Partial controllability of parabolic systems

Michel Duprez

Laboratoire de Mathématiques de Besançon

Sept. 30th 2014

Contrôle, Problème inverse et Applications.

Motivations

Let be $T > 0$ and $\omega \subset \Omega \subset \mathbb{R}^N$.
If we consider, for example, the following parabolic system

$$\text{Find } y := (y_1, y_2)^* : Q_T \to \mathbb{R}^2 \text{ such that}$$
$$\begin{cases} \partial_t y_1 = \Delta y_1 + y_2 + \mathbb{1}_\omega u & \text{in } Q_T := \Omega \times (0, T), \\ \partial_t y_2 = \Delta y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T := \partial\Omega \times (0, T), \\ y(0) = y_0 & \text{in } \Omega, \end{cases}$$

can we find, for all initial condition y_0, a control u such that

$$y_1(T; y_0, u) = 0 \ ?$$

Setting

We consider here the following system of **n** linear parabolic equations with **m** controls

$$\begin{cases} \partial_t y = \Delta y + Ay + B\mathbb{1}_\omega u & \text{in } Q_T := \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma_T := \partial\Omega \times (0,T), \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \quad (1)$$

where $A := (a_{ij})_{ij}$ and $B := (b_{ik})_{ik}$ with $a_{ij},\ b_{ik} \in L^\infty(Q_T)$ for all $1 \leqslant i,j \leqslant n$ and $1 \leqslant k \leqslant m$.

We denote by

$$\begin{array}{rccc} P: & \mathbb{R}^p \times \mathbb{R}^{n-p} & \to & \mathbb{R}^p, \\ & (y_1, y_2)^* & \mapsto & y_1. \end{array}$$

Setting

We consider here the following system of **n** linear parabolic equations with **m** controls

$$\begin{cases} \partial_t y = \Delta y + Ay + B\mathbb{1}_\omega u & \text{in } Q_T := \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma_T := \partial\Omega \times (0,T), \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \quad (1)$$

where $A := (a_{ij})_{ij}$ and $B := (b_{ik})_{ik}$ with $a_{ij},\ b_{ik} \in L^\infty(Q_T)$ for all $1 \leqslant i,j \leqslant n$ and $1 \leqslant k \leqslant m$.

We denote by

$$\begin{array}{rccc} P: & \mathbb{R}^p \times \mathbb{R}^{n-p} & \to & \mathbb{R}^p, \\ & (y_1, y_2)^* & \mapsto & y_1. \end{array}$$

Definitions

We will say that system (1) is

- **partially approximately controllable** on $(0, T)$ if for all $\epsilon > 0$ and $y_0,\ y_T \in L^2(\Omega)^n$ there exists a control $u \in L^2(Q_T)^m$ such that

$$\|Py(\cdot, T; y_0, u) - Py_T(\cdot)\|_{L^2(\Omega)^p} \leqslant \epsilon.$$

- **partially null controllable** on $(0, T)$ if for all $y_0 \in L^2(\Omega)^n$, there exists a $u \in L^2(Q_T)^m$ such that

$$Py(\cdot, T; y_0, u) = 0 \text{ in } \Omega.$$

Definitions

We will say that system (1) is

- **partially approximately controllable** on $(0, T)$ if for all $\epsilon > 0$ and $y_0,\ y_T \in L^2(\Omega)^n$ there exists a control $u \in L^2(Q_T)^m$ such that

$$\|Py(\cdot, T; y_0, u) - Py_T(\cdot)\|_{L^2(\Omega)^p} \leqslant \epsilon.$$

- **partially null controllable** on $(0, T)$ if for all $y_0 \in L^2(\Omega)^n$, there exists a $u \in L^2(Q_T)^m$ such that

$$Py(\cdot, T; y_0, u) = 0 \text{ in } \Omega.$$

Plan

1. Partial null controllability
   - Autonomous case for differential equations
   - Constant matrices
   - Time dependent matrices
2. Open problem
3. Conclusions and perspectives

