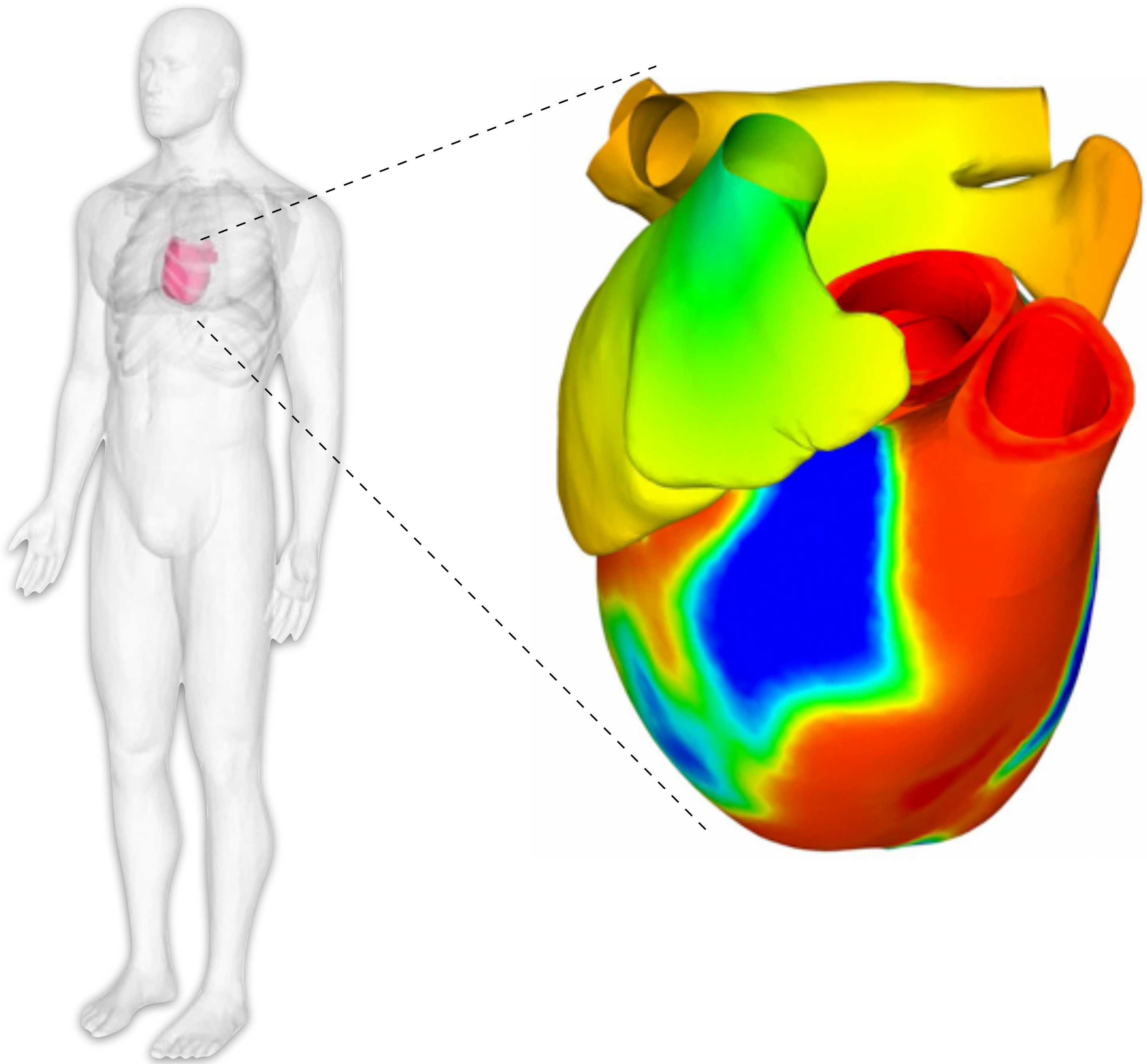
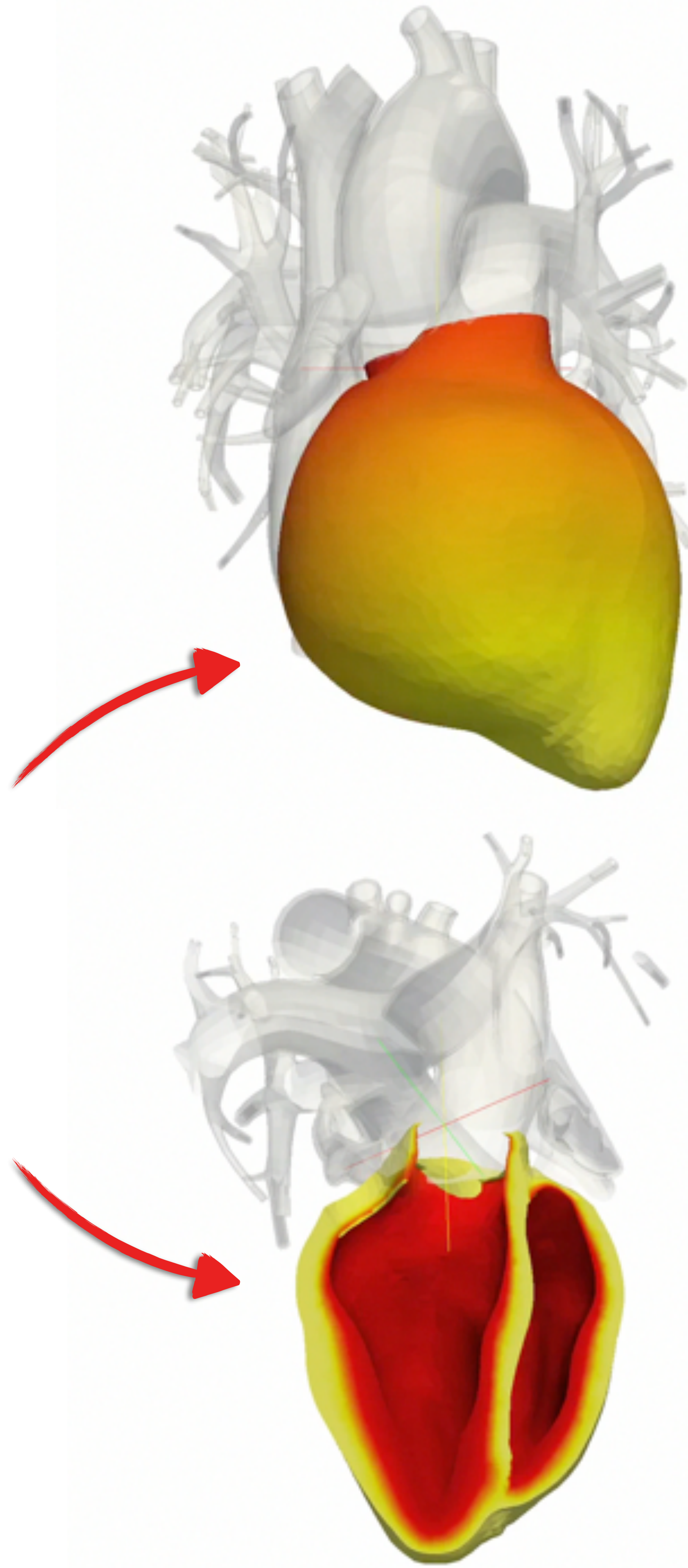
Observers based on front data for PDE models of propagative phenomena.

P. Moireau, Inria Saclay - Ile de France, Team MΞDISIM
— Joint work with D. Chapelle, A. Collin, S. Imperiale

Electrophysiology





From electrophysiology to reaction-diffusion

- Asymptotic Bidomain model

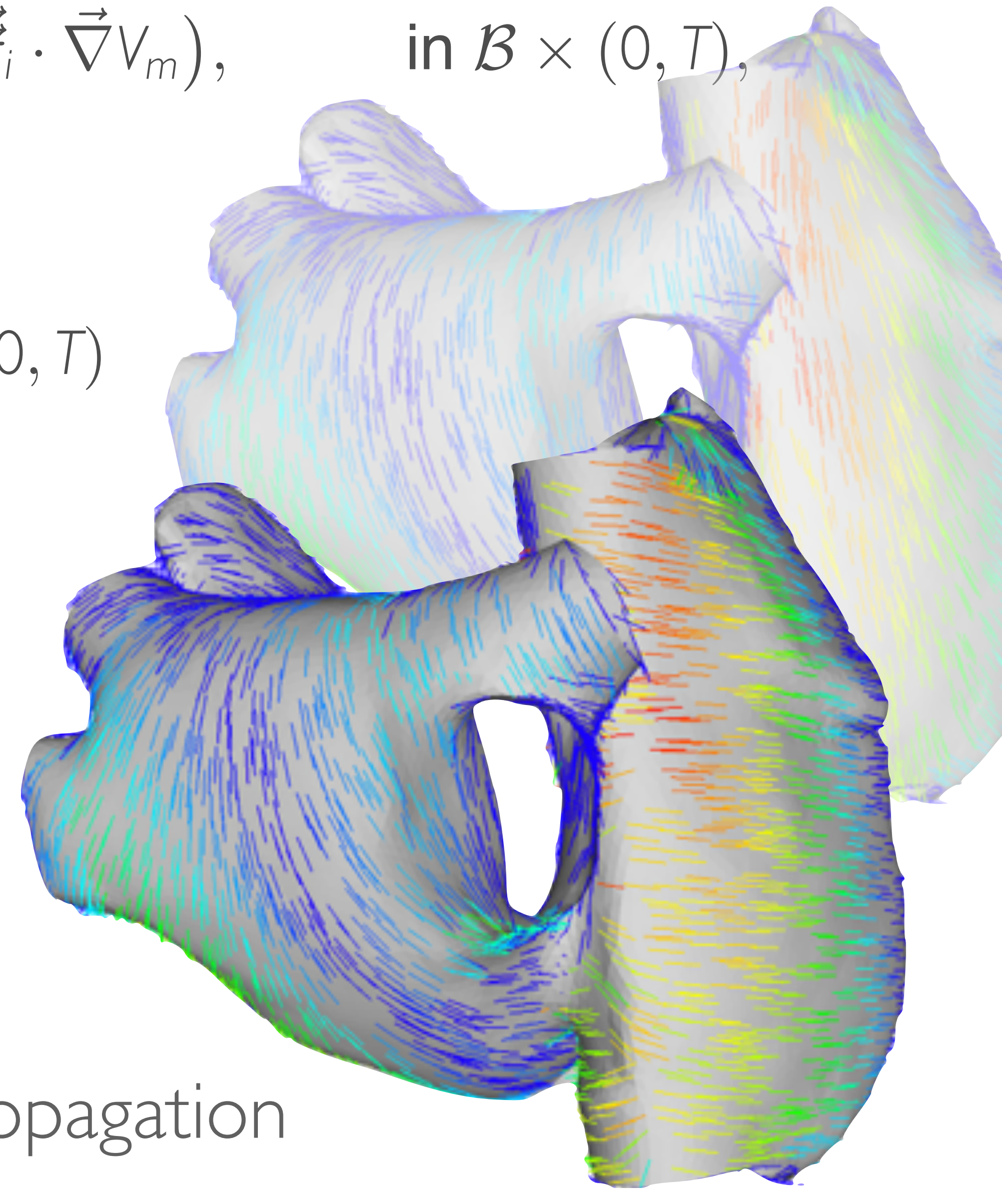
$$\begin{cases} A_m\left(C_m \dfrac{\partial V_m}{\partial t} + I_{ion}(V_m, \cdots)\right) - \mathbf{div}(\vec{\vec{\sigma}}_i \cdot \vec{\nabla} V_m) & = & \mathbf{div}(\vec{\vec{\sigma}}_i \cdot \vec{\nabla} u_e) + I_{app}, & \text{in } \mathcal{B} \times (0,T), \\ \mathbf{div}\left((\vec{\vec{\sigma}}_i + \vec{\vec{\sigma}}_e) \cdot \vec{\nabla} u_e\right) & = & -\mathbf{div}(\vec{\vec{\sigma}}_i \cdot \vec{\nabla} V_m), & \text{in } \mathcal{B} \times (0,T), \end{cases}$$

- Monodomain model

$$A_m\left(C_m \frac{\partial V_m}{\partial t} + I_{ion}(V_m, \cdots)\right) - \mathbf{div}(\vec{\vec{\sigma}} \cdot \vec{\nabla} V_m) = I_{app}, \quad \text{in } \mathcal{B} \times (0,T)$$

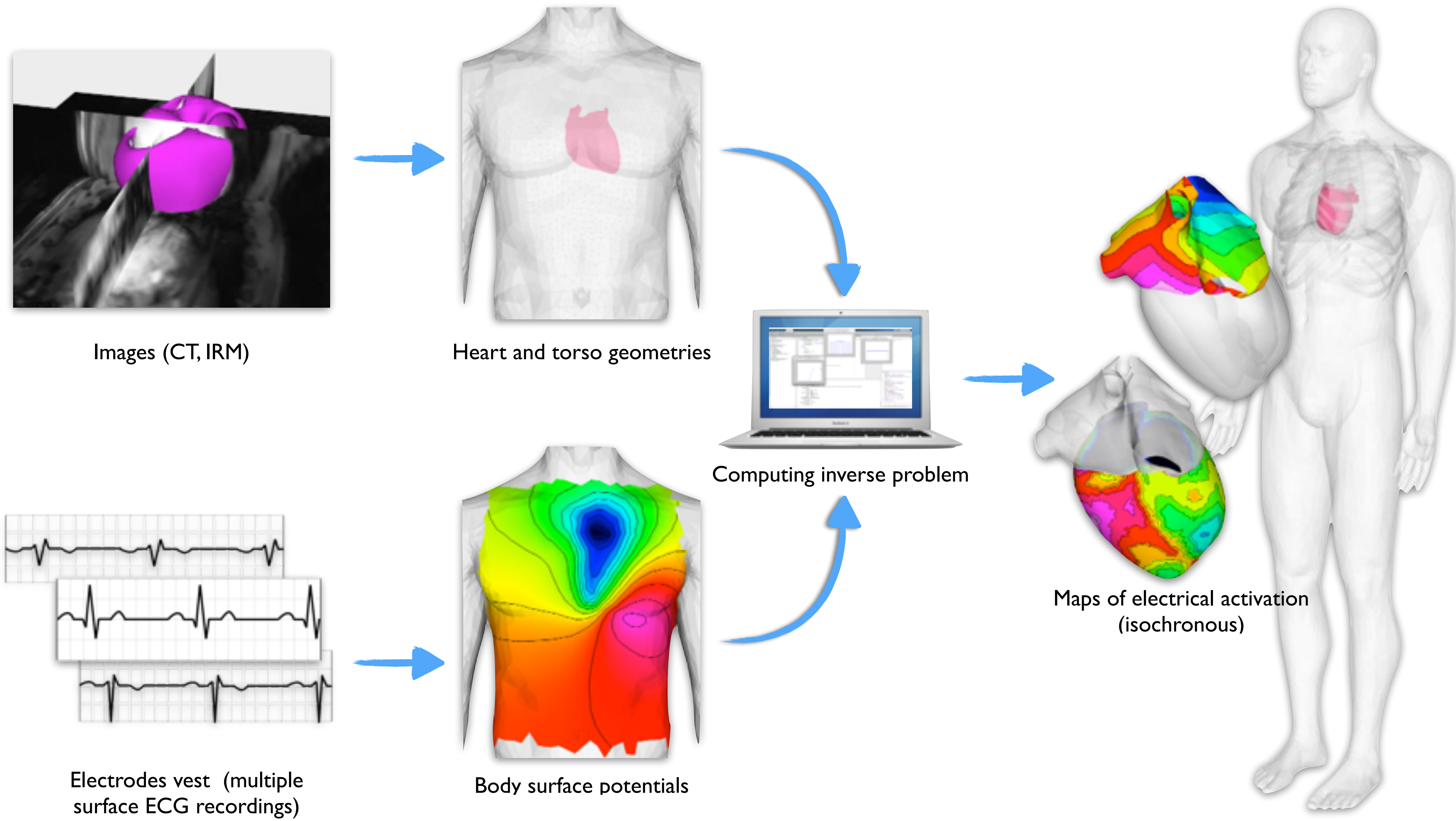
- Reaction diffusion system

$$\begin{cases} \partial_t u - \underline{\nabla} \cdot (D\, \underline{\nabla} u) & = & f(u), & \text{in } \mathcal{B} \times (0,T), \\ (D\, \underline{\nabla} u) \cdot \underline{n} & = & 0, & \text{on } \partial\mathcal{B} \times (0,T), \\ u(\underline{x}, 0) & = & u_0(\underline{x}), & \text{in } \mathcal{B}. \end{cases}$$

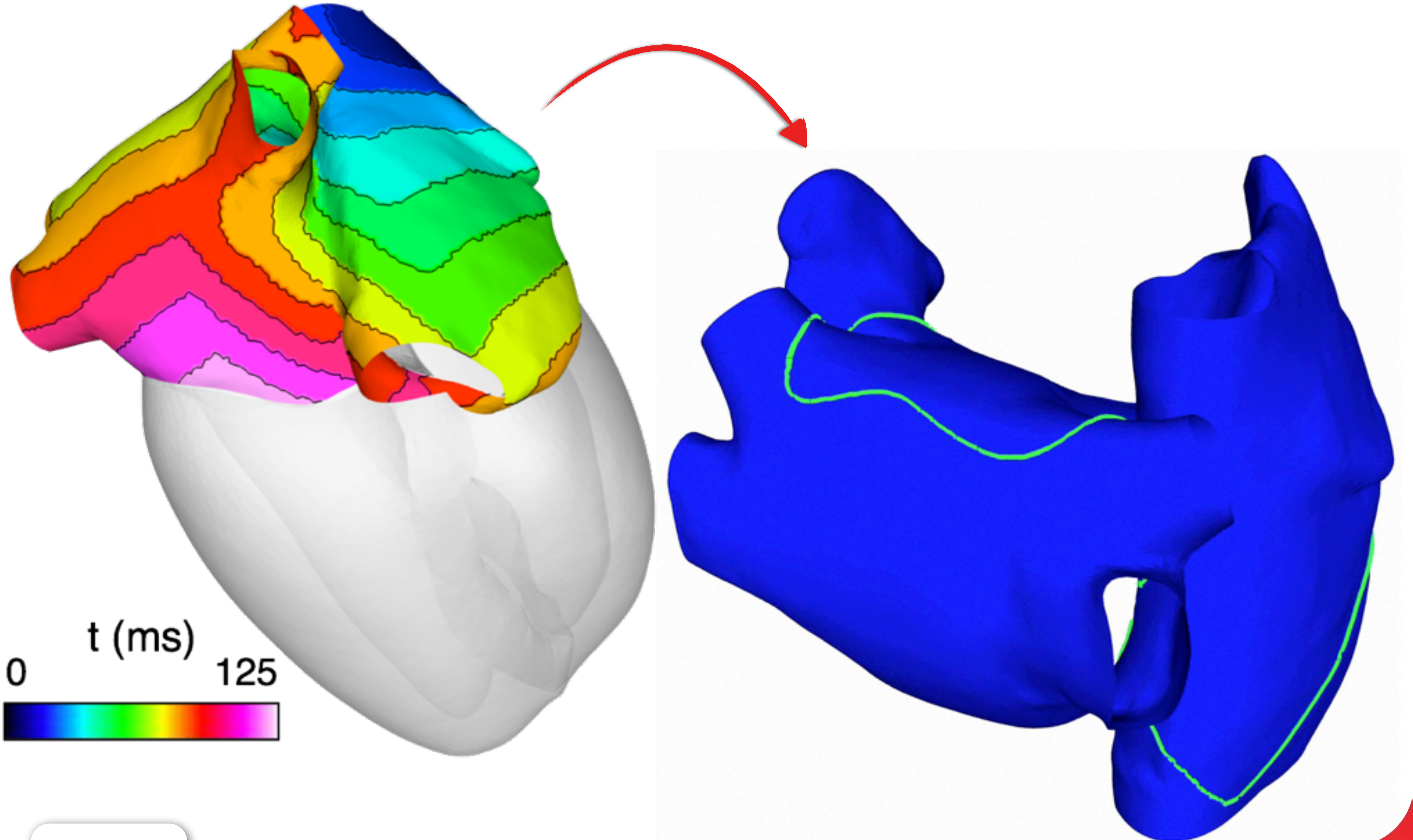


- Other application fields: Tumor growth, wildfire propagation

Electrocardiographic Imaging



From activation map to front propagation



Motivation

Traveling wave propagation phenomena modelled by reaction-diffusion systems to be compared with front propagation observations.

