(\cdot, 0) = \mathbf{y}_0 - \overline{\mathbf{y}} \quad \text{in } \; \Omega. \tag{23}$$

We add the boundary condition $\mathbf{z} = \mathbf{0}$ on Σ_T.
For any $\mathbf{y}_0$ such that $\mathbf{y}_0 - \overline{\mathbf{y}} \in \mathbf{H}$, we determine $\mathbf{f}$ such that $\mathbf{z}(\cdot, T) = 0$ on Q_T.
$\mathbf{y} := \mathbf{z} + \overline{\mathbf{y}}$ is then controlled to the trajectory $\overline{\mathbf{y}}$ at time T.

$\Omega = (0,5) \times (0,1)$, $\omega = (1,2) \times (0,1)$, $T = 2$ and $\nu = 1/40$ and

$$\mathbf{y}_0 = \overline{\mathbf{y}} + \nabla \times \psi, \quad \psi = K(1-x_2)^2 x_2^2 (5-x_1)^2 x_1^2, \quad m \in \mathbb{N} \tag{24}$$

We take K such that $\|\nabla \times \psi\|_{L^2(\Omega)} = 2$.

Numerical application : controllability to trajectory

The Poiseuille flow $\overline{\mathbf{y}} = \left(-\frac{c}{2\nu}\, x_2(1-x_2), 0\right), \overline{\pi} = c\, x_1$ solves the stationary homogeneous Stokes eq.

$$-\nu\Delta\overline{\mathbf{y}} + \nabla\pi = \mathbf{0}, \quad \nabla\cdot\overline{\mathbf{y}} = 0 \quad \text{in } Q_T. \tag{21}$$

We introduce $(\mathbf{z}, \sigma) = (\mathbf{y} - \overline{\mathbf{y}}, \pi - \overline{\pi})$ where $(\mathbf{y}, \pi)$ solves the state equations of (13):

$$\mathbf{y}_t - \nu\Delta\mathbf{y} + \nabla\pi = \mathbf{f}\, 1_\omega, \quad \nabla\cdot\mathbf{y} = 0 \quad \text{in } Q_T, \quad \mathbf{y}(\cdot, 0) = \mathbf{y}_0 \quad \text{in } \Omega \tag{22}$$

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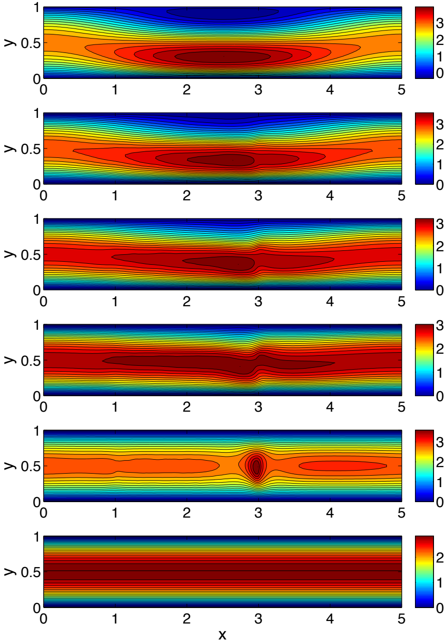

$\nu = 1/40$ - Iso-values of the first component $y_{1,h}(\cdot,t) = \overline{y}_1 + z_h(\cdot,t)$ of the velocity on Ω for $t = t_i \in \frac{i}{5}T$, $i = 0, \cdots, 5$

Reduce the L^2-norm of the control

The method avoids duality arguments and therefore ill-posedness: on the contrary, the controls obtained from the minimization of E does not minimize a priori any particular norm :
Two options :

- for any solution $(u, f) \in \mathcal{A}$, compute a collection of solution $(u_j, f_j) \in \mathcal{A}_0$, $j \in J$ and then solve

$$\min \|f + \sum_{j \in J} \alpha_j f_j\|_{L^2(q_T)} \quad w.r.t. \{\alpha_j\}_{j \in J} \tag{25}$$

- solve the saddle problem

$$\sup_{\lambda \in \mathbb{R}} \inf_{(\mathbf{y}, \pi, \mathbf{f}) \in \mathcal{A}} \mathcal{L}((\mathbf{y}, \pi, \mathbf{f}), \lambda) := \frac{1}{2} \|\mathbf{f}\|^2_{\mathbf{L}^2(q_T)} + \lambda E(\mathbf{y}, \pi, \mathbf{f}). \tag{26}$$