Plan

1. Partial null controllability
   - Autonomous case for differential equations
   - Constant matrices
   - Time dependent matrices
2. Open problem
3. Conclusions and perspectives

System of differential equations

Assume that $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ and consider the system

$$\begin{cases} \partial_t y = Ay + Bu & \text{in } (0,T), \\ y(0) = y_0 \in \mathbb{R}. \end{cases} \quad (2) \qquad \begin{array}{rcl} P: \mathbb{R}^p \times \mathbb{R}^{n-p} & \to & \mathbb{R}^p, \\ (y_1, y_2)^* & \mapsto & y_1. \end{array}$$

THEOREM (Kalman, 1969)

System (2) is ***controllable*** *on $(0,T)$ if and only if*

$$\text{rank}(B|AB|...|A^{n-1}B) = n.$$

THEOREM

System (2) is ***partially controllable*** *on $(0,T)$ if and only if*

$$\text{rank}(PB|PAB|...|PA^{n-1}B) = p.$$

System of differential equations

Assume that $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ and consider the system

$$\left\{\begin{array}{ll} \partial_t y = Ay + Bu & \text{in } (0,T), \\ y(0) = y_0 \in \mathbb{R}. \end{array}\right. \qquad (2)$$

$$P: \begin{array}{ccc} \mathbb{R}^p \times \mathbb{R}^{n-p} & \to & \mathbb{R}^p, \\ (y_1, y_2)^* & \mapsto & y_1. \end{array}$$

THEOREM (Kalman, 1969)

*System (2) is **controllable** on $(0,T)$ if and only if*

$$\text{rank}(B|AB|...|A^{n-1}B) = n.$$

THEOREM

*System (2) is **partially controllable** on $(0,T)$ if and only if*

$$\text{rank}(PB|PAB|...|PA^{n-1}B) = p.$$

System of differential equations

Assume that $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ and consider the system

$$\begin{cases} \partial_t y = Ay + Bu & \text{in } (0,T), \\ y(0) = y_0 \in \mathbb{R}. \end{cases} \quad (2) \qquad P: \begin{array}{ccc} \mathbb{R}^p \times \mathbb{R}^{n-p} & \to & \mathbb{R}^p, \\ (y_1, y_2)^* & \mapsto & y_1. \end{array}$$

THEOREM (Kalman, 1969)

System (2) is ***controllable*** *on $(0,T)$ if and only if*

$$\mathrm{rank}(B|AB|...|A^{n-1}B) = n.$$

THEOREM

System (2) is ***partially controllable*** *on $(0,T)$ if and only if*

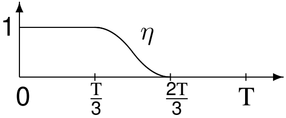
$$\mathrm{rank}(PB|PAB|...|PA^{n-1}B) = p.$$

Sketch of proof for the null controllability for n=3, m=1

$\boxed{\Leftarrow}$ • Let us first consider the variable change : $z := y - \eta\overline{y}$, where

$$\begin{cases} \partial_t \overline{y} = A\overline{y} & \text{in } [0,T], \\ \overline{y}(0) = y_0 \in \mathbb{R} \end{cases}$$

and $\eta \in \mathcal{C}^\infty[0,T]$ with the following form



Thus, if $h := -\partial_t \eta \overline{y}$, z is solution to

$$\begin{cases} \partial_t z = Az + Bu + h & \text{in } [0,T], \\ z(0) = 0. \end{cases}$$

Sketch of proof for the null controllability for n=3, m=1

- We recall that

$$\begin{cases} \partial_t z = Az + Bu + h & \text{in } [0,T], \\ z(0) = 0. \end{cases} \quad (2)$$

If we search a solution of the form $z := Kw$ with $K := (B|AB|A^2B)$ invertible, w satisfies

$$K\partial_t w = AKw + Bu + h.$$

Using the Cayley Hamilton Theorem, $A^3 := \alpha_0 I + \alpha_1 A + \alpha_2 A^2$.