Other examples: Wild Fire propagation, Tumour growth, Chemotaxis

Objective: Data assimilation, namely state and parameter estimation, using observer

Outline

- Introduction to observer theory
- Observer for reaction-diffusion system based on front observations
  - Formulation
  - Mathematical analysis
  - Discretization & Numerical illustrations
  - Adaptive observers
  - Perspectives

Introduction to observer theory

Observer: definition

- Class of dynamical systems

$$\begin{cases} \dot{y}_{|\zeta,\omega}(t) = A(y_{|\zeta,\omega}(t), t) + B(t)\omega(t), & t \in \mathbb{R}^+ \\ y_{|\zeta,\omega}(0) = y_\diamond + \zeta \end{cases}$$

- Trajectory $\{y_\boxdot(t), t \in \mathbb{R}^+\}$
- Observations

$$z(t) = C(y_\boxdot(t), t) + \chi(t) \qquad (\forall y, D(y,t) = z(t) - C(y,t))$$

- Observer or (state) estimator

$$\begin{cases} \dot{\hat{y}}(t) = A(\hat{y}, t) + G(\hat{y}, t) D(z, C(\hat{y}, t)), & t \in \mathbb{R}^+ \\ \hat{y}(0) = y_\diamond \end{cases}$$

- *Objective* here $\tilde{y}(t) = y_\boxdot(t) - \hat{y}(t)$

$$\forall t, \omega(t) = 0, \chi(t) = 0 \Rightarrow \tilde{y}(t) \xrightarrow[t \to +\infty]{} 0$$

or at least $\exists$ contracting map $\beta \mid \|\tilde{y}(t)\| \leq \beta(\|\tilde{y}(s)\|, t-s), \quad \forall t \geq s.$

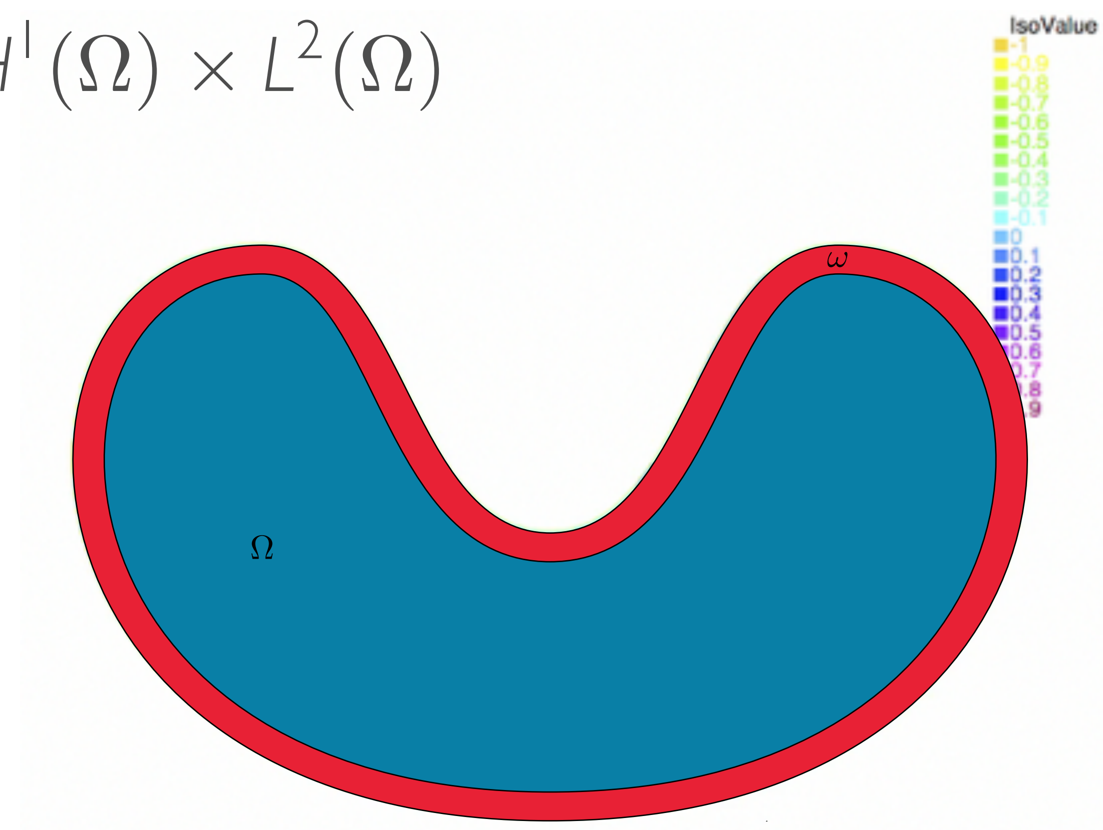
Observer: the wave example

- Wave equation $y(t) = (u(t) \quad \dot{u}(t))^\intercal \in H^1(\Omega) \times L^2(\Omega)$

$$\begin{cases} \ddot{u} - c^2 \Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

$$u(0) = u_\diamond + \zeta_u, \quad \dot{u}(0) = v_\diamond + \zeta_v$$

- Observations $z = \dot{u}_\omega$



- Observer

$$\begin{cases} \ddot{\hat{u}} - c^2 \Delta \hat{u} = f + \gamma(z - \chi_\omega \dot{\hat{u}}), & \text{in } \Omega \\ \hat{u} = g, & \text{on } \partial\Omega \end{cases}$$

$$\hat{u}(0) = u_\diamond, \quad \dot{\hat{u}}(0) = v_\diamond$$

- Error

$$\begin{cases} \ddot{\tilde{u}} + \chi_\omega \dot{\tilde{u}} - c^2 \Delta \tilde{u} = 0 & \text{in } \Omega \\ \tilde{u} = 0, & \text{on } \partial\Omega \end{cases}$$

$$\tilde{u}(0) = \zeta_u, \quad \dot{\tilde{u}}(0) = \zeta_v$$

Theorem. The observer converges to the target trajectory.
When the GCC is satisfied the convergence is exponential.

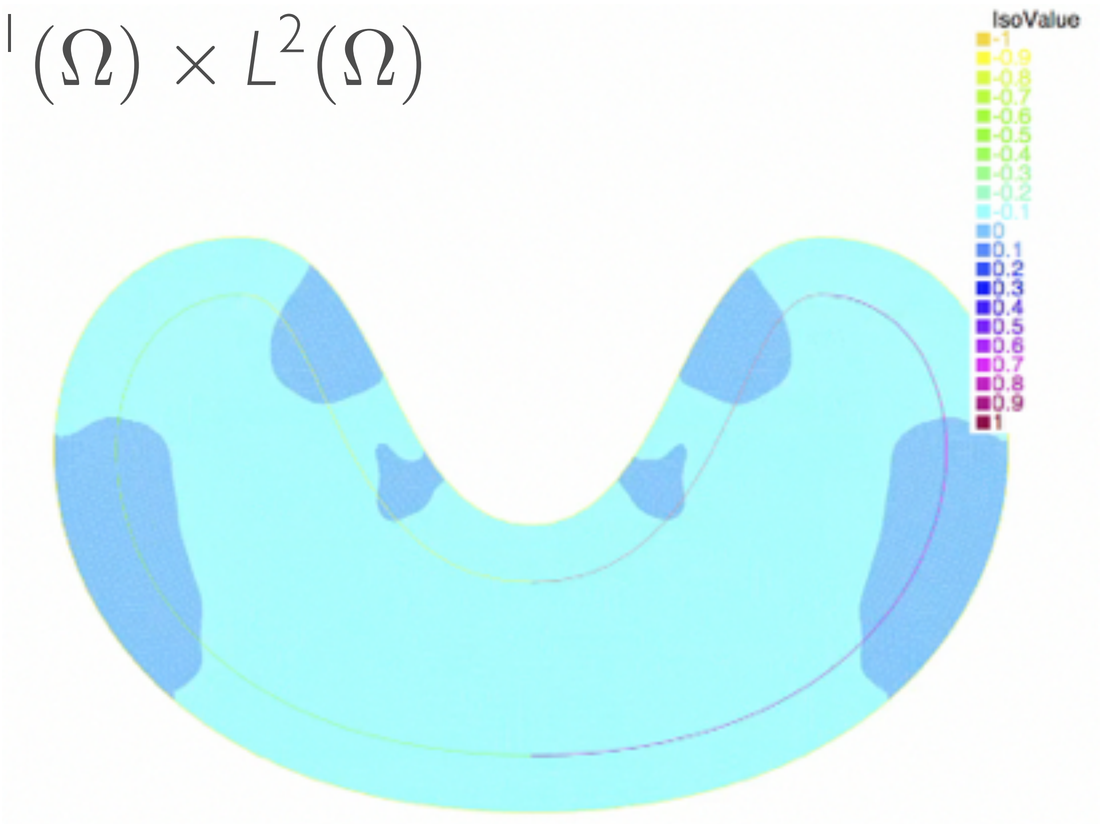
Observer: the wave example

- Wave equation $y(t) = (u(t) \quad \dot{u}(t))^\intercal \in H^1(\Omega) \times L^2(\Omega)$

$$\begin{cases} \ddot{u} - c^2 \Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

$$u(0) = u_\diamond + \zeta_u, \quad \dot{u}(0) = v_\diamond + \zeta_v$$

- Observations $z = \dot{u}_\omega$



- Observer

$$\begin{cases} \ddot{\hat{u}} - c^2 \Delta \hat{u} = f + \gamma(z - \chi_\omega \dot{\hat{u}}), & \text{in } \Omega \\ \hat{u} = g, & \text{on } \partial\Omega \end{cases}$$

$$\hat{u}(0) = u_\diamond, \quad \dot{\hat{u}}(0) = v_\diamond$$

- Error

$$\begin{cases} \ddot{\tilde{u}} + \chi_\omega \dot{\tilde{u}} - c^2 \Delta \tilde{u} = 0 & \text{in } \Omega \\ \tilde{u} = 0, & \text{on } \partial\Omega \end{cases}$$

$$\tilde{u}(0) = \zeta_u, \quad \dot{\tilde{u}}(0) = \zeta_v$$

Theorem. The observer converges to the target trajectory.
When the GCC is satisfied the convergence is exponential.

Observer: the Saint-Venant system example

- Saint-Venant system

$$\begin{cases} \partial_t h + \partial_x hu = 0, & \text{in } \Omega \\ \partial_t(hu) + \partial_x\left(hu^2 + \frac{gh^2}{2}\right) = 0, & \text{in } \Omega \end{cases}$$

$$h(0) = h_\diamond + \zeta_h, \quad u(0) = u_\diamond + \zeta_u$$

- Observations $z = h$

- Observer

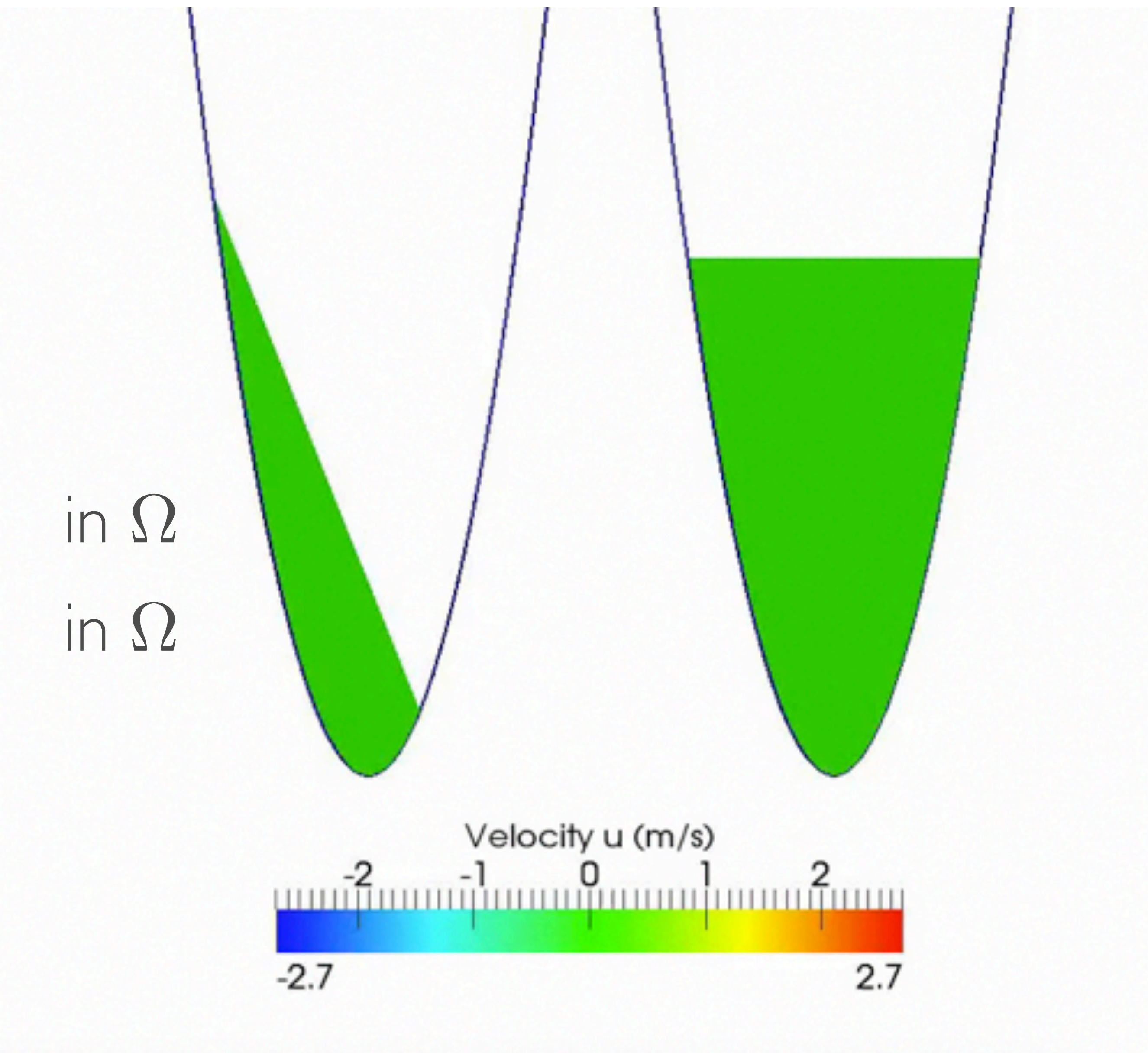
$$\begin{cases} \partial_t \hat{h} + \partial_x \hat{h}\hat{u} = \gamma(z - \hat{h}), & \text{in } \Omega \\ \partial_t(\hat{h}\hat{u}) + \partial_x(\hat{h}\hat{u}^2 + \frac{g\hat{h}^2}{2}) = \gamma\hat{u}(z - \hat{h}), & \text{in } \Omega \end{cases}$$

$$\hat{h}(0) = h_\diamond, \quad \hat{u}(0) = u_\diamond$$

- Error (non-linear)

$$\partial_t \hat{\mathcal{E}} + \partial_x\left(\hat{u}(\hat{\mathcal{E}} + g\frac{\hat{h}^2}{2})\right) \leq \gamma(\mathcal{E} - \hat{\mathcal{E}})$$

or use kinetic representation



A.C. Boulanger, P. Moireau, B. Perthame, J. Sainte-Marie Data Assimilation for hyperbolic conservation laws — A Luenberger observer approach based on a kinetic description. Comm Math Sci **2015**

Observer: the Saint-Venant system example

- Saint-Venant system

$$\begin{cases} \partial_t h + \partial_x hu = 0, & \text{in } \Omega \\ \partial_t (hu) + \partial_x \left(hu^2 + \frac{gh^2}{2}\right) = 0, & \text{in } \Omega \end{cases}$$

$$h(0) = h_\diamond + \zeta_h, \quad u(0) = u_\diamond + \zeta_u$$

- Observations $z = h$

- Observer

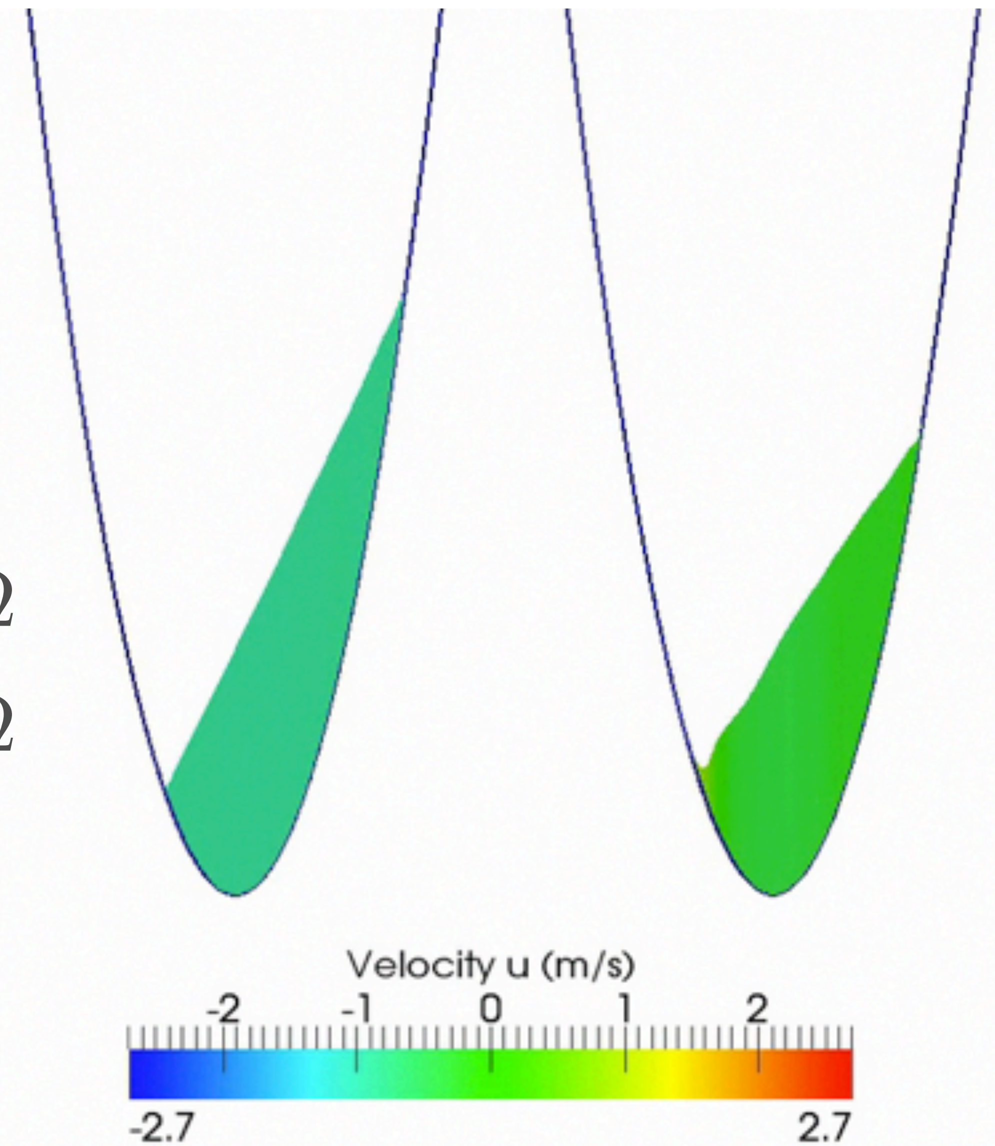
$$\begin{cases} \partial_t \hat{h} + \partial_x \hat{h}\hat{u} = \gamma(z - \hat{h}), & \text{in } \Omega \\ \partial_t (\hat{h}\hat{u}) + \partial_x (\hat{h}\hat{u}^2 + \frac{g\hat{h}^2}{2}) = \gamma\hat{u}(z - \hat{h}), & \text{in } \Omega \end{cases}$$

$$\hat{h}(0) = h_\diamond, \quad \hat{u}(0) = u_\diamond$$

- Error (non-linear)

$$\partial_t \hat{\mathcal{E}} + \partial_x \left(\hat{u}(\hat{\mathcal{E}} + g\frac{\hat{h}^2}{2})\right) \leq \gamma(\mathcal{E} - \hat{\mathcal{E}})$$

or use kinetic representation



A.C. Boulanger, P. Moireau, B. Perthame, J. Sainte-Marie Data Assimilation for hyperbolic conservation laws — A Luenberger observer approach based on a kinetic description. Comm Math Sci **2015**

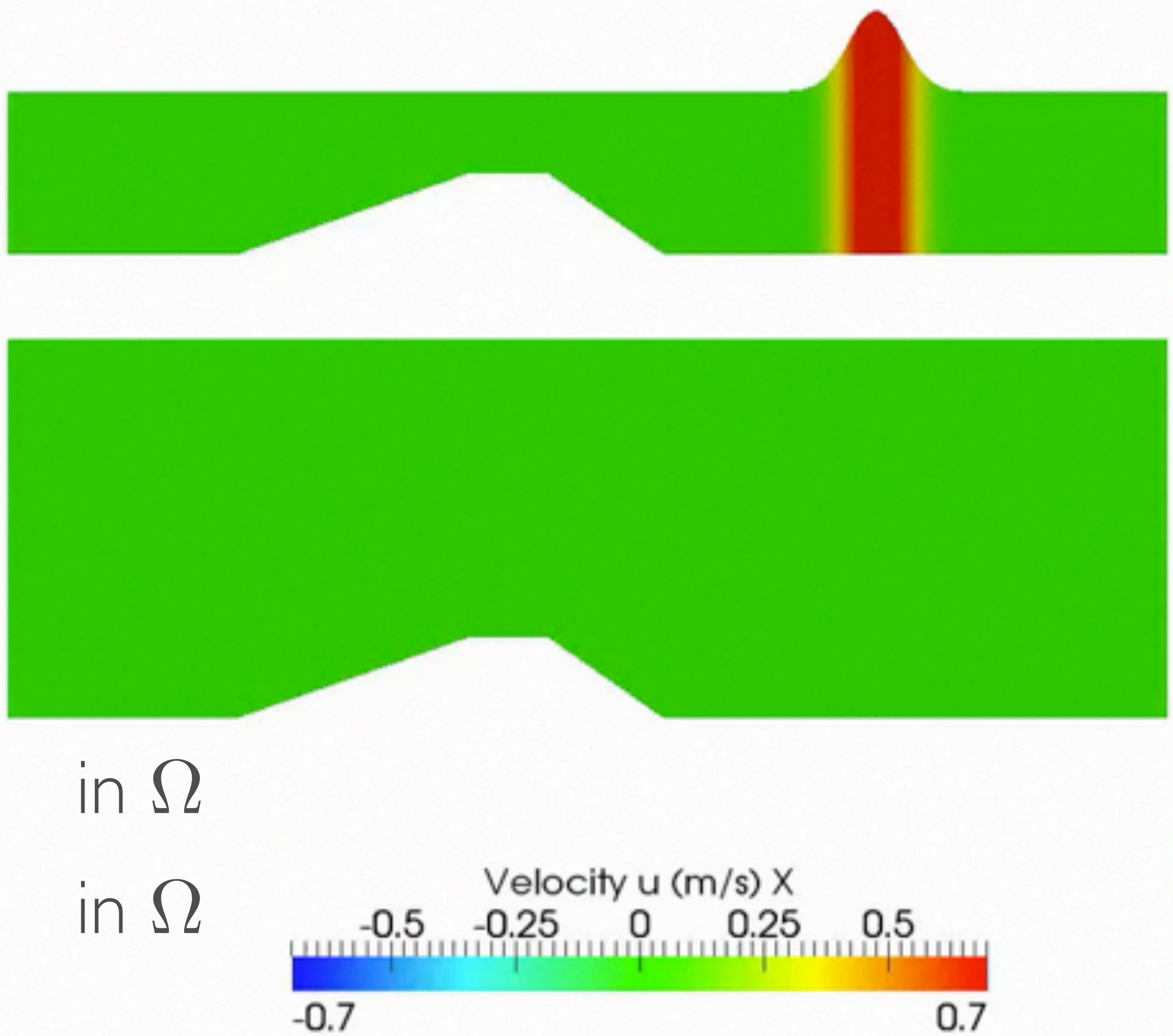
Observer: the Saint-Venant system example

- Saint-Venant system

$$\begin{cases} \partial_t h + \partial_x hu = 0, & \text{in } \Omega \\ \partial_t(hu) + \partial_x\left(hu^2 + \frac{gh^2}{2}\right) = 0, & \text{in } \Omega \end{cases}$$

$$h(0) = h_\diamond + \zeta_h, \quad u(0) = u_\diamond + \zeta_u$$

- Observations $z = h$



- Observer

$$\begin{cases} \partial_t \hat{h} + \partial_x \hat{h}\hat{u} = \gamma(z - \hat{h}), & \text{in } \Omega \\ \partial_t(\hat{h}\hat{u}) + \partial_x(\hat{h}\hat{u}^2 + \frac{g\hat{h}^2}{2}) = \gamma\hat{u}(z - \hat{h}), & \text{in } \Omega \end{cases}$$

$$\hat{h}(0) = h_\diamond, \quad \hat{u}(0) = u_\diamond$$

- Error (non-linear)

$$\partial_t \hat{\mathcal{E}} + \partial_x\left(\hat{u}(\hat{\mathcal{E}} + g\frac{\hat{h}^2}{2})\right) \leq \gamma(\mathcal{E} - \hat{\mathcal{E}})$$

or use kinetic representation

A.C. Boulanger, P. Moireau, B. Perthame, J. Sainte-Marie Data Assimilation for hyperbolic conservation laws — A Luenberger observer approach based on a kinetic description. Comm Math Sci **2015**