  The set $\{(\mathbf{y}, \pi, \mathbf{f}) \in \mathcal{A}, E(\mathbf{y}, \pi, \mathbf{f}) = 0\}$ is convex so Uzawa type algorithm converges :

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Non linear case ? Example : NS steady case

$$\begin{cases} -\nu\Delta \mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla\pi = \mathbf{f}, \quad \nabla\cdot\mathbf{y} = 0 \quad \text{in } \Omega \\ \mathbf{y} = \mathbf{0} \quad \text{on } \partial\Omega \end{cases} \tag{27}$$

$\forall \mathbf{f} \in \mathbf{H}^{-1}(\Omega), \exists(\mathbf{y},\pi) \in \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega)$.

$$\begin{cases} \mathcal{A} = \mathbf{H}^1_0(\Omega) \times L^2(\Omega), \quad E : \mathcal{A} \to \mathbb{R}^+ \\ E(\mathbf{y},\pi) := \frac{1}{2}\int_\Omega (|\nabla \mathbf{v}|^2 + |\nabla\cdot\mathbf{y}|^2)\, d\mathbf{x} \end{cases} \tag{28}$$

Least-Squares problem :

$$\inf_{(\mathbf{y},\pi)\in\mathcal{A}} E(\mathbf{y},\pi). \tag{29}$$

where the corrector $\mathbf{v}$ is the unique solution in $\mathbf{H}^1_0(Q_T)$ of the (elliptic) boundary value problem

$$\begin{cases} -\Delta\mathbf{v} + (-\nu\Delta\mathbf{y} + div(\mathbf{y}\otimes\mathbf{y}) + \nabla\pi - \mathbf{f}) = 0, \quad \text{in} \quad \Omega, \\ \mathbf{v} = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{30}$$

Main issue

$$E(\mathbf{y}_j,\pi_j) \to 0 \quad \text{as} \quad j\to\infty \Longrightarrow (\mathbf{y}_j,\pi_j) \to (\mathbf{y}, f) \in \mathcal{A} \quad \text{with} \quad E(\mathbf{y},\pi) = 0 \quad ?? \tag{31}$$

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Non linear case. Abstract framework

Consider a non-negative, smooth functional $E : H \to \mathbb{R}$ defined on a Hilbert space H.

Definition

A functional E is an error functional if over bounded sets of H, $\lim_{E'(u)=0} E(u) = 0$.

Proposition (Pedregal 2014- Non existence of mountain-pass points)

Suppose $E : H \to \mathbb{R}$ is an error functional and $Z = \{E \equiv 0\} = \{u_0\}$. Then, the functional $\rho : [0, \infty) \to [0, \infty)$ defined by

$$\rho(r) := \inf_{\|u-u_0\|=r} E(u) \qquad \textit{is non-decreasing.}$$

Definition ("Coercivity" of E at its zero set)

The zero set Z of E is regular if it is non-empty and if $\lim_{E(u)\to 0} dist(u, Z) = 0$.

Theorem

Every integral curve of the flow

$$u(0) \in H; \quad u'(t) = -E'(u(t)), \quad t > 0 \tag{32}$$

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Proposition

E is an error functional: over bounded sets of $\mathcal{A}$, $\lim_{E'(\mathbf{y},\pi)=0} E(\mathbf{y}, \pi) = 0$.

Proof

$$E'(\mathbf{y}, \pi) \cdot (\mathbf{Y}, \Pi) = \int_\Omega -\nu \nabla \mathbf{v} \cdot \nabla \mathbf{Y} + (\mathbf{y} \otimes \mathbf{Y} + \mathbf{Y} \otimes \mathbf{y}) : \nabla \mathbf{v} + (\nabla \cdot \mathbf{v}) \Pi d\mathbf{x} \\ + \int_\Omega (\nabla \cdot \mathbf{y})(\nabla \cdot \mathbf{Y}) d\mathbf{x} \tag{34}$$

We easily get that

$$\|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)} \le C(\|\mathbf{y} \otimes \mathbf{y}\|_{L^2(\Omega)} + \|\mathbf{y}\|_{\mathbf{H}_0^1(\Omega)} + \|\pi\|_{L^2(\Omega)} + \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}). \tag{35}$$

so that we can take $\mathbf{Y} = \mathbf{v}$ leading to

$$E'(\mathbf{y}, \pi) \cdot (\mathbf{v}, \Pi) = \int_\Omega -\nu |\nabla \mathbf{v}|^2 - (\mathbf{v} \otimes \mathbf{v}) : \nabla \mathbf{y} + \frac{1}{2} (\nabla \cdot \mathbf{y}) |\mathbf{v}|^2 d\mathbf{x} \\ + \int_\Omega (\nabla \cdot \mathbf{v})(\nabla \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{v} + \Pi) d\mathbf{x} \tag{36}$$