$$\text{Thus } \begin{cases} AK = (AB|A^2B|A^3B) = KC, \\ Ke_1 = B, \end{cases} \text{ with } C = \begin{pmatrix} 0 & 0 & \alpha_0 \\ 1 & 0 & \alpha_1 \\ 0 & 1 & \alpha_2 \end{pmatrix}.$$

Sketch of proof for the null controllability for n=3, m=1

We recall that $h := -\partial_t \eta \overline{y}$.
Thus w is solution to

$$\begin{cases} \partial_t w_1 & = & & & \alpha_0 w_3 + h_1 + u, \\ \partial_t w_2 & = & w_1 + & & \alpha_1 w_3 + h_2, \\ \partial_t w_3 & = & & w_2 + & \alpha_2 w_3 + h_3, \\ w(0) & = & 0. & & \end{cases}$$

We choose

$$\begin{cases} w_3 & = & 0, \\ w_2 & = & -h_3, \\ w_1 & = & \partial_t w_2 - \alpha_1 w_2 - h_2, \\ u & = & \partial_t w_1 - \alpha_0 w_1 - h_1. \end{cases}$$

Conclusion :

$$w_i(0) = w_i(T) = 0 \text{ for all } i \in \{1, 2, 3\}. \qquad \square$$

Constant matrices

Assume that $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ and consider the system

$$\begin{cases} \partial_t y = \Delta y + Ay + B\mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{2}$$

THEOREM (Ammar-Khodja et al, 2009)

*System (2) is **null controllable** on $(0, T)$ if and only*

$$\operatorname{rank}(B|AB|...|A^{n-1}B) = n.$$

THEOREM

*System (2) is **partially null controllable** on $(0, T)$ if and only*

$$\operatorname{rank}(PB|PAB|...|PA^{n-1}B) = p.$$

Constant matrices

Assume that $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ and consider the system

$$\begin{cases} \partial_t y = \Delta y + Ay + B\mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{2}$$

THEOREM (Ammar-Khodja et al, 2009)

*System (2) is **null controllable** on $(0, T)$ if and only*

$$\text{rank}(B|AB|...|A^{n-1}B) = n.$$

THEOREM

*System (2) is **partially null controllable** on $(0, T)$ if and only*

$$\text{rank}(PB|PAB|...|PA^{n-1}B) = p.$$

Constant matrices

Assume that $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ and consider the system

$$\begin{cases} \partial_t y = \Delta y + Ay + B\mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{2}$$

THEOREM (Ammar-Khodja et al, 2009)

System (2) is ***null controllable*** *on* $(0, T)$ *if and only*

$$\mathrm{rank}(B|AB|...|A^{n-1}B) = n.$$

THEOREM

System (2) is ***partially null controllable*** *on* $(0, T)$ *if and only*

$$\mathrm{rank}(PB|PAB|...|PA^{n-1}B) = p.$$

Proof in the case n=3,m=1, p=1, P=(1,0,0)

Let us consider the system

$$\begin{cases} \partial_t y = \Delta y + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} y + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } \Omega. \end{cases}$$

Goal : Find $u \in L^2(Q_T)$ such that

$$y_1(T; y_0, u) = 0.$$

Strategy : Find a variable change $y = M(t)w$ with w solution of a cascade system and use

[1] González-Burgos M, de Teresa L. : Controllability results for cascade systems of m coupled parabolic PDEs by one control force

Proof in the case n=3,m=1, p=1, P=(1,0,0)

Let us consider the system

$$\left\{\begin{array}{rcll} \partial_t y & = & \Delta y + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} y + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \mathbb{1}_\omega u & \text{in } Q_T, \\ y & = & 0 & \text{on } \Sigma_T, \\ y(0) & = & y_0 & \text{in } \Omega. \end{array}\right.$$

Goal : Find $u \in L^2(Q_T)$ such that

$$y_1(T; y_0, u) = 0.$$

Strategy : Find a variable change $y = M(t)w$ with w solution of a cascade system and use