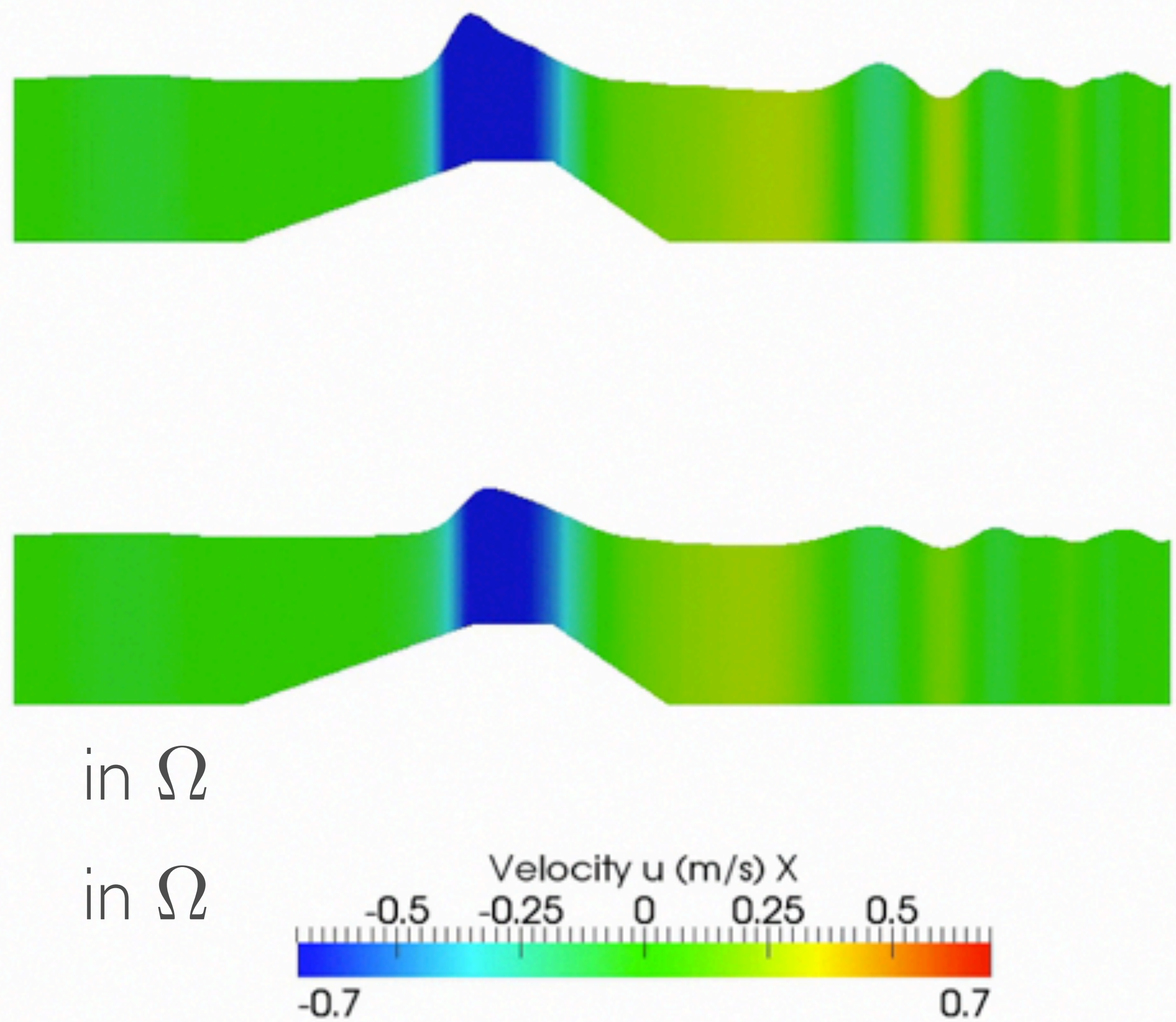
Observer: the Saint-Venant system example

- Saint-Venant system

$$\begin{cases} \partial_t h + \partial_x hu = 0, & \text{in } \Omega \\ \partial_t (hu) + \partial_x \left(hu^2 + \frac{gh^2}{2} \right) = 0, & \text{in } \Omega \end{cases}$$

$$h(0) = h_\diamond + \zeta_h, \quad u(0) = u_\diamond + \zeta_u$$

- Observations $z = h$



- Observer

$$\begin{cases} \partial_t \hat{h} + \partial_x \hat{h}\hat{u} = \gamma(z - \hat{h}), & \text{in } \Omega \\ \partial_t (\hat{h}\hat{u}) + \partial_x (\hat{h}\hat{u}^2 + \frac{g\hat{h}^2}{2}) = \gamma \hat{u}(z - \hat{h}), & \text{in } \Omega \end{cases}$$

$$\hat{h}(0) = h_\diamond, \quad \hat{u}(0) = u_\diamond$$

- Error (non-linear)

$$\partial_t \hat{\mathcal{E}} + \partial_x \left(\hat{u}(\hat{\mathcal{E}} + g\frac{\hat{h}^2}{2}) \right) \leq \gamma(\mathcal{E} - \hat{\mathcal{E}})$$

or use kinetic representation

A.C. Boulanger, P. Moireau, B. Perthame, J. Sainte-Marie Data Assimilation for hyperbolic conservation laws — A Luenberger observer approach based on a kinetic description. Comm Math Sci **2015**

Observer for reaction-diffusion system based on front observations

Part 1: Observer formulation

The "depolarization" front

- Reaction diffusion model

$$\begin{cases} \partial_t u - \underline{\nabla} \cdot (D\,\underline{\nabla} u) = kf(u), & \text{in } \mathcal{B} \times (0,T), \\ (D\,\underline{\nabla} u) \cdot \underline{n} = 0, & \text{on } \partial\mathcal{B} \times (0,T), \\ u(\underline{x}, 0) = u_0(\underline{x}), & \text{in } \mathcal{B}. \end{cases}$$

- We define c_{th} as the depolarisation constant, namely the depolarisation area is given by $\Omega_u(t) = \{\underline{x} \in \mathcal{B} \,|\, u(\underline{x}, t) > c_{th}\}$

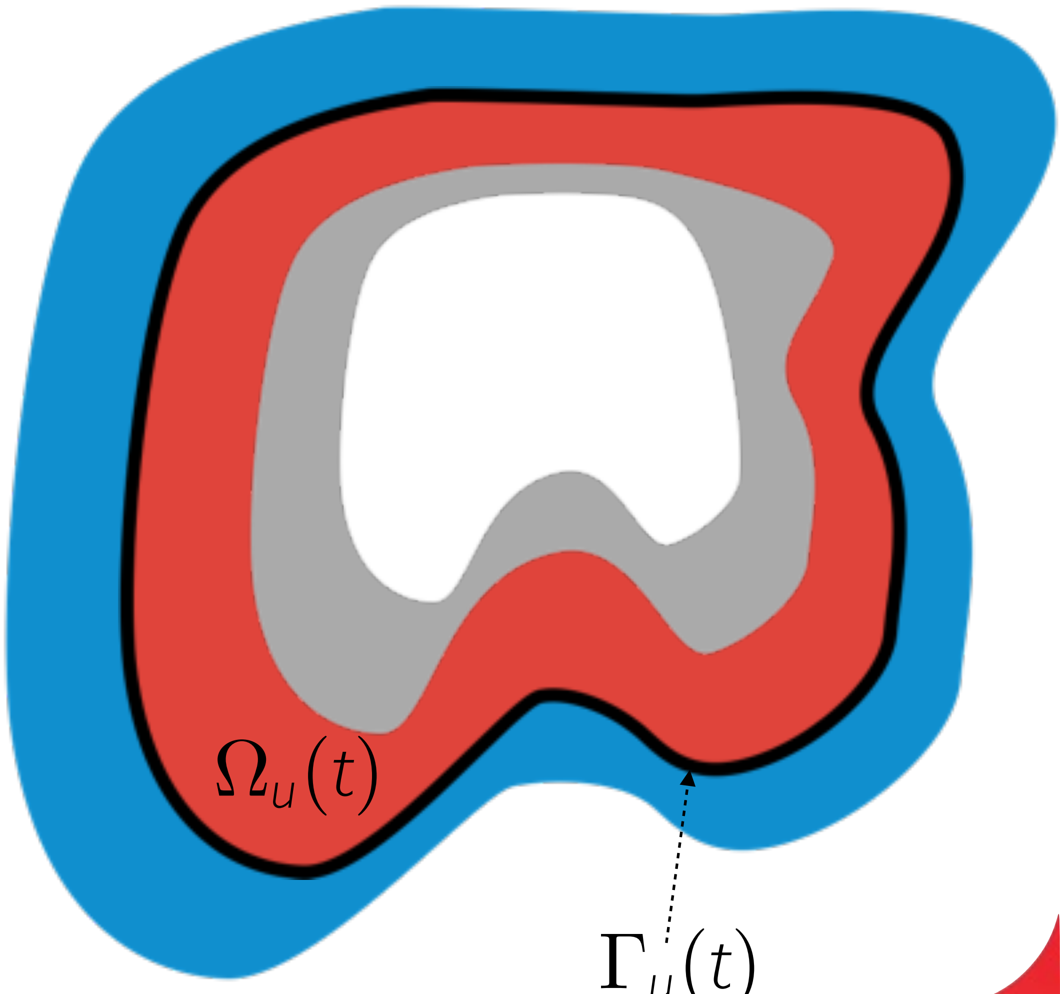
- We can represents by a level-set $\phi_u(t)$

$$\Omega_u(t) = \{\underline{x} \in \mathcal{B} \,|\, \phi_u(t) > 0\}$$
$$\Gamma_u(t) = \{\underline{x} \in \mathcal{B} \,|\, \phi_u(t) = 0\}$$

- And this level-set is solution of an Eikonal model

$$\dot{\phi}_u = |\underline{\nabla}\phi_u| \left(D\,\underline{\nabla} \cdot \left(\frac{\underline{\nabla}\phi_u}{|\underline{\nabla}\phi_u|} \right) + \sqrt{Dk}c_0 \right),$$

in $\mathcal{B} \times (0,T)$



Keener, An eikonal-curvature equation for action potential propagation in myocardium. 1991.

Eikonal equation from asymptotic analysis

- We define a time-dependent coordinate system (ξ_1, ξ_2, ξ_3) designed to follow the front over time

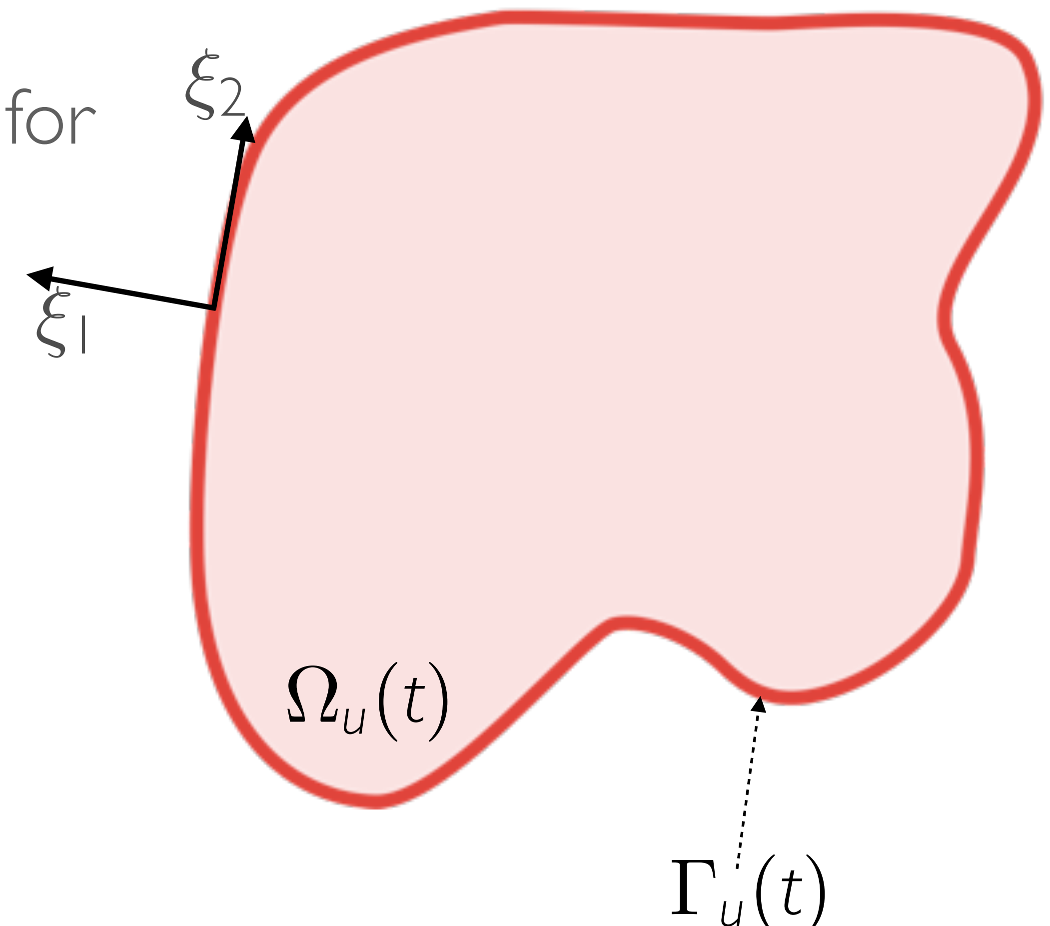
$$\partial_t u = \sigma a_{il} a_{ik} \frac{\partial^2 u}{\partial \xi_l \partial \xi_k} + \sigma \frac{\partial a_{il}}{\partial x_i} \frac{\partial u}{\partial \xi_l} + a_{jk} (\partial_t x_j) \frac{\partial u}{\partial \xi_k} + kf, \qquad a_{ij} = \frac{\partial \xi_i}{\partial x_j}$$

- The front is sharp $O(\epsilon) \Rightarrow a_{i1} = O(\epsilon^{-1})$,

$$\sigma |\underline{a}|^2 \frac{\partial^2 u}{\partial \xi_1^2} + (\sigma \underline{\nabla}_{\underline{x}} \cdot \underline{a} + \underline{a} \cdot \partial_t \underline{x}) \frac{\partial u}{\partial \xi_1} + kf(u) = 0.$$

which is a second order ODE where we seek for decreasing solutions $u'' + c_0 u' + f(u) = 0$

$$\left| \begin{aligned} & kc_0 = \sigma \underline{\nabla}_{\underline{x}} \cdot \underline{a} + \underline{a} \cdot \partial_t \underline{x} \\ & \underline{a} = -\sqrt{\frac{k}{\sigma}} \frac{\underline{\nabla}_{\underline{x}} \phi_u}{|\underline{\nabla}_{\underline{x}} \phi_u|}. \\ & \underline{\nabla} \phi_u \cdot \partial_t \underline{x} + \partial_t \phi_u = 0 \end{aligned} \right.$$



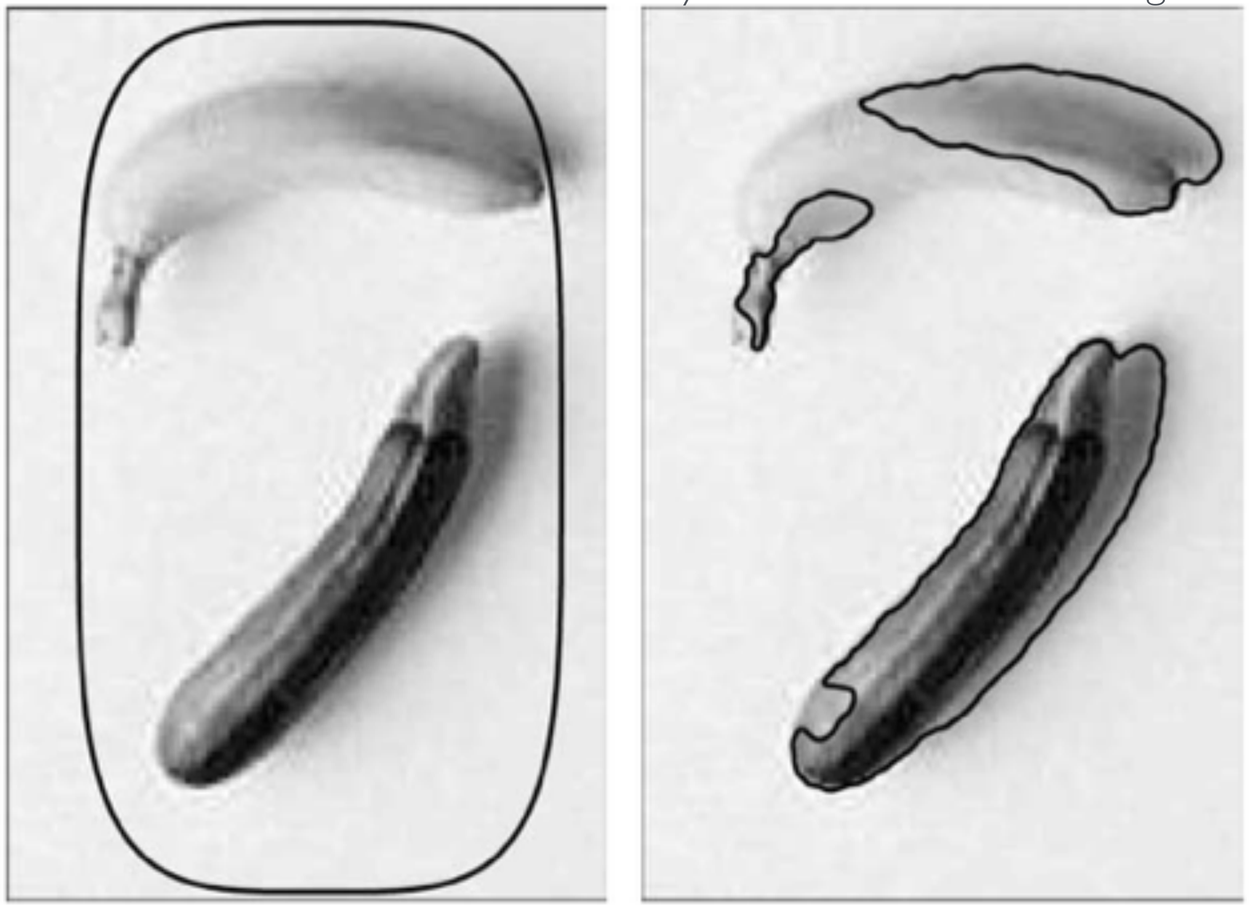
Observer for the Eikonal equation

- In image processing theory (in segmentation) there are Eikonal models associated with data-fitting terms.
- For instance using the Mumford-Shah functional or its Chan-Vese version

$$\mathcal{J}(\phi) = \alpha|\Omega_\phi| + \beta|\Gamma_\phi| + \gamma \int_{\Omega_\phi} (z - C_1(\phi))^2 dx + \gamma \int_{\mathcal{B}\setminus\Omega_\phi} (z - C_2(\phi))^2$$

$$\left| \begin{array}{l} C_1 = \dfrac{1}{|\Omega_\phi|} \displaystyle\int_{\Omega_\phi} z dx \\ C_2 = \dfrac{1}{|\mathcal{B} \setminus \Omega_\phi|} \displaystyle\int_{\mathcal{B}\setminus\Omega_\phi} z dx \end{array} \right.$$

Courtesy, Hintermüller, M. and Ring, W.



