Optimal boundary observation domain

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We consider the wave equation on $\Omega \subset \mathbb{R}^n$, open, connected and convex :

$$\begin{cases} y_{tt} - \Delta y = 0 & (x,t) \in \Omega \times (0,T), \\ y = 0 & (x,t) \in \partial\Omega \times [0,T], \\ y(t=0,\cdot) = y_0 & x \in \Omega, \\ y_t(t=0,\cdot) = y_1 & x \in \Omega, \end{cases} \tag{1}$$

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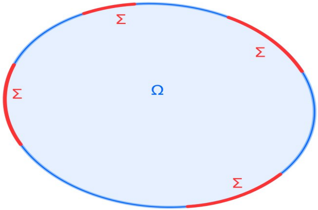
with $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$.

We define the observation domain $\Sigma \subset \partial\Omega$, such that $|\Sigma| = L|\partial\Omega|$.



The purpose is to find the optimal shape and position for Σ.

The system (1) is said observable on Σ in time T if there exist a positive constant $C_T(\chi_\Sigma) > 0$ such that

$$C_T(\chi_\Sigma)\|(y_0, y_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq \int_0^T \int_\Sigma \left|\frac{\partial y}{\partial n}(t,x)\right|^2 d\sigma dt, \tag{2}$$

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We want to maximize $C_T(\chi_\Sigma)$ over all possible subdomains Σ of $\partial\Omega$ of Hausdorff measure $L|\partial\Omega|$.

$$\boxed{C_T(\chi_\Sigma) = \inf_{\substack{y \text{ solution of } (1)\\ \text{with } (y_0,y_1)\neq(0,0)}} \frac{\int_0^T \int_\Sigma \left|\frac{\partial y}{\partial n}(t,x)\right|^2 d\sigma dt}{\| (y_0, y_1) \|^2_{H^1_0\times L^2}}}$$

In practise, this observability constant appears to be very pessimistic. In fact, it takes into account every initial condition, whereas usually only a large number of them can be considered.
Let us introduce a Hilbert basis of $L^2(\Omega)$ consisting in $(\phi_j)_{j\geq 1}$ the eigenfunctions of $-\Delta$, and the associated eigenvalues $(\lambda_j)_{j\geq 1}$.
In this case, the solution of the wave equation y writes :

$$y \quad (t,x) = \sum_{j=1}^{+\infty} \left(\quad a_j e^{i\sqrt{\lambda_j}t} + \quad b_j e^{-i\sqrt{\lambda_j}t} \right) \phi_j(x),$$

with, for all $j \in \mathbb{N}^*$:

$$a_j = \int_\Omega \frac{1}{2} \left(y^0(x) - \frac{i}{\sqrt{\lambda_j}} y^1(x) \right) \phi_j(x)\, dx, \quad b_j = \int_\Omega \frac{1}{2} \left(y^0(x) + \frac{i}{\sqrt{\lambda_j}} y^1(x) \right) \phi_j(x)\, dx.$$

This lead to

$$\boxed{C_T(\chi_\Sigma) = \inf_{\substack{(a_j),(b_j)\in\ell^2(\mathbb{R}) \\ \sum_{j=1}^{+\infty}(a_j^2+b_j^2)=1}} \int_0^T \int_\Sigma \left| \sum_{j=1}^{+\infty} \left(\frac{a_j}{\lambda_j} e^{i\sqrt{\lambda_j}t} + \frac{b_j}{\lambda_j} e^{-i\sqrt{\lambda_j}t} \right) \frac{\partial \phi_j}{\partial n}(t,x) \right|^2 d\sigma dt.}$$