Similarly, $\Pi_s = -(\nabla \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{v}) \in L^2(\Omega)$ remains bounded with respect to $(\mathbf{y}, \pi)$ and we write

$$E'(\mathbf{y}, \pi) \cdot (\mathbf{v}, \Pi_s) = \int_\Omega -\nu |\nabla \mathbf{v}|^2 - (\mathbf{v} \otimes \mathbf{v}) : \nabla \mathbf{y} + \frac{1}{2}(\nabla \cdot \mathbf{y})|\mathbf{v}|^2 d\mathbf{x} \qquad (37)$$

We then use the following result (consequence of the well-posedness of the Oseen equation)

Lemma

For any $\mathbf{y} \in \mathbf{H}^1_0(\Omega)$, $\mathbf{F} \in L^2(\Omega)$, *there exists* $(\mathbf{Y}, \Pi) \in H^1_0(\Omega) \times L^2(\Omega)$ *with* $\nabla \cdot \mathbf{Y} = 0$ *such that*

$$\int_\Omega (\nu \nabla \mathbf{Y} - (\mathbf{Y} \otimes \mathbf{y} + \mathbf{y} \otimes \mathbf{Y})) : \nabla \mathbf{w} - \Pi \nabla \cdot \mathbf{w} - \mathbf{F} \cdot \mathbf{w} = 0, \quad \forall \mathbf{w} \in H^1_0(\Omega) \qquad (38)$$

such that $\|\mathbf{Y}, \Pi\|_{\mathbf{H}^1_0(\Omega) \times L^2(\Omega)} \le C(\|\mathbf{y}\|_{\mathbf{H}^1_0(\Omega)}\| + \|\mathbf{F}\|_{L^2(\Omega)})$ *for some* $C > 0$.

Using this lemma for $\mathbf{F} = \mathbf{v}$ and $\mathbf{w} = \mathbf{v}$ ($\mathbf{v}$ is the corrector associated to the pair $(\mathbf{y}, \pi)$), we obtain that $(\mathbf{Y}, \Pi) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)$ satisfies $\nabla \cdot \mathbf{Y} = 0$ and

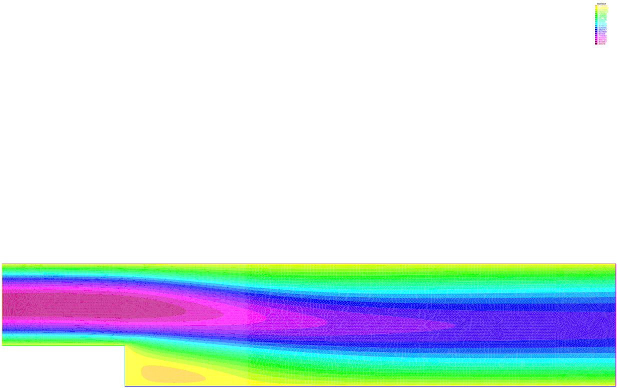
$$\int_\Omega (\nu \nabla \mathbf{Y} - (\mathbf{Y} \otimes \mathbf{y} + \mathbf{y} \otimes \mathbf{Y})) : \nabla \mathbf{v} - \Pi \nabla \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{v} = 0, \quad \forall \mathbf{w} \in H_0^1(\Omega) \tag{39}$$

With this pair $(\mathbf{Y}, \Pi)$ bounded with respect to $\mathbf{v}$ and to $\mathbf{y}$, and so with respect to $(\mathbf{y}, \pi)$, we have from (34), (remind that $\nabla \cdot \mathbf{Y} = 0$)

$$E'(\mathbf{y}, \pi) \cdot (\mathbf{Y}, \Pi) = \int_\Omega -\nu \nabla \mathbf{v} \cdot \nabla \mathbf{Y} + (\mathbf{y} \otimes \mathbf{Y} + \mathbf{Y} \otimes \mathbf{y}) : \nabla \mathbf{v} + (\nabla \cdot \mathbf{v}) \Pi d\mathbf{x} \tag{40}$$

The property $E'(\mathbf{y}, \pi) \cdot (\mathbf{Y}, \Pi) \to 0$ then implies that $\|\mathbf{v}\|_{L^2(\Omega)} \to 0$. Then, from (37), the property $E'(\mathbf{y}, \pi) \cdot (\mathbf{v}, \Pi_s) \to 0$ then implies from the equality (39) that $\|\nabla \mathbf{v}\|_{L^2(\Omega)} \to 0$. Then, 34 implies that $\int_\Omega \nabla \cdot \mathbf{y} \nabla \cdot \mathbf{Y} d\mathbf{x} \to 0$ for all $\mathbf{Y} \in H_0^1(\Omega)$ so that $\|\nabla \cdot \mathbf{y}\|_{L^2(\Omega)} \to 0$.