Observer for the Eikonal equation

- In image processing theory (in segmentation) there are Eikonal models associated with data-fitting terms. For instance the Mumford-Shah functional or its Chan-Vese version

$$\mathcal{J}(\phi) = \alpha|\Omega_\phi| + \beta|\Gamma_\phi| + \gamma \int_{\Omega_\phi} (z - C_1(\phi))^2 dx + \gamma \int_{\mathcal{B}\setminus\Omega_\phi} (z - C_2(\phi))^2$$

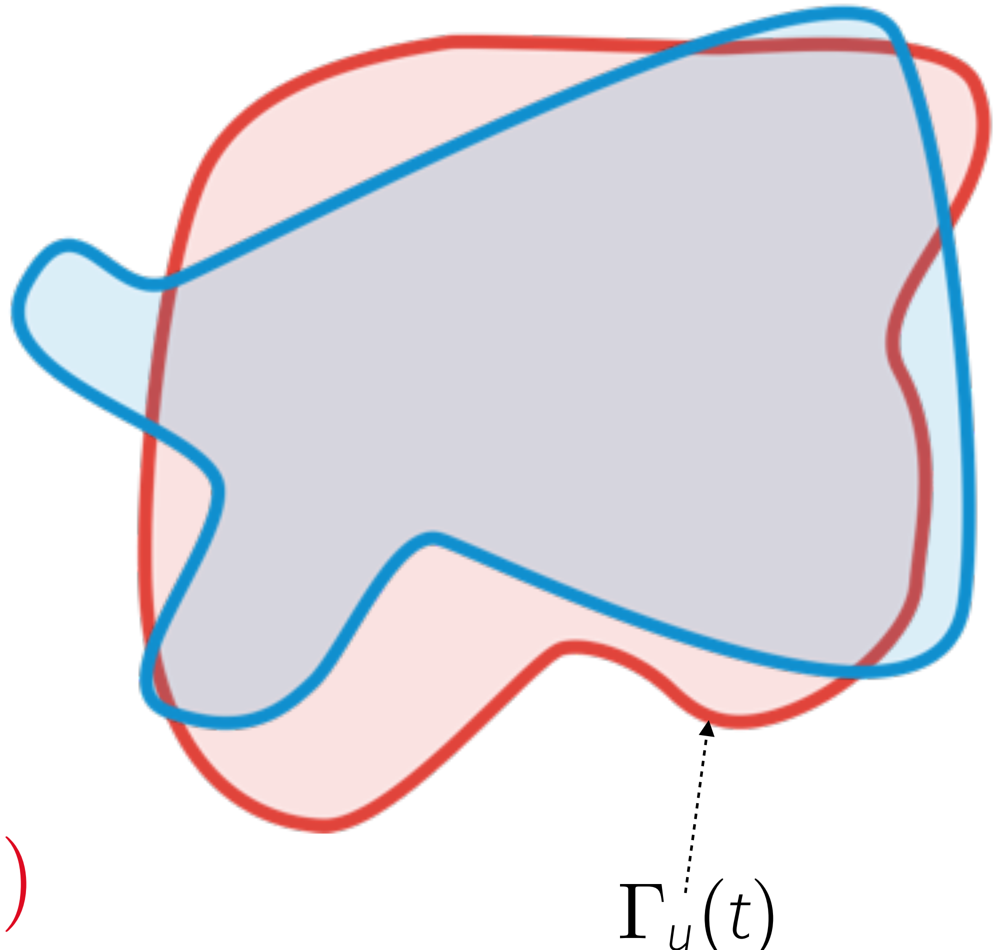
where
$$\left| \begin{aligned} C_1 &= \frac{1}{|\Omega_\phi|} \int_{\Omega_\phi} z dx \\ C_2 &= \frac{1}{|\mathcal{B} \setminus \Omega_\phi|} \int_{\mathcal{B}\setminus\Omega_\phi} z dx \end{aligned} \right.$$

- Minimization by gradient projection method

$$\dot{\phi}_u = -\Upsilon \nabla \mathcal{J}(\phi_u)$$

- Eikonal observer

$$\dot{\phi} = -\alpha\delta(\phi) + \beta\delta(\phi)\underline{\nabla} \cdot \left(\frac{\underline{\nabla}\phi}{|\underline{\nabla}\phi|}\right) + \gamma\delta(\phi)\left(-(z - C_1)^2 + (z - C_2)^2\right)$$



Observer for the Reaction-diffusion model

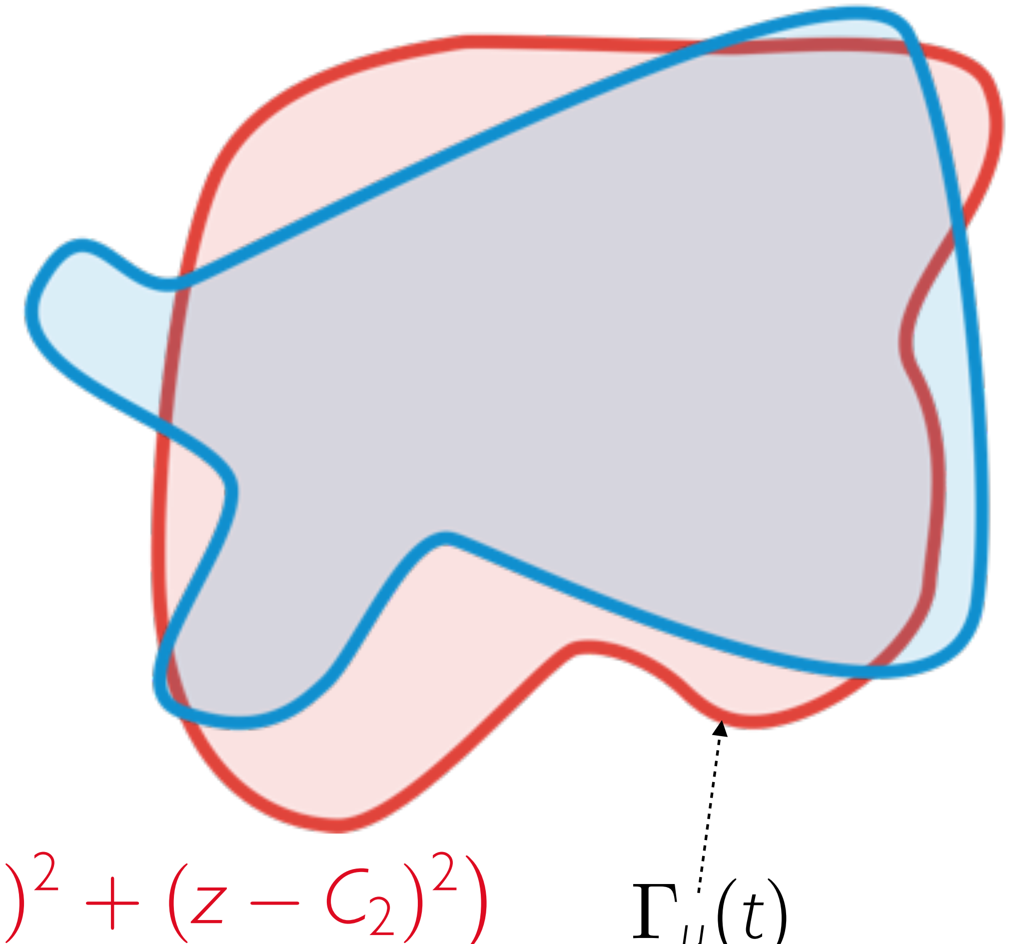
- Find a correction on the reaction-diffusion system so that the asymptotic analysis leads to the Eikonal observer

$$\dot{\hat{u}} = \underline{\nabla}\cdot(D\underline{\nabla\hat{u}}) + kf(\hat{u})$$
$$+ \gamma\delta(\phi_{\hat{u}})\frac{1}{|\underline{\nabla}\phi_{\hat{u}}|}\frac{\underline{\nabla}\phi_{\hat{u}}}{|\underline{\nabla}\phi_{\hat{u}}|}\cdot\underline{\nabla}u\left(-(z-C_1(\Omega_{\hat{u}}))^2+(z-C_2(\Omega_{\hat{u}}))^2\right)$$

$$\gamma(|\underline{\nabla}\hat{\phi}_{\hat{u}}|) = \frac{\gamma}{|\underline{\nabla}\hat{\phi}_{\hat{u}}|}$$

- Eikonal observer

$$\dot{\hat{\phi}} = \sqrt{kDc_0}|\underline{\nabla}\hat{\phi}_{\hat{u}}| + |\underline{\nabla}\hat{\phi}_{\hat{u}}|\underline{\nabla}\cdot\left(\frac{\underline{\nabla}\hat{\phi}_{\hat{u}}}{|\underline{\nabla}\hat{\phi}_{\hat{u}}|}\right)$$
$$+ \gamma(|\underline{\nabla}\hat{\phi}_{\hat{u}}|)\delta(\hat{\phi}_{\hat{u}})\left(-(z-C_1)^2+(z-C_2)^2\right)$$



Observer for the Reaction-diffusion model

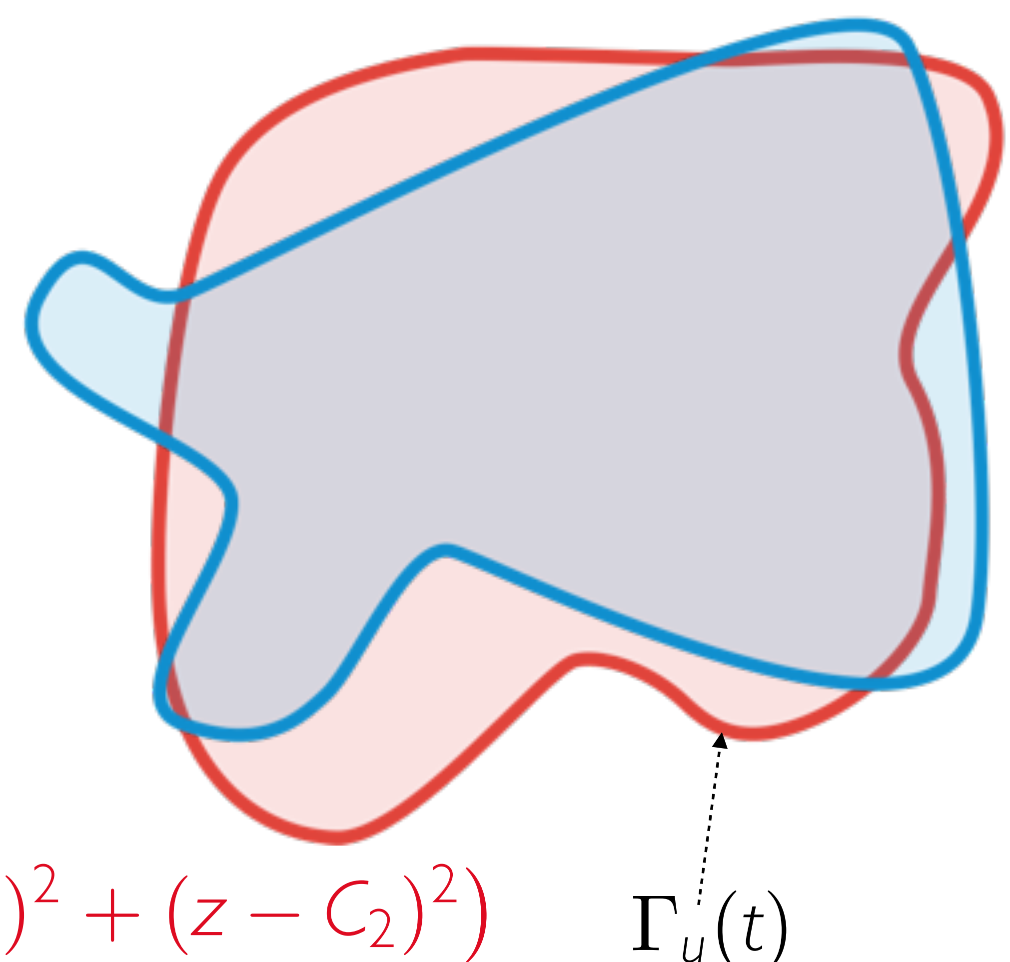
- Find a correction on the reaction-diffusion system so that the asymptotic analysis leads to the Eikonal observer

$$\dot{\hat{u}} = \underline{\nabla} \cdot (D\underline{\nabla \hat{u}}) + kf(\hat{u}) + \gamma\delta(\hat{u} - c_{th})\left(-(z - C_1(\Omega_{\hat{u}}))^2 + (z - C_2(\Omega_{\hat{u}}))^2\right)$$

$$\phi_{\hat{u}} = \hat{u} - c_{th}$$

- Eikonal observer

$$\dot{\hat{\phi}} = \sqrt{kDc_0}|\underline{\nabla}\hat{\phi}_{\hat{u}}| + |\underline{\nabla}\hat{\phi}_{\hat{u}}|\underline{\nabla} \cdot \left(\frac{\underline{\nabla}\hat{\phi}_{\hat{u}}}{|\underline{\nabla}\hat{\phi}_{\hat{u}}|}\right) + \gamma(|\underline{\nabla}\hat{\phi}_{\hat{u}}|)\delta(\hat{\phi}_{\hat{u}})\left(-(z - C_1)^2 + (z - C_2)^2\right)$$



Observer for reaction-diffusion system based on front observations

Part II: Elements of analysis

Error Analysis (I)

- System

$$\begin{cases} \partial_t u - \underline{\nabla} \cdot (D\,\underline{\nabla} u) = kf(u), & \mathcal{B} \times (0,T), \\ (D\,\underline{\nabla} u) \cdot \underline{n} = 0, & \partial\mathcal{B} \times (0,T), \\ u(\underline{x},0) = u_\diamond(\underline{x}) + \zeta(\underline{x}), & \mathcal{B}. \end{cases}$$

- Observer

$$\begin{cases} \partial_t \hat{u} - \underline{\nabla} \cdot (D\,\underline{\nabla} \hat{u}) = kf(\hat{u}) \\ \quad + \gamma\hat{\delta}(\Gamma_{\hat{u}}, \underline{x}) \dfrac{1}{|\underline{\nabla}\hat{u}|} \Big(-\big(z - C_1(\Omega_{\hat{u}})\big)^2 + \big(z - C_2(\Omega_{\hat{u}})\big)^2 \Big), & \mathcal{B} \times (0,T), \\ (D\,\underline{\nabla} \hat{u}) \cdot \underline{n} = 0, & \partial\mathcal{B} \times (0,T), \\ \hat{u}(\underline{x},0) = u_\diamond(\underline{x}), & \mathcal{B}. \end{cases}$$

- Error ?

$$\begin{cases} \partial_t \tilde{u} - \underline{\nabla} \cdot (D\,\underline{\nabla} \tilde{u}) = k(f(u) - f(\hat{u})) \\ \quad + \gamma\hat{\delta}(\Gamma_{\hat{u}}, \underline{x}) \dfrac{1}{|\underline{\nabla}\hat{u}|} \Big(\big(z - C_1(\Omega_{\hat{u}})\big)^2 - \big(z - C_2(\Omega_{\hat{u}})\big)^2 \Big), & \mathcal{B} \times (0,T), \\ (D\,\underline{\nabla} \tilde{u}) \cdot \underline{n} = 0, & \partial\mathcal{B} \times (0,T), \\ \tilde{u}(\underline{x},0) = \zeta(\underline{x}), & \mathcal{B}. \end{cases}$$

- or Linearized error around the target trajectory ?

Error Analysis (2)

- Error in variational form

$$\int_{\mathcal{B}} \partial_t \tilde{u}\, v\, dx = -\int_{\mathcal{B}} D\, \vec{\nabla}\tilde{u} \cdot \vec{\nabla} v\, dx \quad + \quad \int_{\mathcal{B}} k(f(u) - f(\hat{u}))\, v\, dx$$

$$\boxed{+\gamma \int_{\Gamma_{\hat{u}}} \frac{1}{|\vec{\nabla}\hat{u}|}\Big(\big(z - C_1(\Omega_{\hat{u}})\big)^2 - \big(z - C_2(\Omega_{\hat{u}})\big)^2\Big)\, v\, d\Gamma}$$

Stabilizing term ?

- Energy

$$\int_{\mathcal{B}} \partial_t \tilde{u}^2\, dx = -\int_{\mathcal{B}} D\, \vec{\nabla}\tilde{u} \cdot \vec{\nabla}\, \tilde{u}\, dx \quad + \quad \int_{\mathcal{B}} k(f(u) - f(\hat{u}))\, v\, dx$$

$$\boxed{+\gamma \int_{\Gamma_{\hat{u}}} \frac{1}{|\vec{\nabla}\hat{u}|}\Big(\big(z - C_1(\Omega_{\hat{u}})\big)^2 - \big(z - C_2(\Omega_{\hat{u}})\big)^2\Big)\, \tilde{u}\, d\Gamma}$$

Dissipative ?

- Dissipative property at least around the target trajectory

$$\mathcal{Q}_{\hat{u}}(\tilde{u}) = \int_{\Gamma_{\hat{u}}} \frac{\gamma}{|\vec{\nabla}\hat{u}|}\Big(\big(z - C_1(\Omega_{\hat{u}})\big)^2 - \big(z - C_2(\Omega_{\hat{u}})\big)^2\Big)\, \tilde{u}\, d\Gamma.$$

Shape optimisation based computations

Assuming that the contour is closed

$$
\begin{aligned}
(d_{\hat u}\mathcal{Q}_{\hat u}(\chi)\,;\psi) = &\int_{\Gamma_{\hat u}} \kappa\,\gamma(|\underline{\nabla}\hat u|)\Big(\big(z_u - C_1(\Omega_{\hat u})\big)^2 - \big(z_u - C_2(\Omega_{\hat u})\big)^2\Big)\,\chi\,\frac{\psi}{|\underline{\nabla}\hat u|}\,d\Gamma \\
&- \int_{\Gamma_{\hat u}} \gamma'(|\underline{\nabla}\hat u|)\Big(\big(z_u - C_1(\Omega_{\hat u})\big)^2 - \big(z_u - C_2(\Omega_{\hat u})\big)^2\Big)\,\chi\,\partial_n\psi\,d\Gamma \\
&- \int_{\Gamma_{\hat u}} \gamma'(|\underline{\nabla}\hat u|)\,\partial_n^2\hat u\Big(\big(z_u - C_1(\Omega_{\hat u})\big)^2 - \big(z_u - C_2(\Omega_{\hat u})\big)^2\Big)\,\chi\,\frac{\psi}{|\underline{\nabla}\hat u|}\,d\Gamma \\
&+2\int_{\Gamma_{\hat u}} \gamma(|\underline{\nabla}\hat u|)\,\partial_n z_u\big(C_2(\Omega_{\hat u}) - C_1(\Omega_{\hat u})\big)\,\chi\,\frac{\psi}{|\underline{\nabla}\hat u|}\,d\Gamma \\
&-2\frac{1}{|\Omega_{\hat u}|}\int_{\Gamma_{\hat u}} \big(z_u - C_1(\Omega_{\hat u})\big)\,\frac{\psi}{|\underline{\nabla}\hat u|}\,d\Gamma \times \int_{\Gamma_{\hat u}} \gamma(|\underline{\nabla}\hat u|)\big(z_u - C_1(\Omega_{\hat u})\big)\chi\,d\Gamma \\
&-2\frac{1}{|\mathcal{B}\setminus\overline{\Omega_{\hat u}}|}\int_{\Gamma_{\hat u}} \big(z_u - C_2(\Omega_{\hat u})\big)\,\frac{\psi}{|\underline{\nabla}\hat u|}\,d\Gamma \times \int_{\Gamma_{\hat u}} \gamma(|\underline{\nabla}\hat u|)\big(z_u - C_2(\Omega_{\hat u})\big)\chi\,d\Gamma \\
&+ \int_{\Gamma_{\hat u}} \gamma(|\underline{\nabla}\hat u|)\Big(\big(z_u - C_1(\Omega_{\hat u})\big)^2 - \big(z_u - C_2(\Omega_{\hat u})\big)^2\Big)\,\partial_n\chi\,\frac{\psi}{|\underline{\nabla}\hat u|}\,d\Gamma,
\end{aligned}
$$

$$
\begin{aligned}
(d_{\hat u}\mathcal{Q}_u(\tilde u)\,;\tilde u) = &\,2\big(C_2(\Omega_u) - C_1(\Omega_u)\big)\int_{\Gamma_u} \gamma(|\underline{\nabla}u|)\,\partial_n z_u\,\frac{\tilde u}{|\underline{\nabla}u|}\,\tilde u\,d\Gamma \\
&- 2\left(\frac{C_2(\Omega_u) - C_1(\Omega_u)}{2}\right)^2\left(\frac{1}{|\Omega_u|} + \frac{1}{|\mathcal{B}\setminus\overline{\Omega_u}|}\right)\int_{\Gamma_u}\frac{\tilde u}{|\underline{\nabla}u|}\,d\Gamma \times \int_{\Gamma_u}\gamma(|\underline{\nabla}u|)\,\tilde u\,d\Gamma
\end{aligned}
$$

Delfour. Zolésio. *Shapes and Geometry.* 2nde edition**. 2011.**

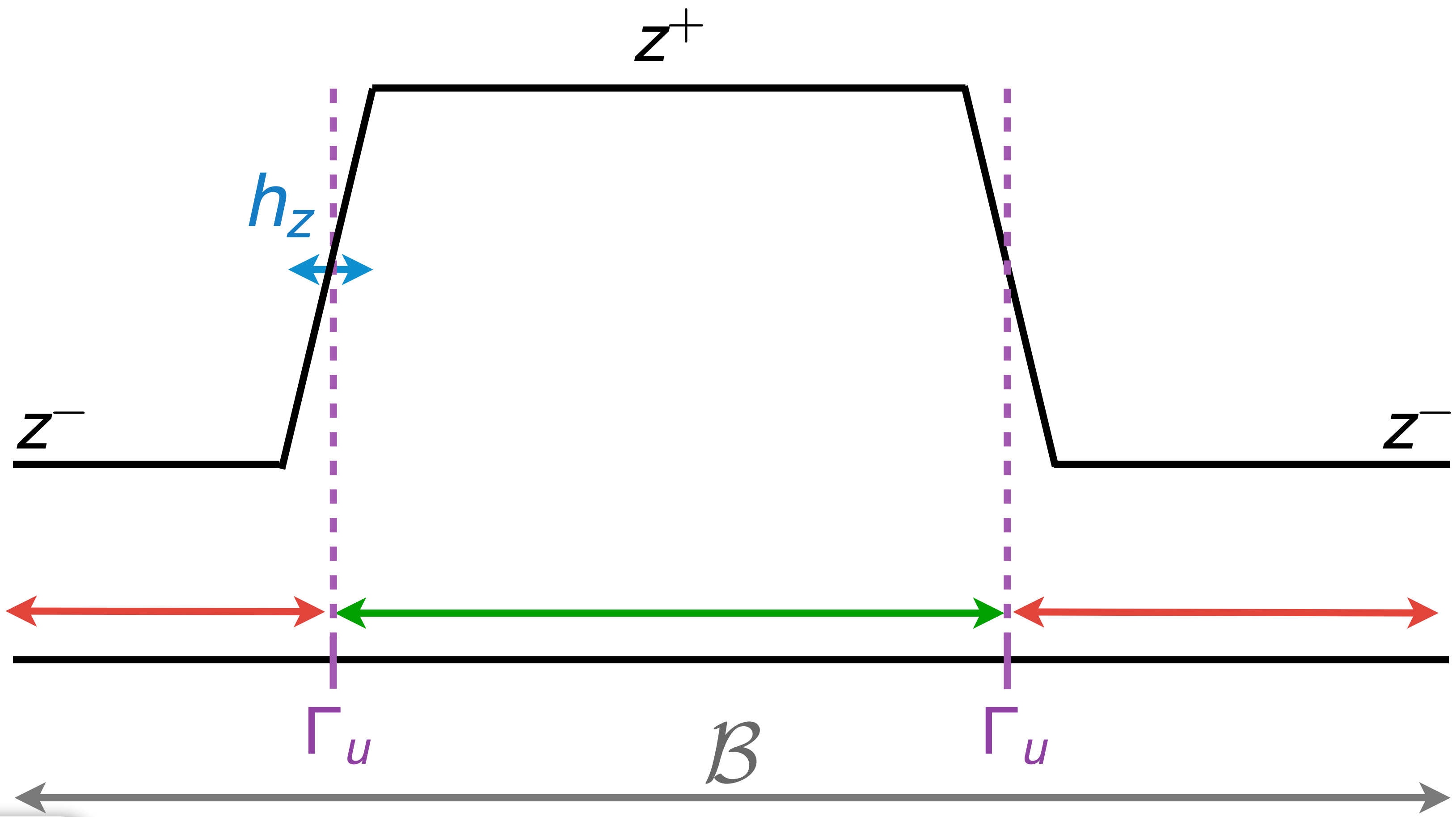
Hintermüller, Ring. *An Inexact Newton-CG-Type Active Contour Approach for the Minimization of the Mumford-Shah Functional*. **2004.**

Stabilization property

Proposition. If the condition

$$\frac{1}{h_z} \geq \frac{1}{4}\left(\frac{1}{|\Omega_u|} + \frac{1}{|\mathcal{B} \setminus \overline{\Omega_u}|}\right)\left(\int_{\Gamma_u} \gamma(|\nabla u|)|\vec{\nabla} u| \, d\Gamma \times \int_{\Gamma_u} \frac{1}{\gamma(|\nabla u|)|\vec{\nabla} u|} \, d\Gamma\right)^{\frac{1}{2}}$$

is satisfied then the observer effect is energy-decreasing in the linearized error



Object z defined from observations

Observer front

Stabilization property

Proposition. If the contrast condition

$$\frac{1}{h_z} \geq \frac{1}{4}\left(\frac{1}{|\Omega_u|} + \frac{1}{|\mathcal{B} \setminus \overline{\Omega_u}|}\right)|\Gamma_u|$$

is satisfied then the observer effect is energy-decreasing in the linearized error