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Let us introduce a Hilbert basis of $L^2(\Omega)$ consisting in $(\phi_j)_{j\geq 1}$ the eigenfunctions of $-\Delta$, and the associated eigenvalues $(\lambda_j)_{j\geq 1}$.
In this case, the "random" solution of the wave equation y_{rand} writes :

$$y_{rand}(t,x) = \sum_{j=1}^{+\infty} \left(\beta_{1,j}^{\omega} a_j e^{i\sqrt{\lambda_j}t} + \beta_{2,j}^{\omega} b_j e^{-i\sqrt{\lambda_j}t} \right) \phi_j(x),$$

with

$$\mathbb{P}(\beta_{j,1}^{\omega} = \pm 1) = \mathbb{P}(\beta_{j,2}^{\omega} = \pm 1) = \frac{1}{2} \qquad \text{and} \qquad \mathbb{E}(\beta_{j,1}^{\omega}\beta_{k,2}^{\omega}) = 0.$$

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In the end, we want to maximize the following function

$$\boxed{J(\chi_\Sigma) = \inf_{j\in\mathbb{N}^*} \frac{1}{\lambda_j} \int_{\partial\Omega} \chi_\Sigma(x) \left(\frac{\partial \phi_j}{\partial n}(x)\right)^2 \, d\sigma,} \tag{3}$$

over all subdomains Σ of $\partial\Omega$ of Hausdorff measure $L|\partial\Omega|$.

We do not know a priori wether Problem (3) is well-posed or not, and we thus propose here a relaxation procedure : Let us define

- $\mathcal{U}_L = \{\chi_\Sigma \mid \Sigma \subset \partial\Omega \text{ of Hausdorff measure } |\Sigma| = L|\partial\Omega|\}$,
- $\overline{\mathcal{U}}_L = \left\{a \in L^\infty(\partial\Omega; (0,1)) \mid \int_{\partial\Omega} a(x)\, d\sigma = L|\partial\Omega|\right\}$.

Problem (3) is now well-posed on $\overline{\mathcal{U}}_L$, and there exist at least one solution !

1. cf A. Hassel, S. Zelditch, *Quantum ergodicity of boundary values of eigenfunctions* Commun. Math. Phys. 248, 119-168 (2004).

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No-Gap Theorem

Under geometric assumptions on Ω, one has

$$\sup_{\chi_\Sigma\in\mathcal{U}_L} J(\chi_\Sigma) = \max_{a\in\overline{\mathcal{U}}_L} J(a). \tag{4}$$

Geometric conditions on Ω :

- $\frac{1}{\lambda_j}\left(\frac{\partial\phi_j}{\partial n}\right)^2$ converges vaguely to $\frac{1}{n|\Omega|}$. (WQEB Condition)
  This hold true for piecewise smooth and ergodic domains [1].
- $\exists A > 0$ such that $\left\|\frac{\partial\phi_j}{\partial n}\right\|^2_{L^\infty(\partial\Omega)} \leq A\lambda_j$.

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A particular solution

If Ω is piecewise smooth, thanks to a Rellich-type formula, one has $\forall x_0 \in \mathbb{R}^n$:

$$\boxed{2\lambda_j = \int_{\partial\Omega} \langle x - x_0 . n \rangle \left(\frac{\partial \phi_j}{\partial n}(x) \right)^2 d\sigma} \tag{5}$$

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Moreover :

- $\displaystyle\int_{\partial\Omega} \langle x - x_0 . n \rangle d\sigma = n|\Omega|.$
- WQEB implies $\displaystyle\frac{1}{\lambda_j} \left(\frac{\partial \phi_j}{\partial n} \right)^2 \rightharpoonup \frac{2}{n|\Omega|}$

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Thus, setting $a^*_{x_0} = \dfrac{L|\partial\Omega|}{n|\Omega|}\langle x - x_0 . n\rangle$ one has

$$J(a^*_{x_0}) = \frac{2L|\partial\Omega|}{n|\Omega|} \leq \max_{a \in \overline{\mathcal{U}}_L} J(a)$$

$$J(a) = \inf_{j\in\mathbb{N}^*} \frac{1}{\lambda_j}\int_{\partial\Omega} a(x)\left(\frac{\partial \phi_j}{\partial n}(x)\right)^2 d\sigma \leq \lim_{j\to+\infty} \frac{1}{\lambda_j}\int_{\partial\Omega} a(x)\left(\frac{\partial \phi_j}{\partial n}(x)\right)^2 d\sigma = \frac{2L|\partial\Omega|}{n|\Omega|}$$