[1] González-Burgos M, de Teresa L. : Controllability results for cascade systems of m coupled parabolic PDEs by one control force

Proof in the case n=3,m=1, p=1, P=(1,0,0)

Let us consider the system

$$\left\{\begin{array}{rcll} \partial_t y & = & \Delta y + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} y + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \mathbb{1}_\omega u & \text{in } Q_T, \\ y & = & 0 & \text{on } \Sigma_T, \\ y(0) & = & y_0 & \text{in } \Omega. \end{array}\right.$$

Goal : Find $u \in L^2(Q_T)$ such that

$$y_1(T; y_0, u) = 0.$$

Strategy : Find a variable change $y = M(t)w$ with w solution of a cascade system and use

[1] González-Burgos M, de Teresa L. : Controllability results for cascade systems of m coupled parabolic PDEs by one control force

Proof in the case n=3,m=1, p=1, P=(1,0,0)

Let be

$$K := [A|B] = (B|AB|A^2B),$$
$$s := \operatorname{rank}(K),$$
$$X := \operatorname{span}\langle B, AB, A^2B\rangle.$$

(i) s=1:

Since $\begin{cases} B \neq 0 \\ \operatorname{rank}(B|AB|A^2B) = 1) \end{cases}$, then $X = \operatorname{span}\langle B\rangle$.

In particular

$$AB := \alpha_1 B \text{ and } A^2B := \alpha_2 B.$$

Proof in the case n=3,m=1, p=1, P=(1,0,0)

Let be

$$K := [A|B] = (B|AB|A^2B),$$
$$s := \mathrm{rank}(K),$$
$$X := \mathrm{span}\langle B, AB, A^2B\rangle.$$

(i) s=1:

Since $\begin{cases} B \neq 0 \\ \mathrm{rank}(B|AB|A^2B) = 1) \end{cases}$, then $X = \mathrm{span}\langle B\rangle$.

In particular

$$AB := \alpha_1 B \ \text{ and } \ A^2B := \alpha_2 B.$$

Proof in the case n=3,m=1, p=1, P=(1,0,0)

$\boxed{\Leftarrow}$ Let us suppose that $\mathrm{rank}(PK) = 1$.
We define

$$M(t) := (B|M_2(t)|M_3(t)),$$

where

$$\begin{cases} \partial_t M_2 = AM_2, \\ M_2(T) = e_2 \end{cases} \text{and} \begin{cases} \partial_t M_3 = AM_3, \\ M_3(T) = e_3. \end{cases}$$

We have $b_1 \neq 0$, indeed

$$\begin{aligned} \mathrm{rank}(PB|PAB|PA^2B) &= \mathrm{rank}(PB|\alpha_1 PB|\alpha_2 PB) \\ &= \mathrm{rank}(b_1|\alpha_1 b_1|\alpha_2 b_1) = 1. \end{aligned}$$

Then $M(T)$ is invertible and there exists T^* such that $M(t)$ is invertible in $[T^*, T]$.

Proof in the case n=3,m=1, p=1, P=(1,0,0)

Matrix M satisfies

$$\begin{cases} -\partial_t M(t) + AM(t) = (AB|0|0) = M(t)C, \\ M(t)e_1 = B, \end{cases}$$

with C given by

$$C := \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If $\mathbf{T}^* = \mathbf{0}$: Since $M(t)$ is invertible on $[0,T]$, y is the solution to system (2) if and only if $w := (w_1, w_2, w_3)^* = M(t)^{-1}y$ is the solution to the cascade system

$$\begin{cases} \partial_t w = \Delta w + Cw + e_1 \mathbb{1}_\omega u & \text{in } Q_T, \\ w = 0 & \text{on } \Sigma_T, \\ w(0) = w_0 & \text{in } \Omega. \end{cases}$$

Proof in the case n=3,m=1, p=1, P=(1,0,0)

We recall that $y = M(t)w$ in Q_T and

$$M(T) = \begin{pmatrix} b_1 & 0 & 0 \\ b_2 & 1 & 0 \\ b_3 & 0 & 1 \end{pmatrix}.$$

Thus $y_1(T) = b_1 w_1(T) = 0$ in Ω.
And it is possible to find $u \in L^2(Q_T)$ such that the solution to cascade system satisfies

$$w_1(T) = 0 \text{ in } \Omega.$$

If now $\mathbf{T^* \neq 0}$:

The system is considered without control until T^* and we control only on $[T^*, T]$.