- Proof

$$\left(\int_{\Gamma_u} \gamma(|\underline{\nabla} u|)|\underline{\nabla} u| \, d\Gamma \times \int_{\Gamma_u} \frac{1}{\gamma(|\underline{\nabla} u|)|\underline{\nabla} u|} \, d\Gamma\right)^{\frac{1}{2}} \geq \int_{\Gamma_u} \frac{\gamma(|\underline{\nabla} u|)|\underline{\nabla} u|}{\gamma(|\underline{\nabla} u|)|\underline{\nabla} u|} \, d\Gamma = |\Gamma_u|,$$

and equality with

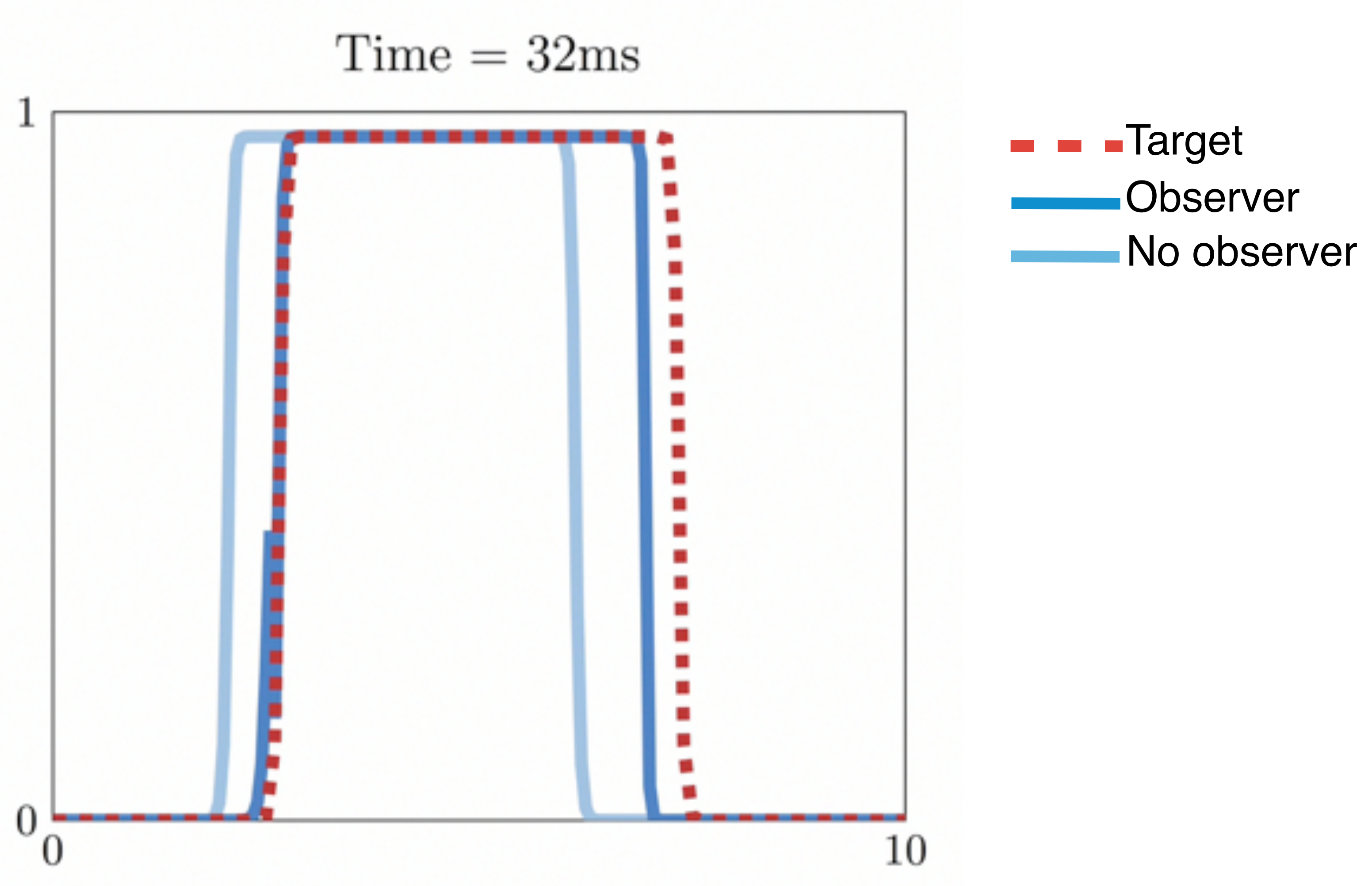
$$\gamma(|\underline{\nabla} u|) = \frac{1}{|\underline{\nabla} u|}.$$

Observer for reaction-diffusion system based on front observations

Part III: Numerical illustrations

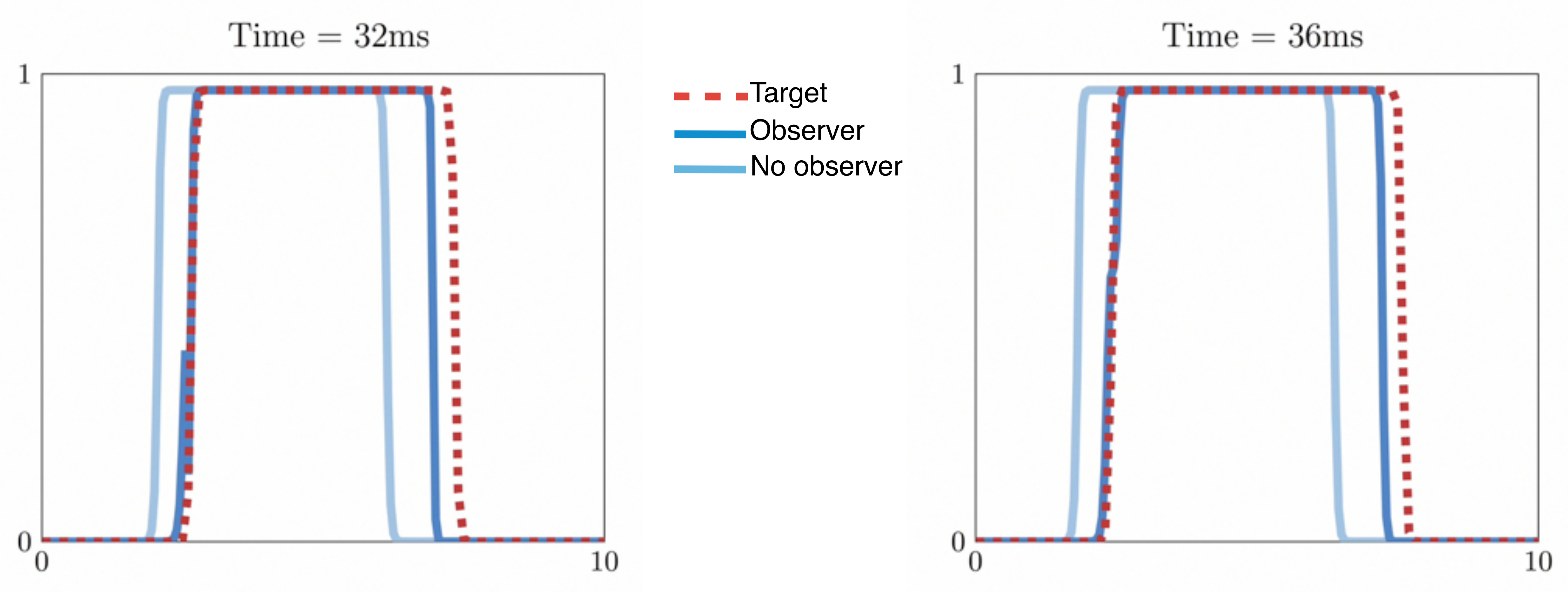
1D Case

- Reaction diffusion system of Mitchell-Schaeffer type (similar to Fitzugh-Nagumo)



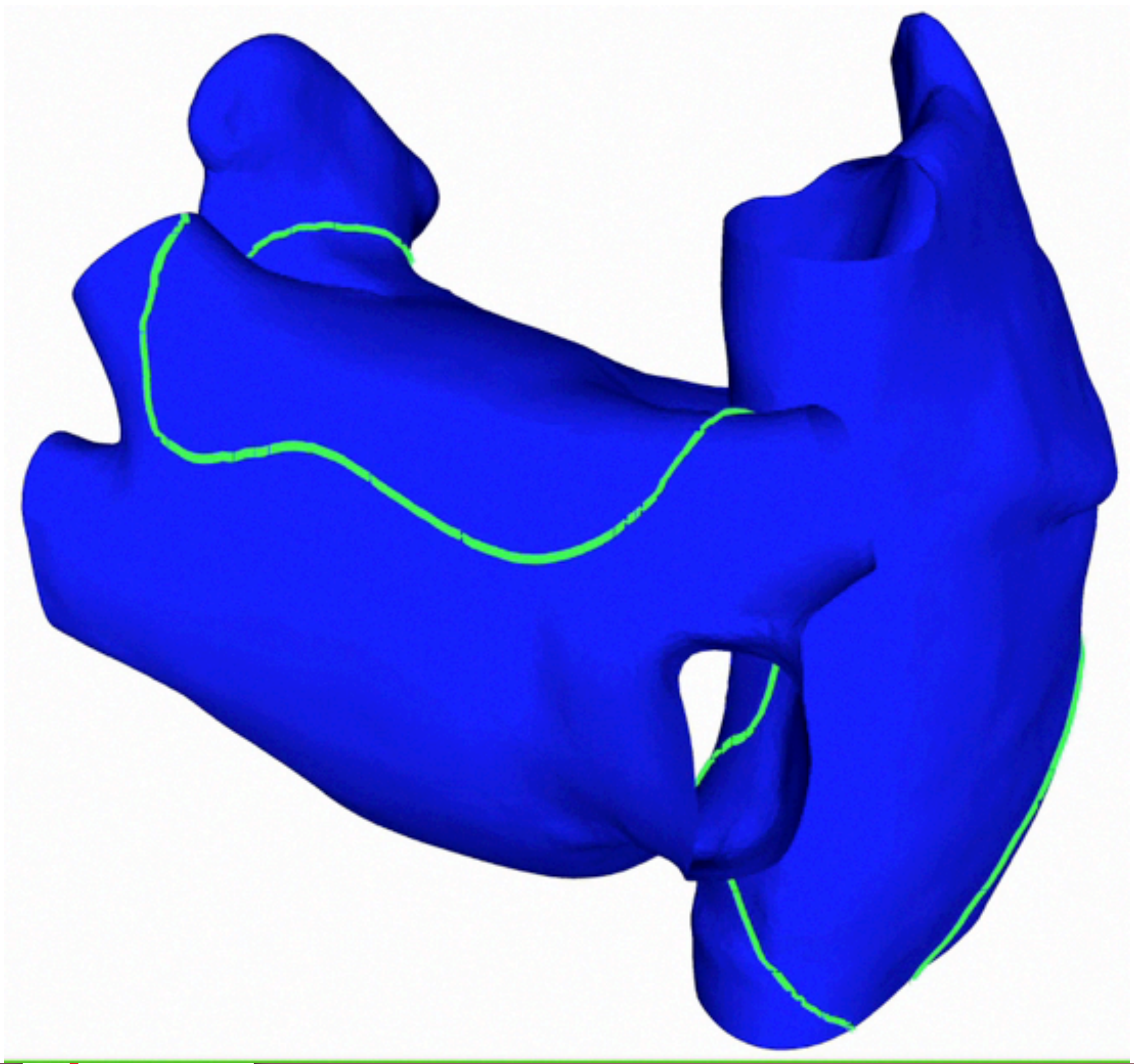
1D Case

- Reaction diffusion system of Mitchell-Schaeffer type (similar to Fitzugh-Nagumo)
- When time-sampling the data



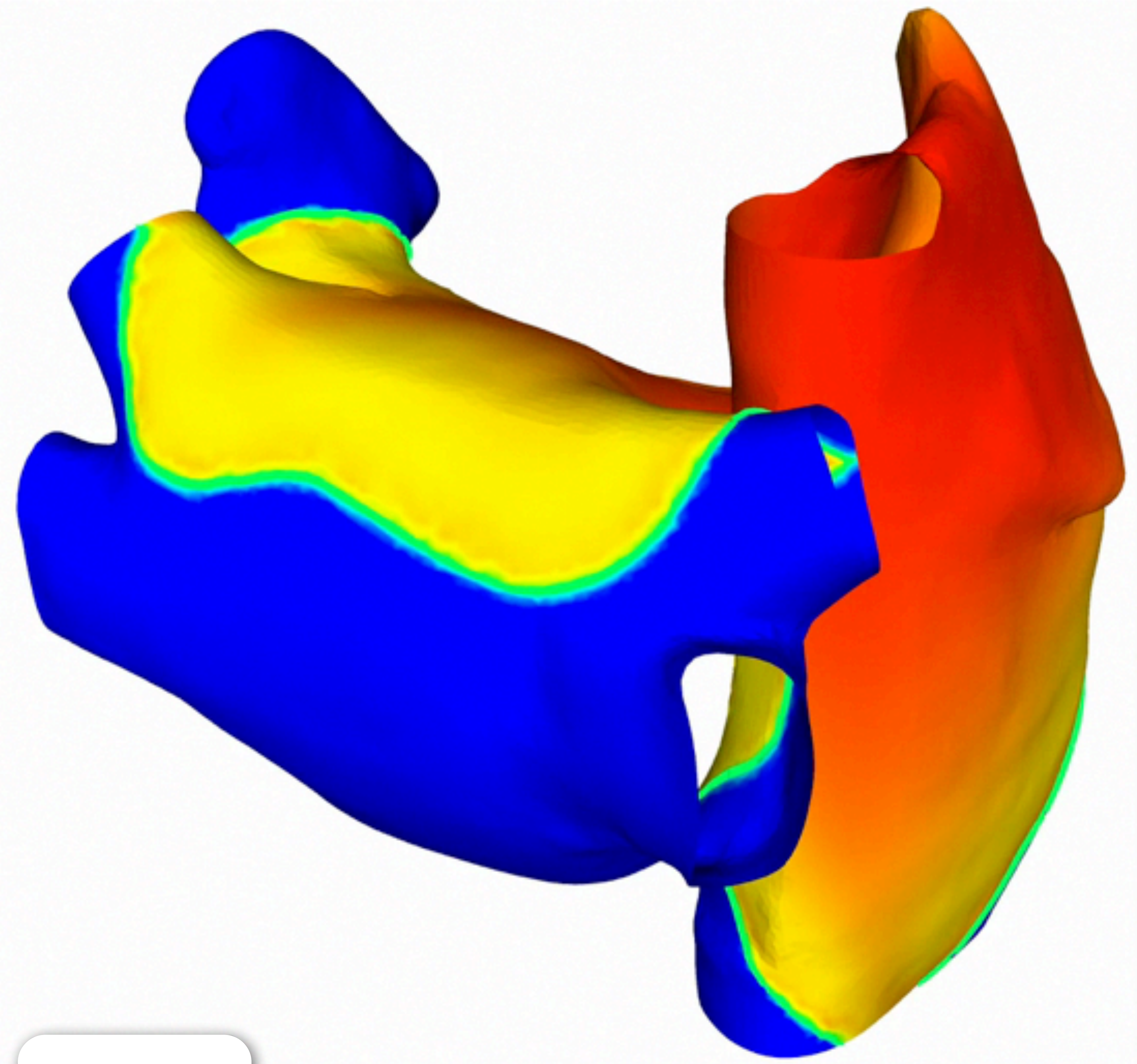
3D (Surfacic) Case

- Surface bidomain model with Courtemanche-Ramirez-Nattel ionic model specifically adapted to the atria (12 ionic currents and 20 other variables)



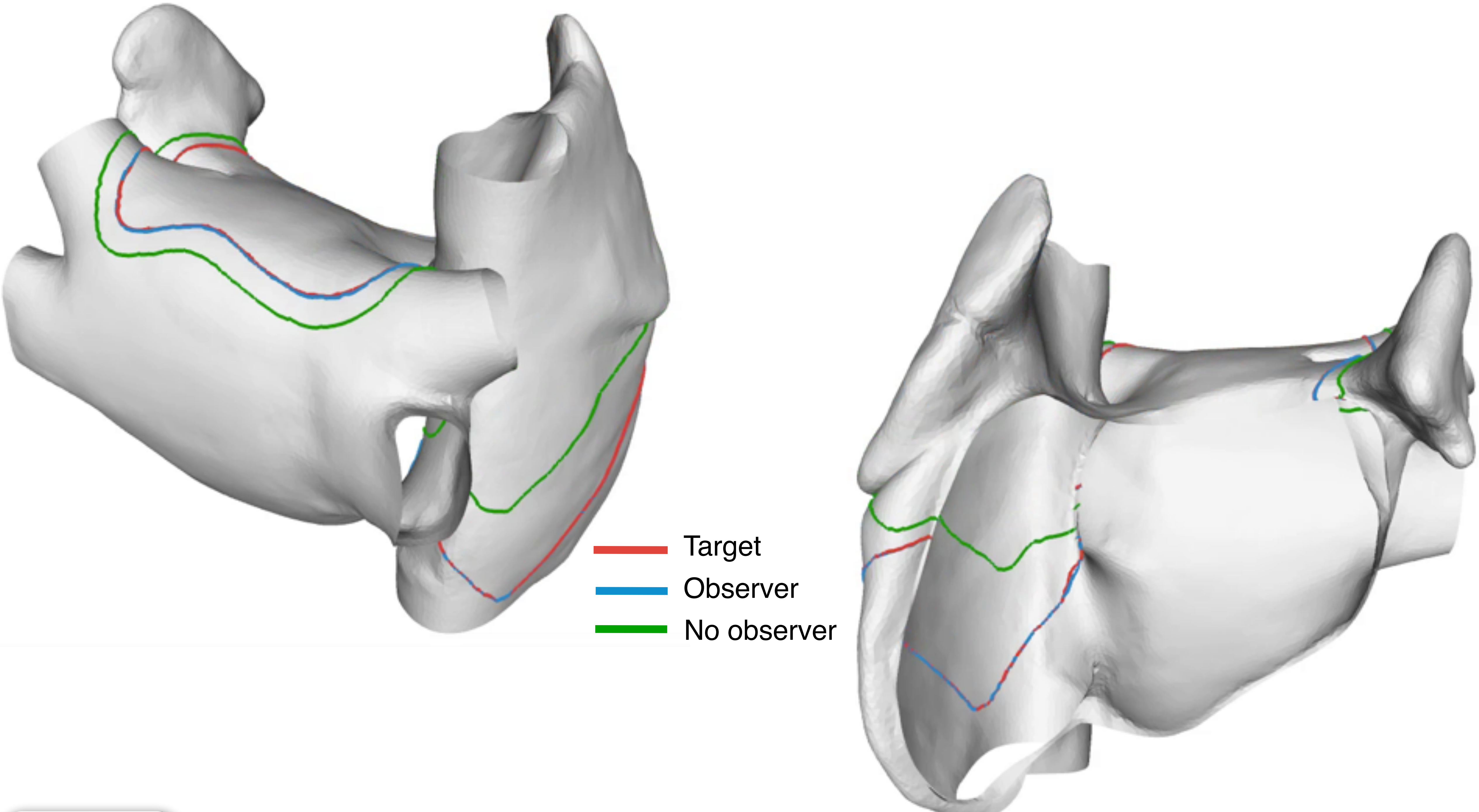
3D (Surfacic) Case

- Surface bidomain model with Courtemanche-Ramirez-Nattel ionic model specifically adapted to the atria (12 ionic currents and 20 other variables)