And thus :

$$\boxed{\max_{a \in \overline{\mathcal{U}}_L} J(a) = J(a^*_{x_0}) = \frac{2L|\partial\Omega|}{n|\Omega|}}$$

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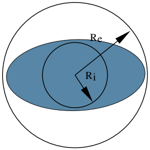
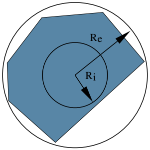
- $0 \leq a^*$: Ok if Ω is star-shaped.
- $a^* \leq 1$: For instance, if Ω is convex and included is a ring of radii such that :

$$\left(\frac{r_e}{r_i}\right)^n \leq \frac{n^2}{4L} :$$





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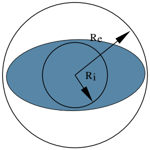

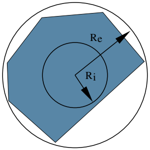

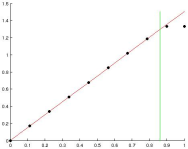

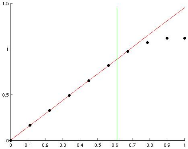

For some particular domains Ω, one can compute directly the optimal value, and thus find particular solution for Problem (3). Here are several examples :

1- The square : If $\Omega = (0,\pi) \times (0,\pi)$ one has

$$\max_{a \in \overline{\mathcal{U}}_L} J(a) = J(L) = \pi L$$

and the equality $J(\chi_\Sigma) = \pi L$ is reached on $\mathcal{U}_L$ iff $L \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$.

2- The disk : If Ω is the disk of radius R one has

$$\max_{a \in \overline{\mathcal{U}}_L} J(a) = \pi R L$$

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The truncated problem

We propose here to truncate the criterion at some order $N \in \mathbb{N}^*$, and thus to maximize

$$\boxed{J_N(\chi_\Sigma) = \min_{1 \leq j \leq N} \frac{1}{\lambda_j} \int_{\partial\Omega} \chi_\Sigma(x) \left(\frac{\partial \phi_j}{\partial n}(x) \right)^2 d\sigma} \tag{6}$$

over all subdomains Σ of $\partial\Omega$ of Hausdorff measure $L|\partial\Omega|$.

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- Because this is the criterion used in practise.
- Because one can prove that, now, for a large class of domains Ω there exist a solution in $\mathcal{U}_L$ denoted by $\chi_{\Sigma_N^*}$.

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- Because one can prove that, now, for a large class of domains Ω there exist a solution in $\mathcal{U}_L$ denoted by $\chi_{\Sigma_N^*}$.
- Because one has the following Γ-convergence of J_N to J in $\overline{\mathcal{U}}_L$:
  - $\chi_{\Sigma_N^*} \rightharpoonup a^*$, with $a^* \in \overline{\mathcal{U}}_L$ maximizing J.
  - $\max_{a \in \overline{\mathcal{U}}_L} J(a) = \lim_{N \to +\infty} J_N(\chi_{\Sigma_N^*})$.

When $a^*_{x_0} \notin \Omega$, or when Ω does not satisfy all the geometric conditions :

The study of the truncated problem gives the optimal value of J on $\overline{\mathcal{U}}_L$, and thus on $\mathcal{U}_L$ (No-Gap). This helps measuring the efficiency of a given position for Σ. This also helps giving a domain Σ accurate for J_N for a given Ω, when it is not possible to build a maximizing sequence for J.

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Nevertheless,

$$\Big(J_N(\chi_{\Sigma_N^*}) \to J(a^*)\Big) \not\Rightarrow \Big(J(\chi_{\Sigma_N^*}) \to J(a^*)\Big)$$

which is the case, for example, of the square, and can be explained by the spillover pheomenon [2].

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How to build $\chi_{\Sigma_N^*}$?