Non linear case ? Backward facing step



Iso-values of the first component of the velocity with Reynolds number $Re = 1/150$

Null controllability of a non linear heat equation

$$\begin{cases} y_t - \Delta y + F(y) = v\,1_\omega, & (x,t) \in Q_T, \\ y(\cdot,0) = y_0, & x \in \Omega, \\ y = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{41}$$

Theorem (Barbu 99, Fernandez-Cara Zuazua 00)

If $F : \mathbb{R} \to \mathbb{R}$ is locally lipschitz-continuous and satisfies

$$\frac{F(s)}{|s| log^{3/2}(1 + |s|)} \to 0 \quad as \quad s \to \infty$$

then the system is uniformly controllable.

Remark -
The controllability is proved by linearization and fixed point argument, useless in practice if the fixed point operator is not a contraction.

[Fernandez-Cara Münch, 2012]

Null control of the non linear heat equation

$$F(s) = -\alpha s \log^p(1+|s|), \quad \alpha = 5, \quad p = 1.4.$$

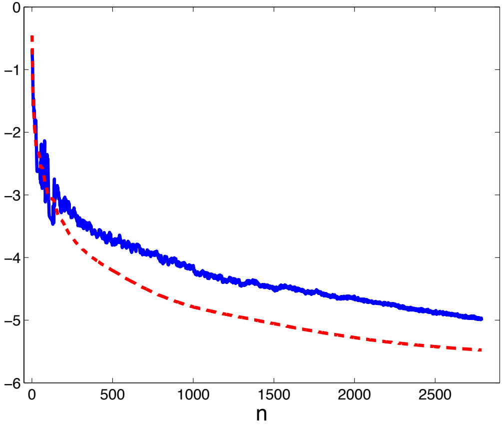
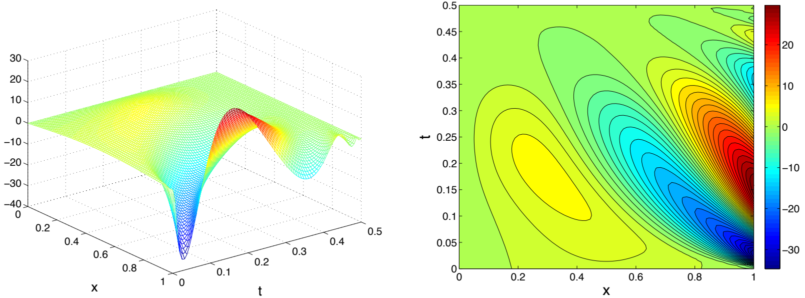


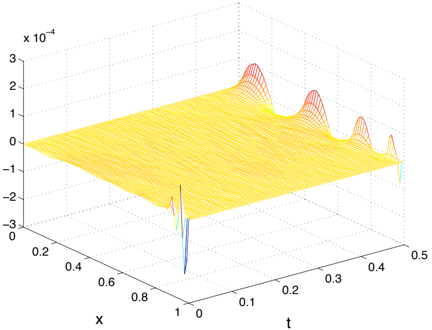
Figure: $u_0(x) = 3\sin(\pi x)$ - $T = 1/2$, $a_0 = 1/4$ - $\log_{10}(E_h(u_h^n))$ (**dashed line**) and $\log_{10}(\|g_h^n\|_{\mathcal{A}})$ (**full line**) vs. the iteration n of the CG algorithm.

Null control of the non linear heat equation



Convergent function $u \in \mathcal{A}$ along $Q_T = (0,1) \times (0,T)$ and its isovalues.

Null control of the non linear heat equation



Corrector function $v \in H^1(Q_T)$ along $Q_T = (0,1) \times (0,T)$

Conclusions

- Use of Least-squares method to controllability seems original
- Construction of strong convergence sequences.
- Can be extended to solve inverse type problems
- General method, numerically robust, simple implementation and (apparently !) fast :
- Open question: speed of convergence ?

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THANK YOU FOR YOUR ATTENTION !!!