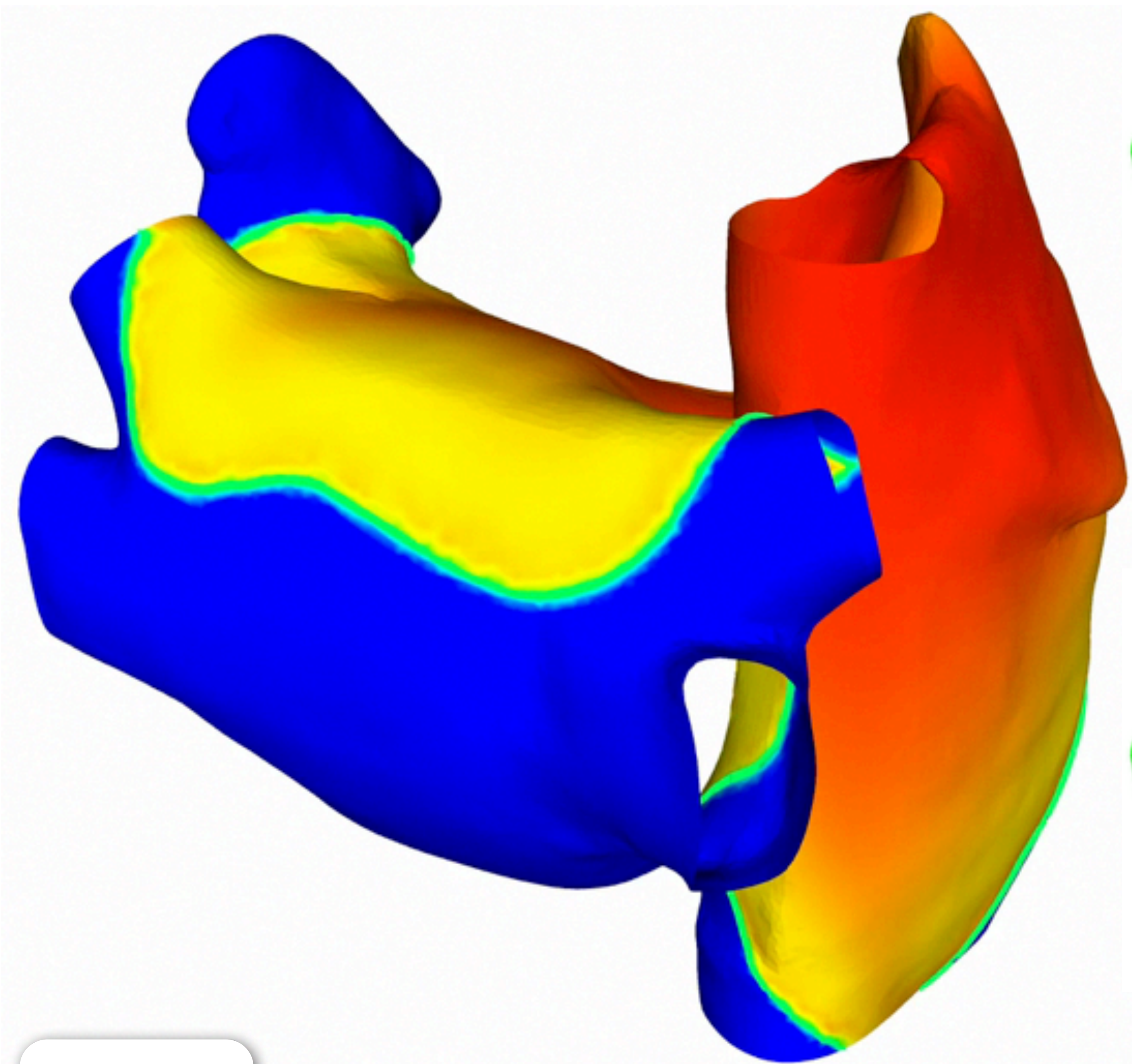
3D (Surfacic) Case

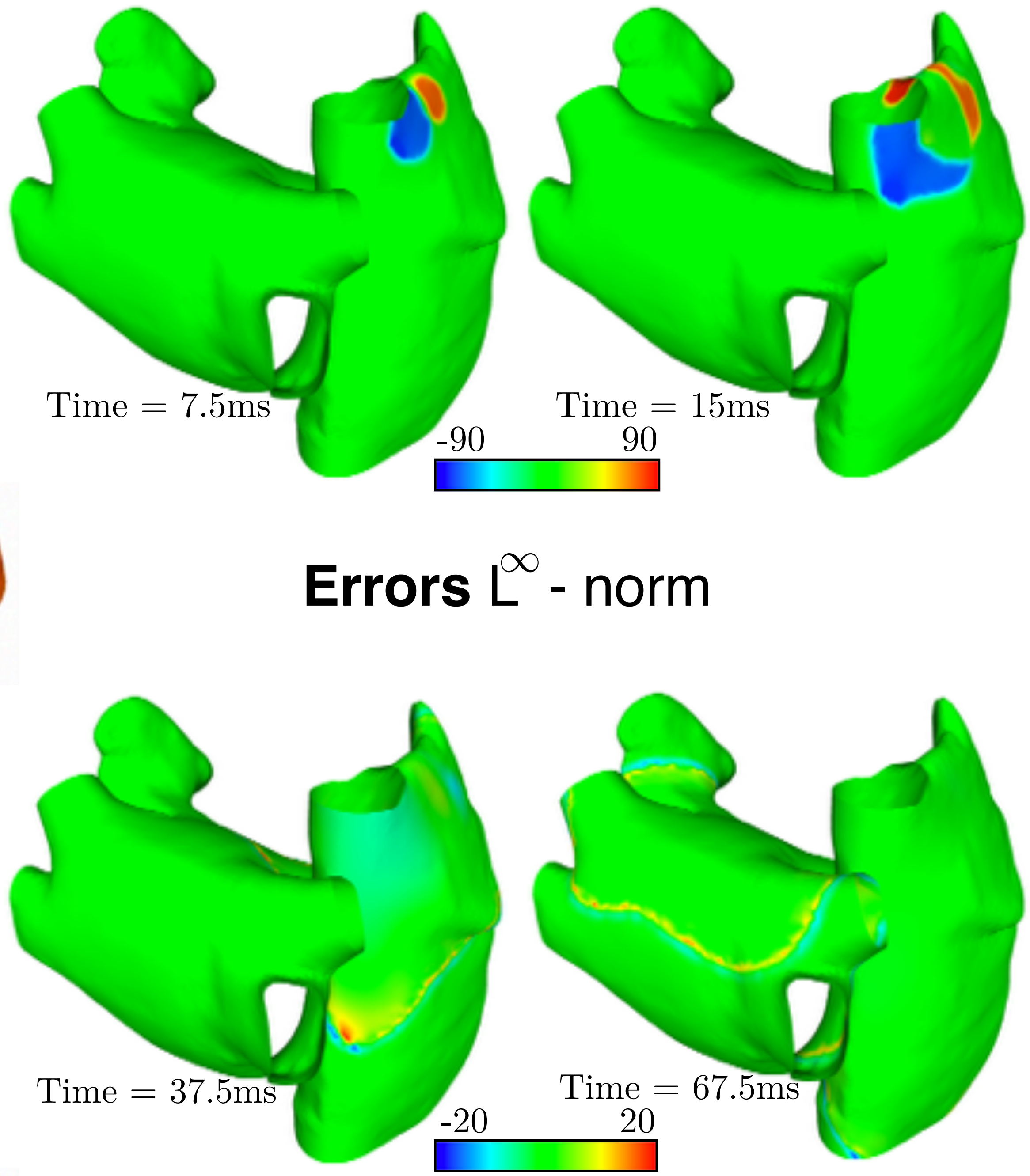
- Surface bidomain model with Courtemanche-Ramirez-Nattel ionic model specifically adapted to the atria (12 ionic currents and 20 other variables)



3D (Surfacic) Case

- Surface bidomain model with Courtemanche-Ramirez-Nattel ionic model specifically adapted to the atria (12 ionic currents and 20 other variables)





Observer for reaction-diffusion system based on front observations

Part IV: Adaptive Observer for Parameter identification

RoEKF applied to the parameter space

Principle: *Apply a Kalman-based filter of reduced rank to the parameters space where the uncertainties are now concentrated.*

- System $\begin{cases} \dot{y} = A(y, \vartheta) \\ \dot{\vartheta} = 0 \end{cases}$
- Observer $\begin{cases} \dot{\hat{y}} = A(\hat{y}, \hat{\vartheta}) + \gamma dC^*(z - C(\hat{y})) + L\dot{\hat{\vartheta}} \\ \dot{\hat{\vartheta}} = U^{-1} L^* dC^* M(z - C(\hat{y})) \\ \dot{U} = L^* dC^* M \, dC \, L \\ \dot{L} = (\partial_y A(\hat{y}, \hat{\vartheta}) - \gamma dC^* dC) L + \partial_\vartheta A(\hat{y}, \hat{\vartheta}) \end{cases}$
- Conditions of application
  - ✓ Use only discrepancies $dC^*(z - C(\hat{y})) = -dD^* D(y)$
  - ✓ Define a discrepancy D (with corresponding gain as dD^*)

$$D = H(\phi_u)(z - C_1(\phi_u)) + (1 - H(\phi_u))(z - C_2(\phi_u))$$

$$dD^* = \delta(\phi_u)\big((z - C_1(\phi_u)) - (z - C_2(\phi_u))\big)$$

Mathematical Analysis

- *Only* after linearization of the error $(\tilde{y}, \tilde{\vartheta})$ small $\Rightarrow (\delta\tilde{y}, \delta\tilde{\vartheta})$

- After a change of variable $(\delta\tilde{y}, \delta\tilde{\vartheta}) \to (\eta = \delta\tilde{y} - L\delta\tilde{\vartheta}, \delta\tilde{\vartheta})$
  the system of the linearised error satisfied

Asymptotically Stable

$$\begin{cases} \dot{\eta} = (\partial_y A - \gamma\, \mathrm{d}C^*\, \mathrm{d}C)\eta \\ \dot{\delta\tilde{\vartheta}} = -U^{-1} L^*\, \mathrm{d}C^* M\, \mathrm{d}C\, L\, \delta\tilde{\vartheta} - U^{-1}\, \mathrm{d}C^* M\, \mathrm{d}C\, \eta \end{cases}$$

0

(Exponentially) Stable if the system is identifiable

- Bibliography

P. Moireau, D. Chapelle, P. Le Tallec Joint state and parameter estimation for distributed mechanical systems Comput. Methods Appl. Mech. Engrg. 197 659–677 **2008**

- This strategy unifies two existing types of results

Q. Zhang, A. Clavel Adaptive observer with exponential forgetting factor for linear time varying systems, Decision and Control, **2001**.

DT Pham, J Verron, and L Gourdeau *Singular evolutive Kaiman filters for data assimilation in oceanography, Comptes Rendus de l'Académie des Sciences-Series IIA-Earth and Planetary Science, 326(4), 255–260* **1996**

Discretized version with RoUKF

- Splitting time-discretization scheme
  - Sampling

$$\begin{cases} \Lambda_n = \sqrt{U_n^{-1}} \\ \hat{Y}^{(i)}_{n|n} = \hat{Y}_{n|n} + L^Y_n \Lambda_n^T I^{(i)}, \quad 1 \leq i \leq \\ \hat{\theta}^{(i)}_{n|n} = \hat{\theta}^+_n + L^\theta_n \Lambda_n^{\mathsf{T}} I^{(i)}, \quad 1 \leq i \leq r \end{cases}$$

  - Prediction

$$\begin{cases} \hat{Y}^{(i)}_{n+1|n} = A_{n+1|n}(\hat{Y}^{(i)}_{n|n}, \hat{\theta}^{(i)}_{n|n}), \quad 1 \leq i \leq r \\ \hat{Y}_{n+1|n} = E_\alpha(\hat{Y}^*_{n+1|n}) = \sum_i \alpha_i \hat{Y}^{(i)}_{n+1|n} \\ \hat{\vartheta}_{n+1|n} = \hat{\vartheta}_{n|n} \end{cases}$$

  - Correction

$$\begin{cases} L^Y_{n+1} = [\hat{Y}^*_{n+1|n}] D_\alpha [I^*]^{\mathsf{T}} \\ L^\theta_{n+1} = [\theta^*_{n+1|n}] D_\alpha [I^*]^{\mathsf{T}} \\ \Gamma^{(i)}_{n+1} = D(Y^{(i)-}_{n+1}, t_{n+1}) \\ \Delta_{n+1} = [\Gamma^*_{n+1}] D_\alpha [I^*]^T \\ U_{n+1} = Id + \Delta^{\mathsf{T}}_{n+1} M_{n+1} \Delta_{n+1} \\ \hat{Y}_{n+1|n+1} = \hat{Y}_{n+1|n} + L^Y_{n+1} U^{-1}_{n+1} \Delta^{\mathsf{T}}_{n+1} M_{n+1} (E_\alpha(\Gamma^*_{n+1})) \\ \hat{\theta}_{n+1|n+1} = \hat{\theta}_{n+1|n} + L^\theta_{n+1} U^{-1}_{n+1} \Delta^{\mathsf{T}}_{n+1} M_{n+1} (E_\alpha(\Gamma^*_{n+1})) \end{cases}$$

Time: t^n
Generate ***r*** particles around the mean value using the covariance reduced to parametric space

Sampling

Apply the model to this particle to compute one time step

Prediction

Compute the discrepancy w.r.t the observations for each particle

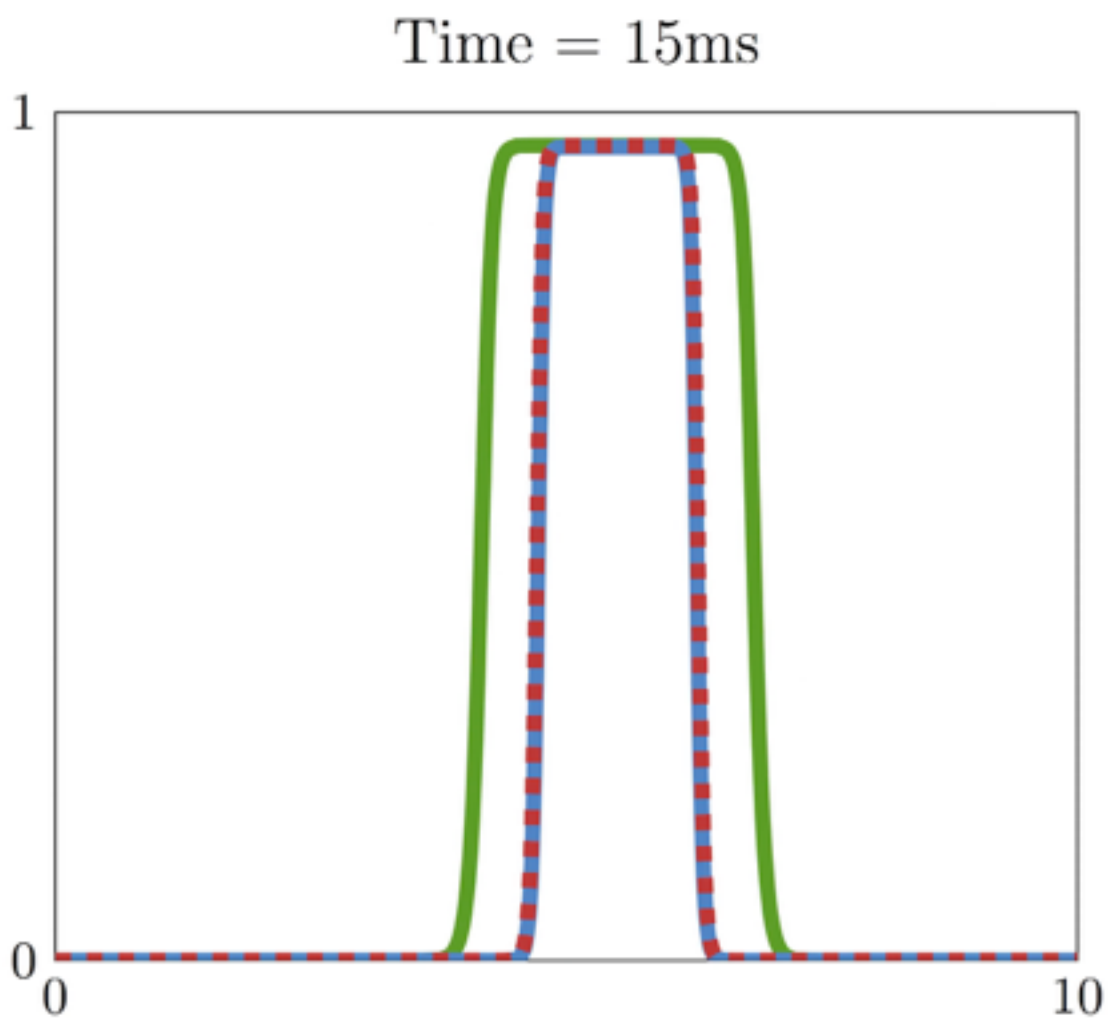
Correction

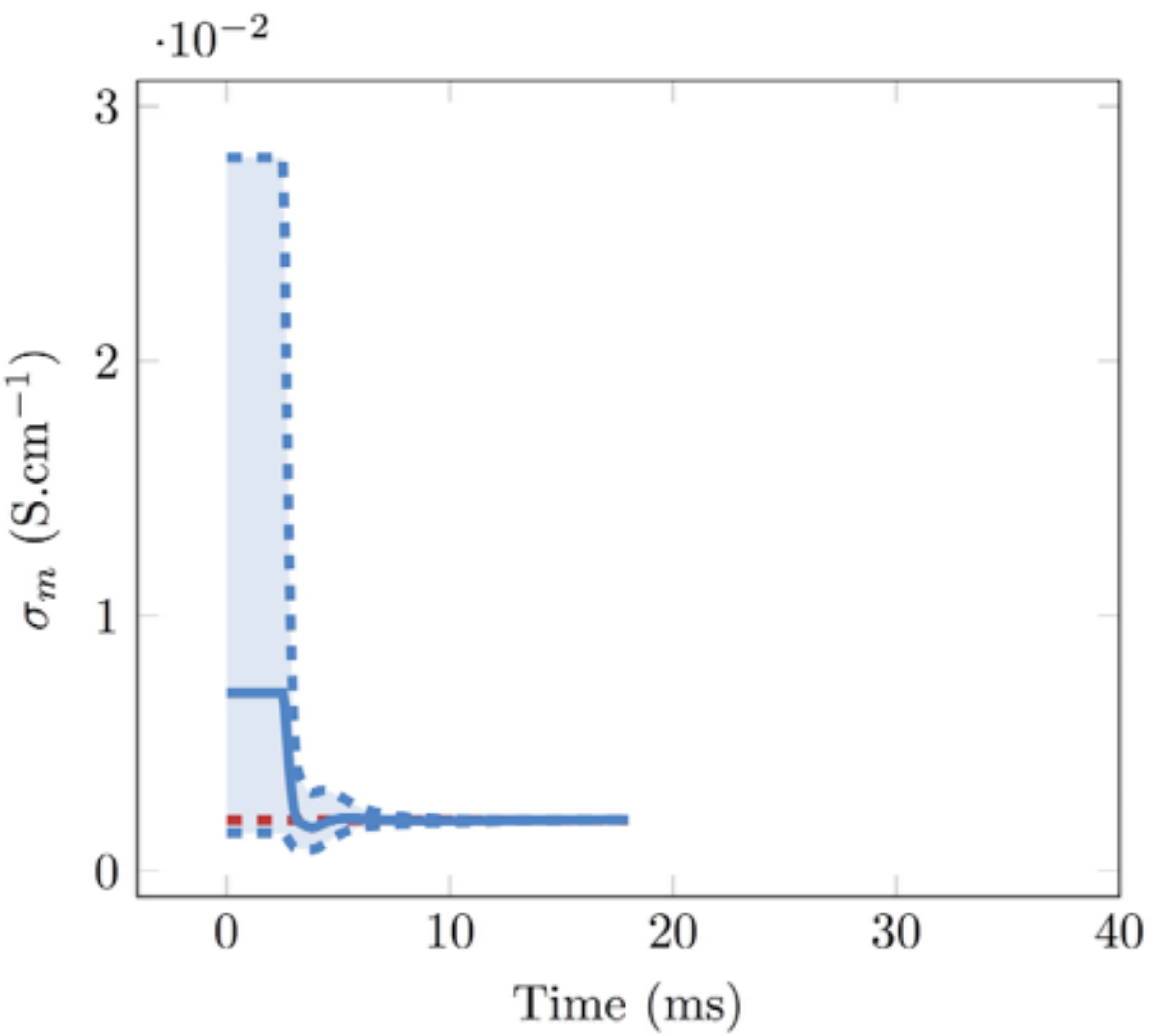
Gather the errors to compute a mean value and a "parametric" covariance

P. Moireau, D. Chapelle, Reduced-Order Unscented Kalman Filtering With Application to Parameter Identification in Large-Dimensional Systems - ESAIM COCV 2011

1D Case

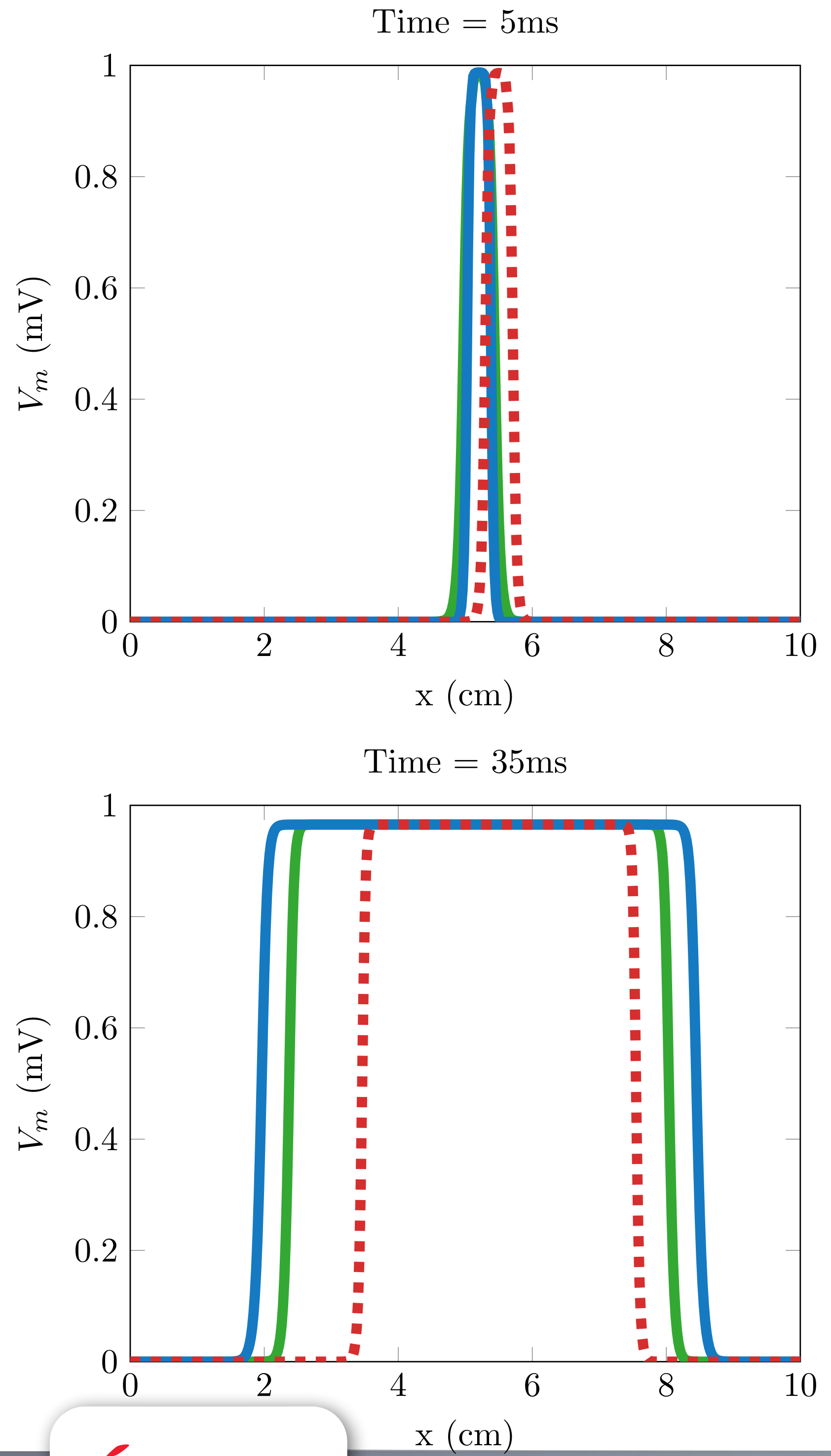
- Isotropic and spacewise-constant diffusion tensor