Through optimality conditions, one can prove that :

$$\Sigma_N^* \subset \{\varphi^* \geq \Lambda^*\}\,.$$

- Λ^* is the Lagrange multiplier associated to the area constraint.
- $\varphi^* = \sum_{1\leq j\leq N} \alpha_j^* \frac{1}{\lambda_j} \left(\frac{\partial \phi_j}{\partial \boldsymbol{n}}\right)^2$, where α_j^* are the Lagrange multipliers associated with the constraint on the infimum.

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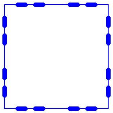
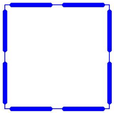
Remark : If Ω has an analytic boundary, then φ^* is analytic on $\partial\Omega$. Thus :

- Ether the measure of $\{\varphi^* = \Lambda^*\}$ is null, and thus Σ_N^* is the following level-set $\Sigma_N^* = \{\varphi^* \geq \Lambda^*\}$.
- Or $\varphi^* = \Lambda^*$ on $\partial\Omega$, and thus one cannot determine a priori Σ_N^*...

Examples of solutions for the square for $L = 1/3$, $L = 2/3$ and for 2 and 5 modes :





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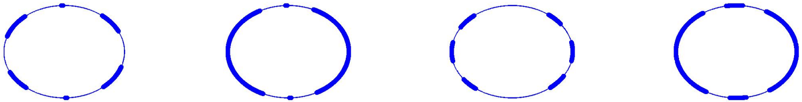

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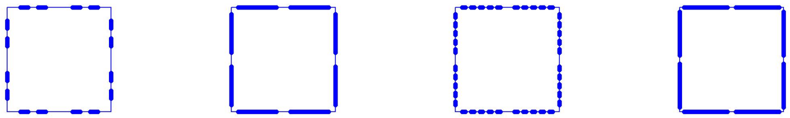

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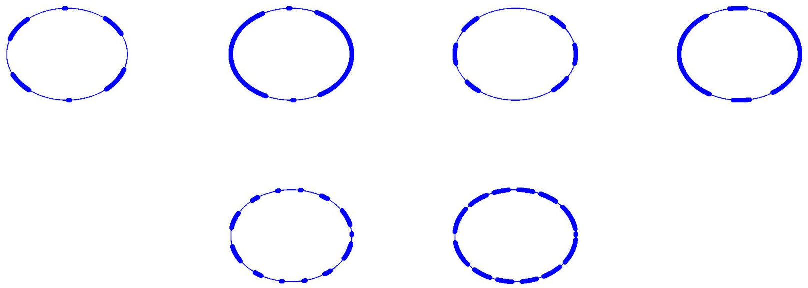
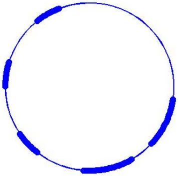


The disk provides a case where φ^* is constant. ($L = 1/3$)



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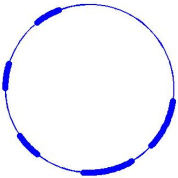

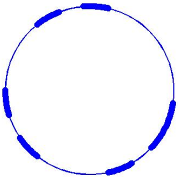

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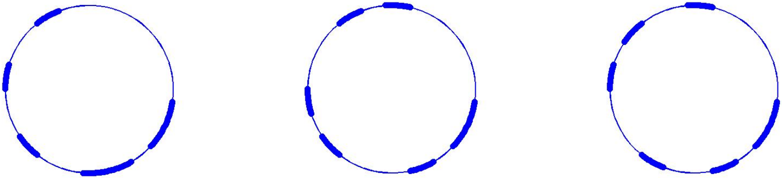