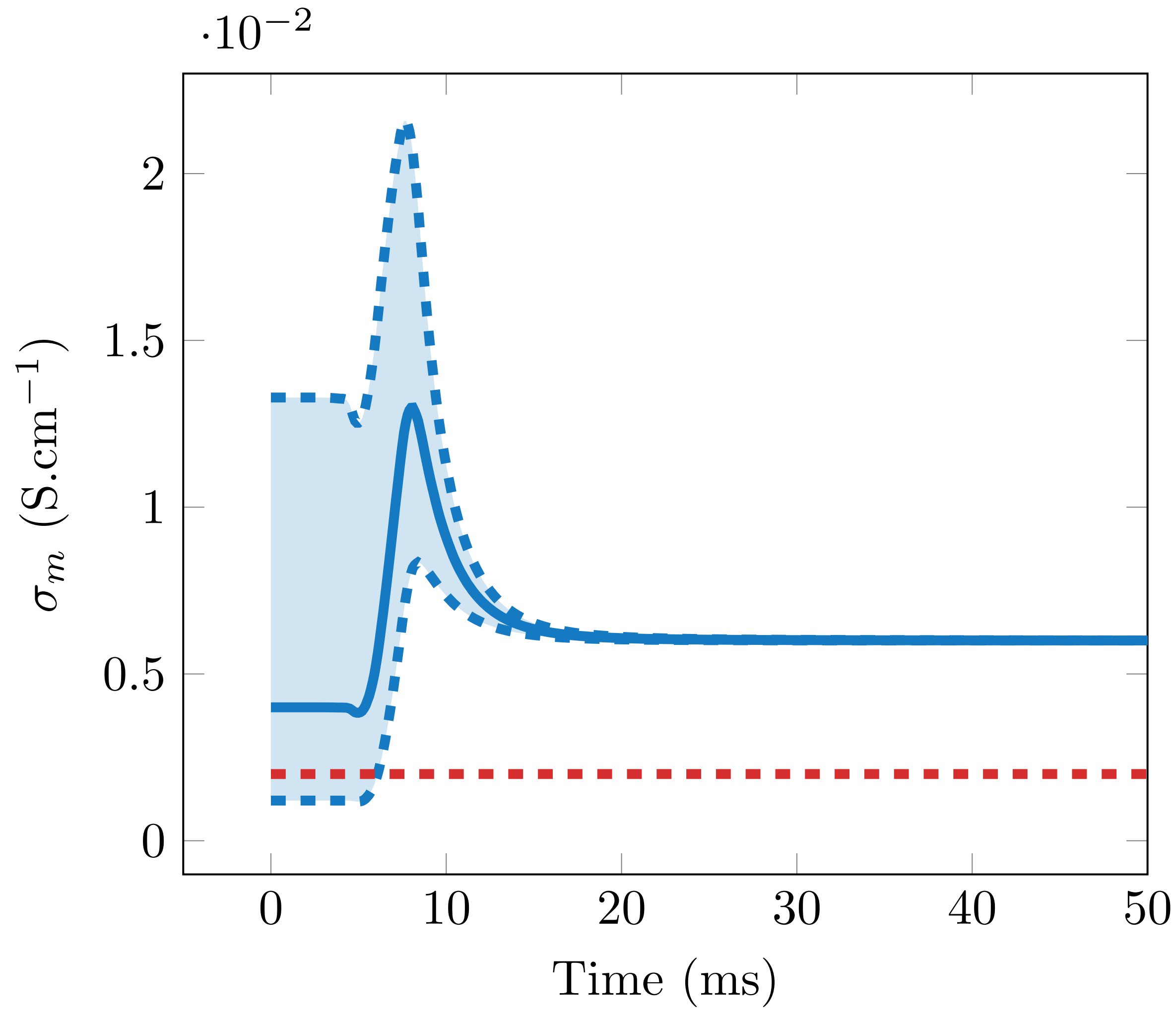


1D Case

- Importance of the **state** estimator during identification

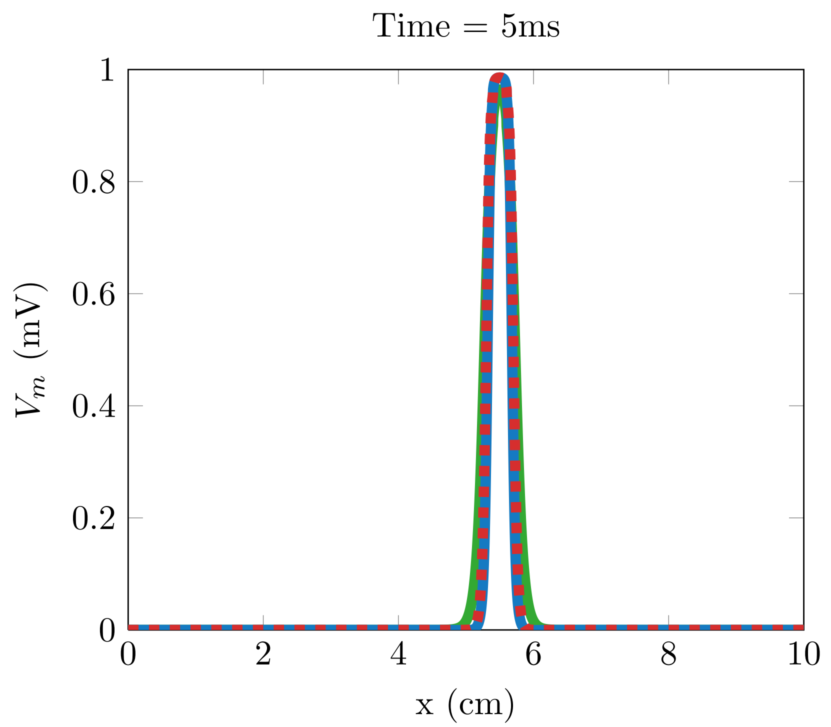


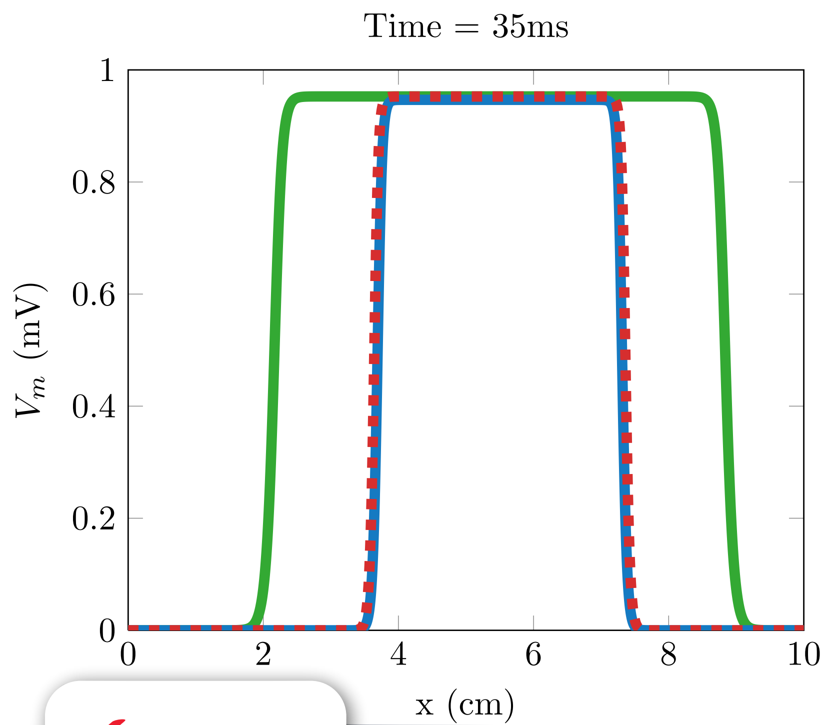
➡ Even for a small error on the initial condition

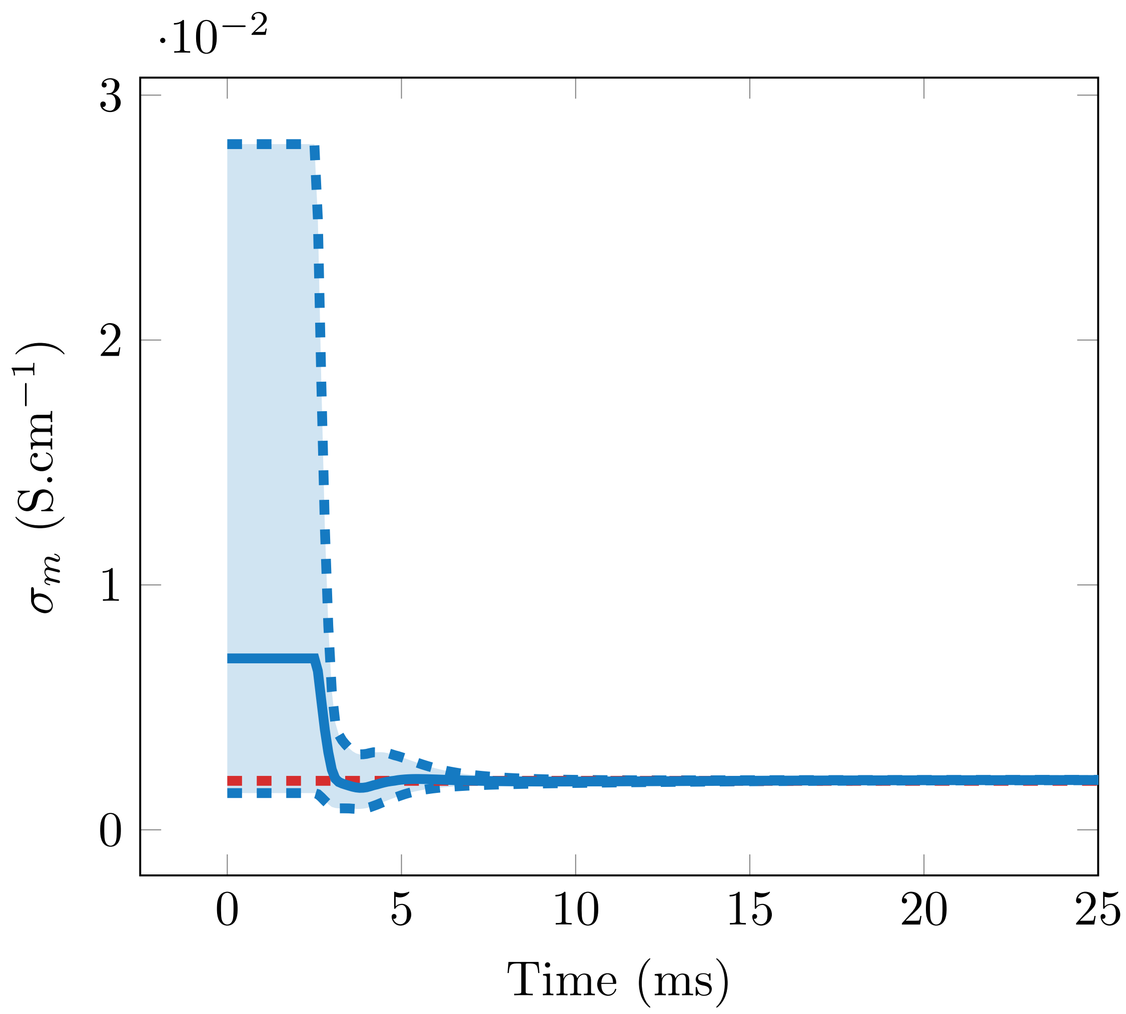


1D Case

- Isotropic and spacewise-constant diffusion tensor
    - Good initial condition - **Wrong parameter**

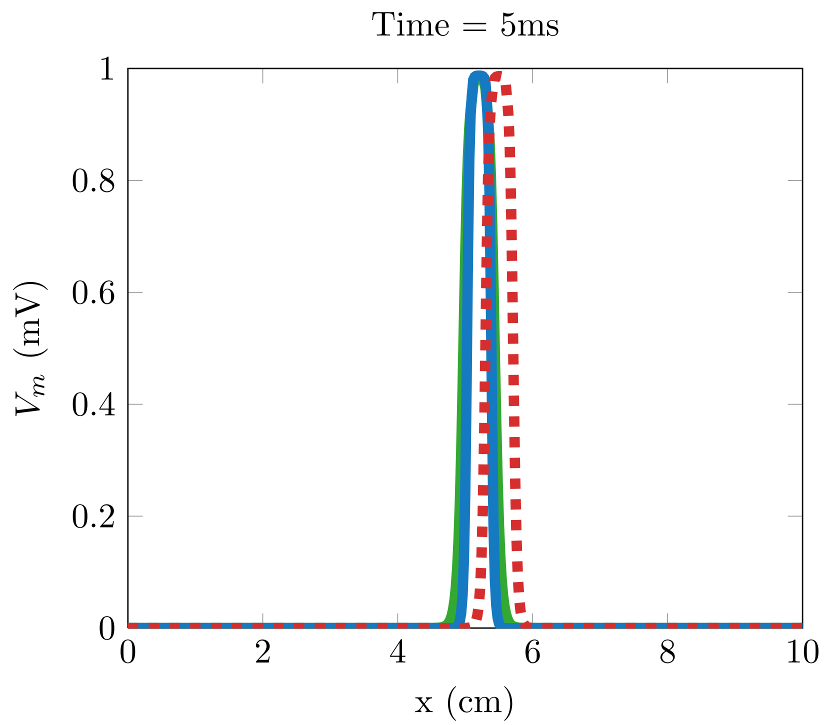


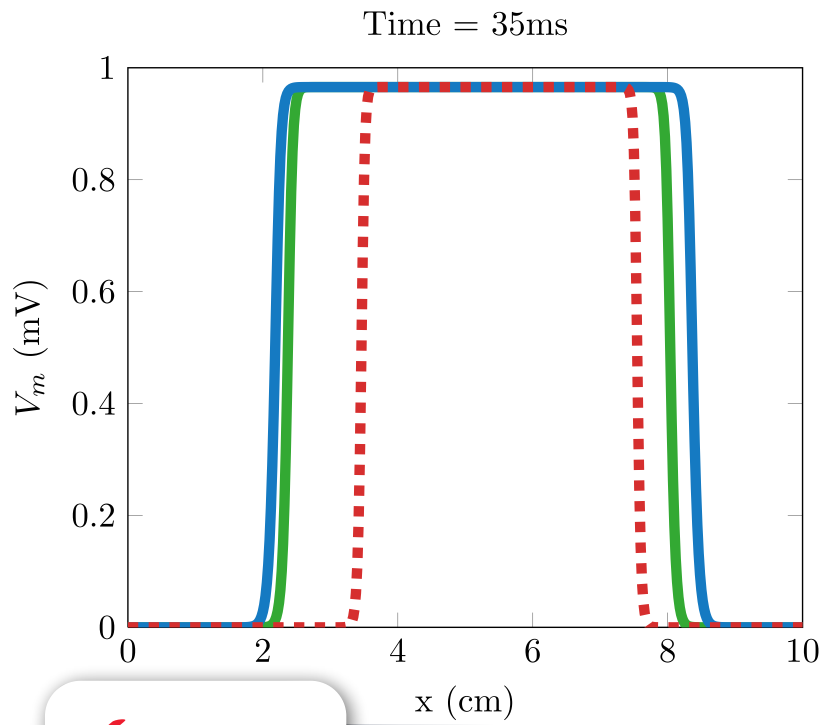


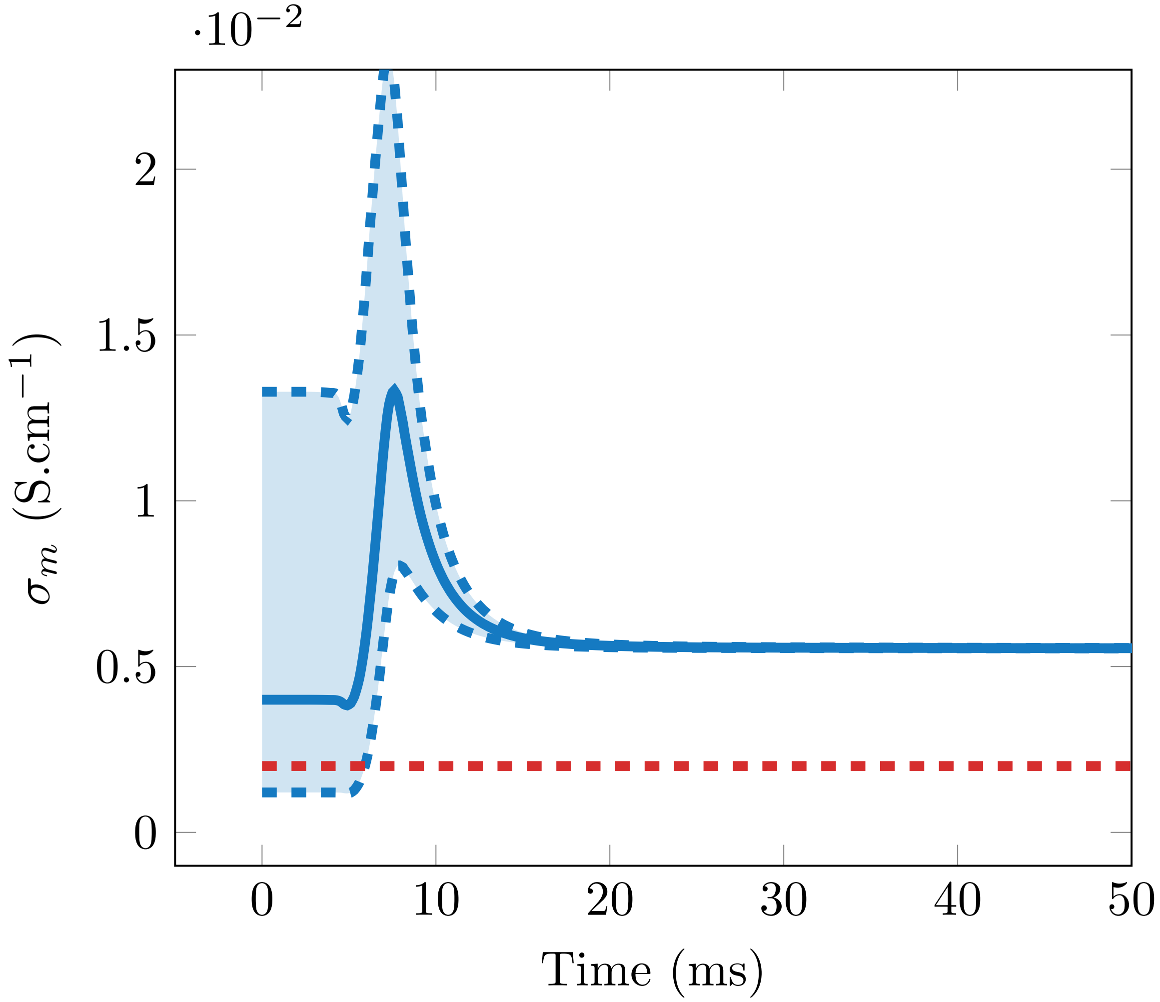


1D Case

- Isotropic and spacewise-constant diffusion tensor
  - ~~Good~~ initial condition - **Wrong parameter**

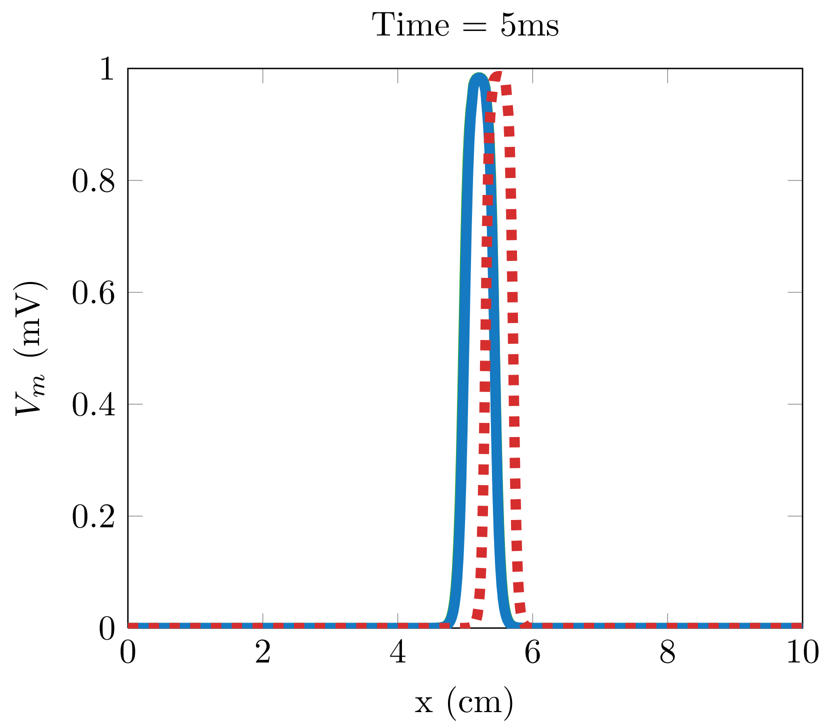


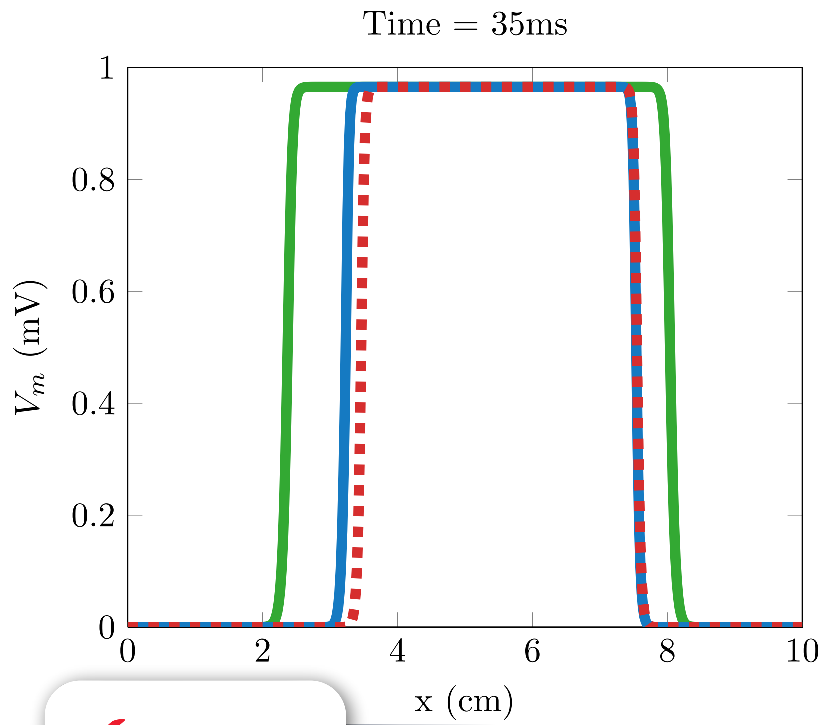


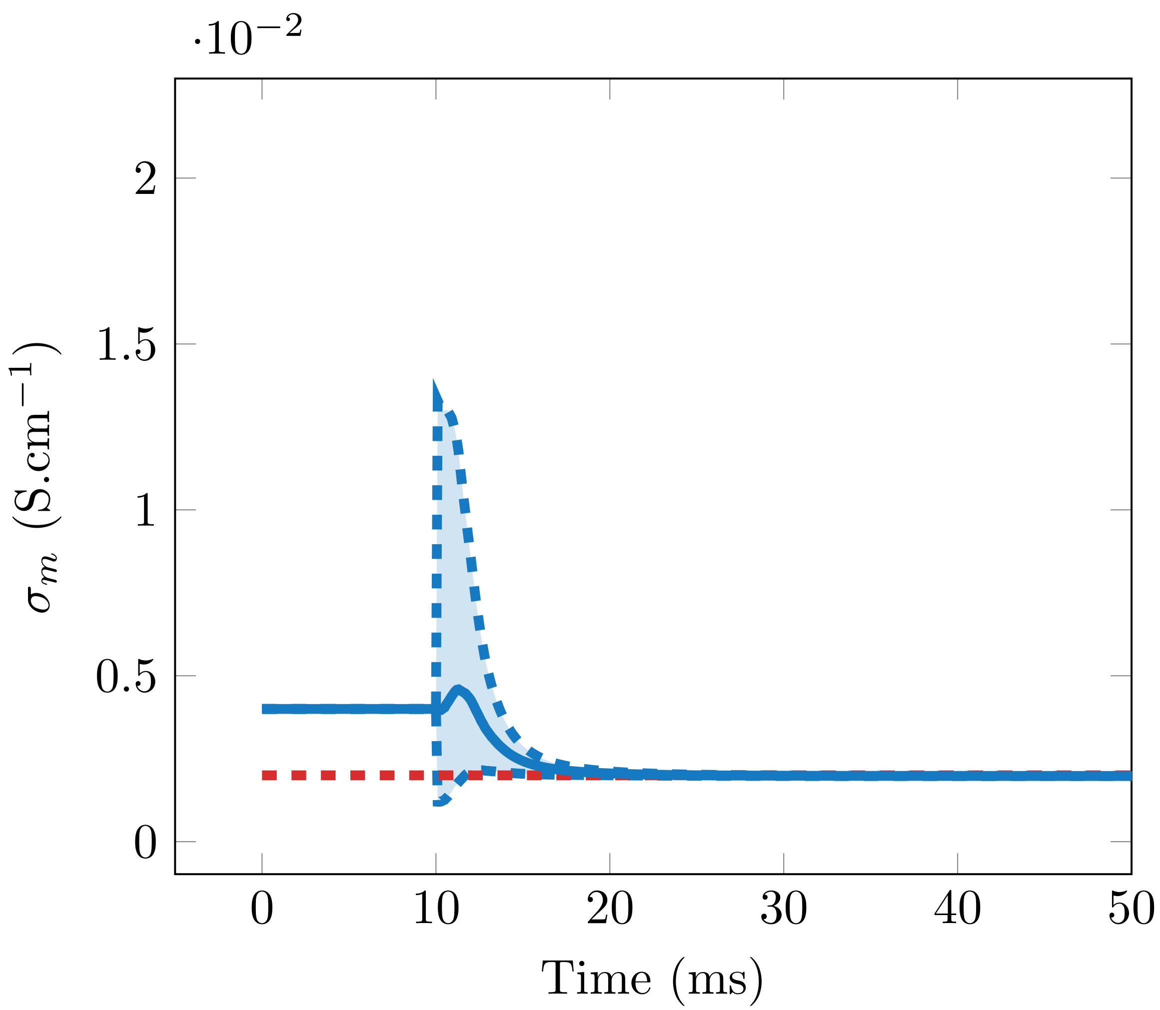


1D Case

- Isotropic and spacewise-constant diffusion tensor
  - ~~Good~~ initial condition - **Wrong parameter**
    - ➡ Delaying the joint state and parameter observer