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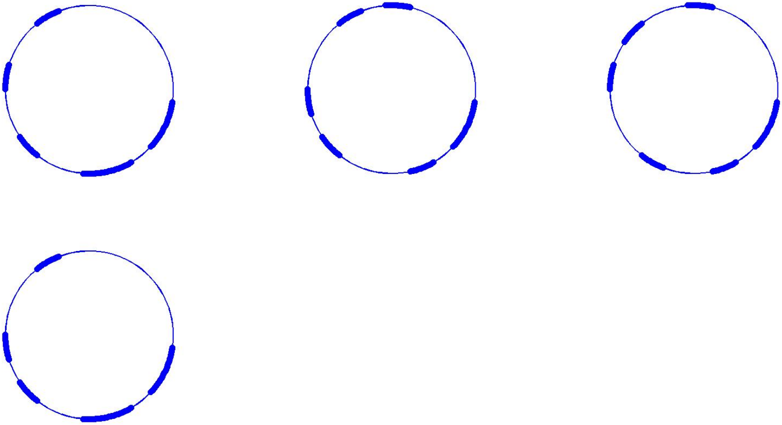

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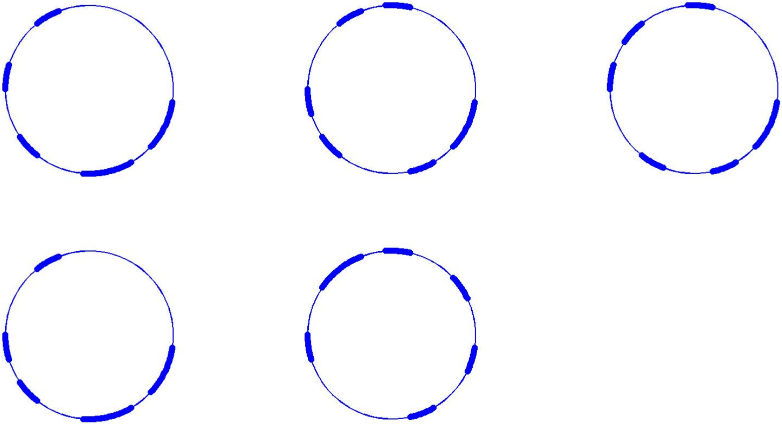

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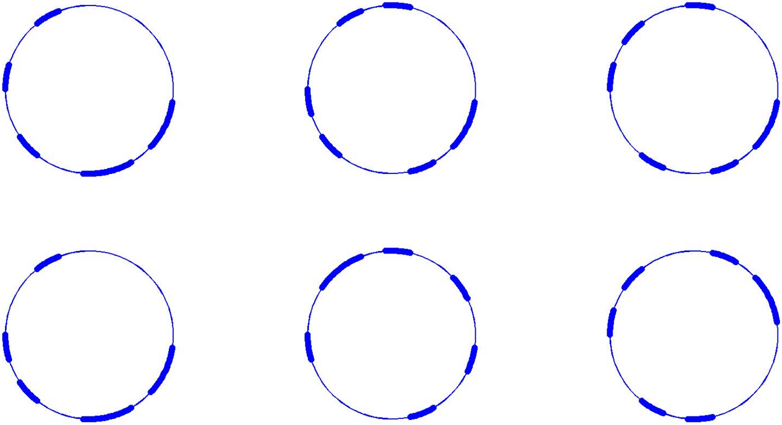

Outlook

- Extend those results to other PDEs.
- Study the optimal value when the Rellich-type solutions $a^*_{x_0}$ are no more admissibles.
- Compute maximizing sequences when $a^*_{x_0} \in \overline{\mathcal{U}}_L$.
- Study the numerical behaviour of the observability constant.

Bibliography :

1- A. Hassel, S. Zelditch, *Quantum ergodicity of boundary values of eigenfunctions* Commun. Math. Phys. 248, 119-168 (2004).

2- P. Hebrard, A. Henrot, *A spillover phenomenon in the optimal location of actuators*, SIAM J. Control Optim. 44 (2005), 349-366.

3- A. Munch, P. Pedregal, F. Periago, *Optimal design of the damping set for the stabilization of the wave equation*, Journal of Differential Equation, 231(1), 331-358 (2006).

4- Y. Privat, E. Trélat, E. Zuazua, *Optimal observability of the multi-dimensional wave and Schrödinger equations in quantum ergodic domains*, to appear in J. Eur. Math. Soc.

Thank you for your attention.