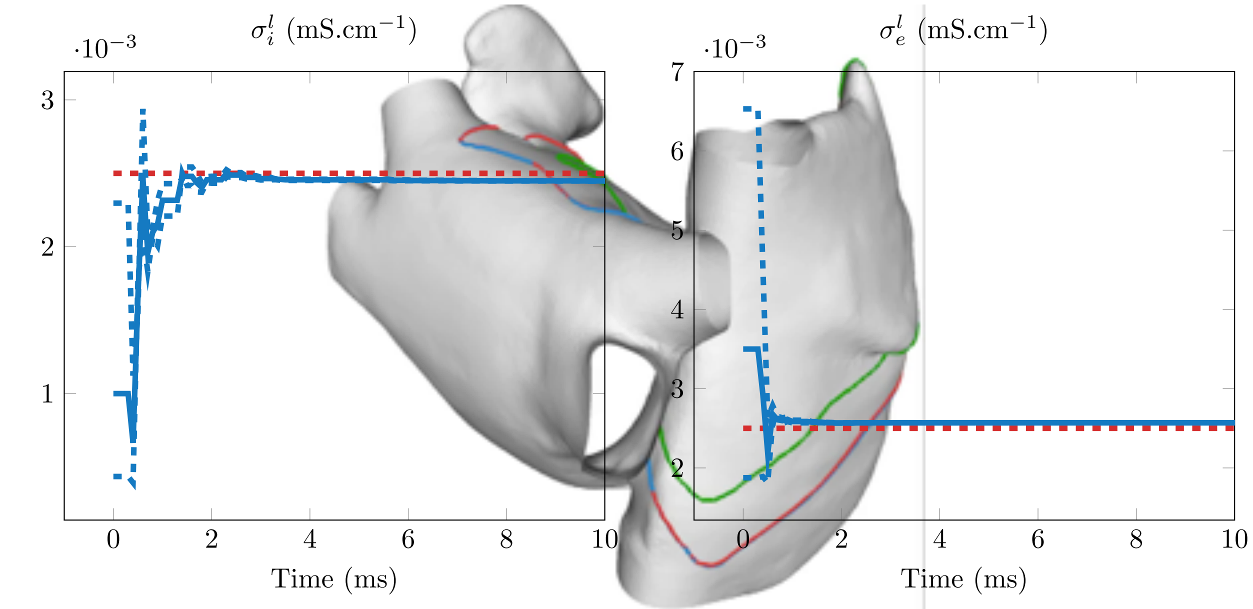




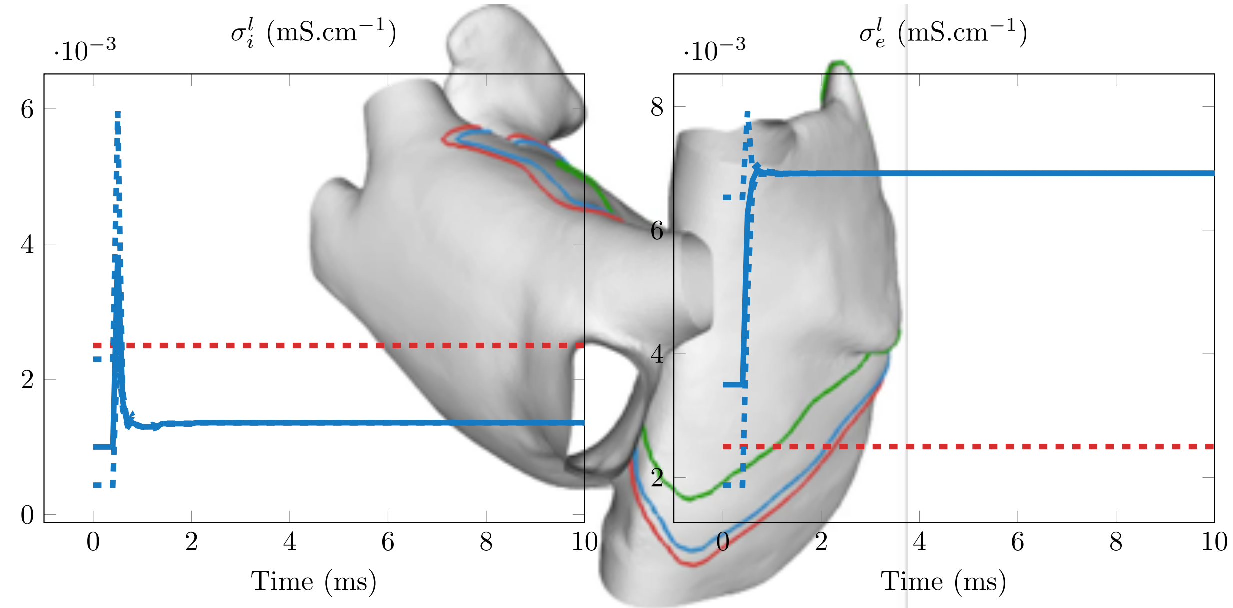
3D (Surfacic) Case

Target
Observer
No observer

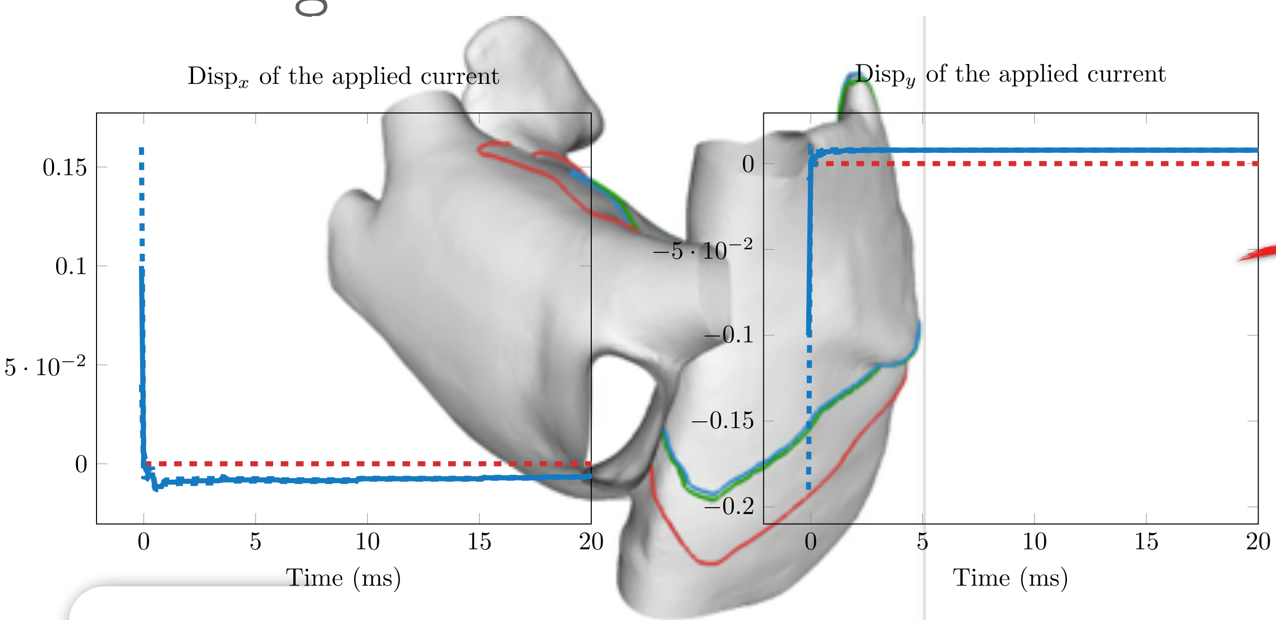
- First tests
  - Good initial condition



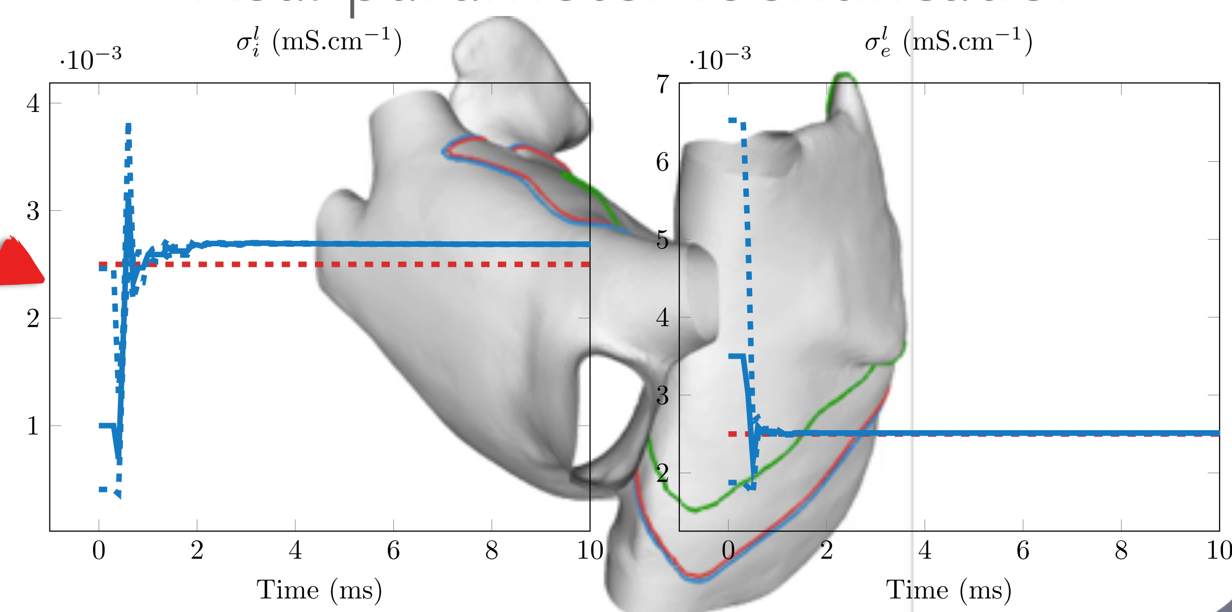
  - ~~Good~~ initial condition



- Complete strategy
  - Stage of initial condition identification



  - Real parameter identification



Observer for propagating phenomena based on front observations

Part V: Perspectives

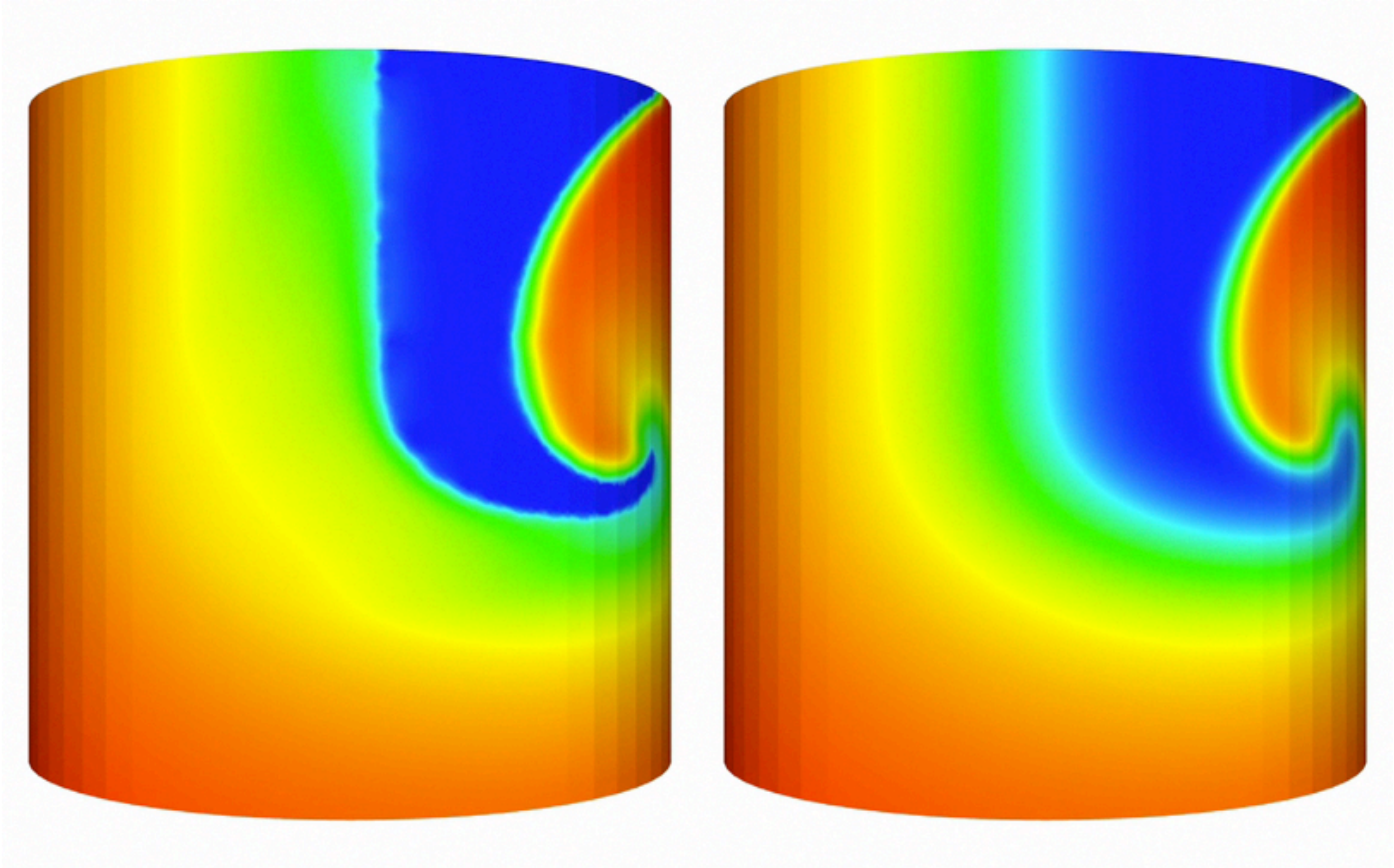
A — Complete analysis

- Finalyze the actual analysis
  - Existence of the observer
  - Observability condition associated to the linearised error

- Via the Mumford Shah functional
  - Consider a stationary pattern
  - Define a functional associated with our observer
  - Use the classical Mumford Shah theory to define a minimizer and show that the observer converges to it

B — Incorporate more information

- Incorporate topological gradient associated to the Chan-Vese data fitting term



- Use multiple phase to handle multiple isocontours

C — Wave equation case

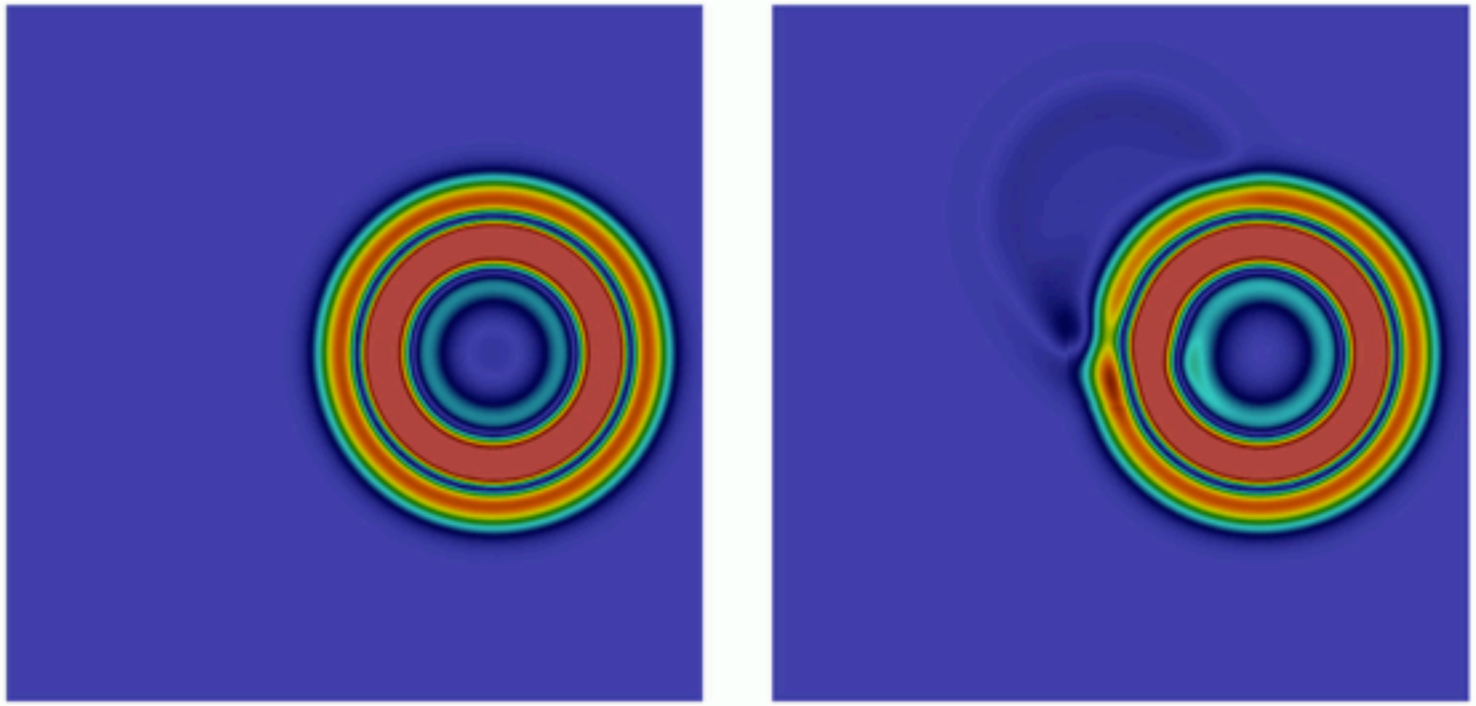
- Wave equation with no b.c.

$$\ddot{u} - c^2 \Delta u = 0, \qquad \text{in } \mathcal{B}$$

- Special first order form

$$\begin{cases} \dot{u} + c\underline{\nabla} \cdot \underline{v} = 0, & \text{in } \mathcal{B} \\ \dot{\underline{v}} + c\underline{\nabla} u = 0, & \text{in } \mathcal{B} \end{cases}$$

with $\mathcal{E} = \int_{\mathcal{B}} u^2 + v^2 dx$



- Observer with a partial measurement $u_{|\omega}$

$$\begin{cases} \dot{\hat{u}} + c\underline{\nabla} \cdot \hat{\underline{v}} = \gamma \chi_\omega (z - \hat{u}_{|\omega}), & \text{in } \mathcal{B} \\ \dot{\hat{\underline{v}}} + c\underline{\nabla} \hat{u} = 0, & \text{in } \mathcal{B} \end{cases}$$

- Observer with a front observation ?

$$\begin{cases} \dot{\hat{u}} + c\underline{\nabla} \cdot \hat{\underline{v}} = \gamma \delta_{\Gamma_{\hat{u}}} \frac{1}{|\underline{\nabla} \hat{u}|} \Big(-\big(z - C_1(\phi_{\hat{u}})\big)^2 + \big(z - C_2(\phi_{\hat{u}})\big)^2 \Big), & \text{in } \mathcal{B} \\ \dot{\hat{\underline{v}}} + c\underline{\nabla} \hat{u} = 0, & \text{in } \mathcal{B} \end{cases}$$

To be continued ...

Analyse asymptotique en électrophysiologie cardiaque. Applications à la modélisation et à l'assimilation de données.

THÈSE DE DOCTORAT

Présentée par

Annabelle COLLIN

pour obtenir le grade de

DOCTEUR DE
L' UNIVERSITÉ PIERRE ET MARIE CURIE - Paris VI

Spécialité : Mathématiques Appliquées

Soutenue publiquement le ? ? ? ? ? ? ? ? ? ? ? ? devant le jury composé de :

? ? ? ?	Président
Dominique Chapelle	Directeur de thèse
Jean-Frédéric Gerbeau	Directeur de thèse
? ? ? ?	Rapporteur
? ? ? ?	Rapporteur
? ? ? ?	Examinateur

Après avis favorables des rapporteurs : ? ? ? ? et ? ? ? ?