Proof in the case n=3,m=1, p=1, P=(1,0,0)

We recall that $y = M(t)w$ in Q_T and

$$M(T) = \begin{pmatrix} b_1 & 0 & 0 \\ b_2 & 1 & 0 \\ b_3 & 0 & 1 \end{pmatrix}.$$

Thus $y_1(T) = b_1 w_1(T) = 0$ in Ω.
And it is possible to find $u \in L^2(Q_T)$ such that the solution to cascade system satisfies

$$w_1(T) = 0 \text{ in } \Omega.$$

If now $\mathbf{T^* \neq 0}$:

The system is considered without control until T^* and we control only on $[T^*, T]$.

Proof in the case n=3,m=1, p=1, P=(1,0,0)

$\boxed{\Rightarrow}$ If now $\text{rank}(PB|PAB|PA^2B) \neq 1$. Let us remark that

$$\text{rank}P[A|B] \leqslant \text{rank}[A|B] = s = 1.$$

Thus $\text{rank}(PB|PAB|PA^2B) = 0$ and $PB = PAB = PA^2B = 0$.
Since $B \neq 0$, let us suppose that $b_2 \neq 0$.
We define

$$M := (B|e_1|e_3) = \begin{pmatrix} 0 & 1 & 0 \\ b_2 & 0 & 0 \\ b_3 & 0 & 1 \end{pmatrix}.$$

We have $Ae_1 := c_{12}B + c_{22}e_1 + c_{32}e_3$, $Ae_3 := c_{13}B + c_{23}e_1 + c_{33}e_3$.
Hence

$$\begin{cases} AM = (\alpha_1 B|Ae_1|Ae_3) = MC, \\ Ke_1 = B, \end{cases}$$

with C given by $C := \begin{pmatrix} \alpha_1 & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & c_{32} & c_{33} \end{pmatrix}$.

Proof in the case n=3,m=1, p=1, P=(1,0,0)

Since K is invertible, y is the solution to (2) if and only if $w := (w_1, w_2, w_3)^* = M^{-1}y$ is the solution to cascade system

$$\begin{cases} \partial_t w = \Delta w + Cw + e_1 \mathbb{1}_\omega u & \text{in } Q_T, \\ w = 0 & \text{on } \Sigma_T, \\ w(0) = M^{-1} y_0 & \text{in } \Omega. \end{cases}$$

Since M is given by

$$M := (B|e_1|e_3) = \begin{pmatrix} 0 & 1 & 0 \\ b_2 & 0 & 0 \\ b_3 & 0 & 1 \end{pmatrix}.$$

Thus we have

$$y_1(T) = 0 \text{ in } \Omega \Leftrightarrow w_2(T) = 0 \text{ in } \Omega.$$

Proof in the case n=3,m=1, p=1, P=(1,0,0)

Since K is invertible, y is the solution to (2) if and only if $w := (w_1, w_2, w_3)^* = M^{-1}y$ is the solution to cascade system

$$\begin{cases} \partial_t w = \Delta w + Cw + e_1 \mathbb{1}_\omega u & \text{in } Q_T, \\ w = 0 & \text{on } \Sigma_T, \\ w(0) = M^{-1}y_0 & \text{in } \Omega. \end{cases}$$

Since M is given by

$$M := (B|e_1|e_3) = \begin{pmatrix} 0 & 1 & 0 \\ b_2 & 0 & 0 \\ b_3 & 0 & 1 \end{pmatrix}.$$

Thus we have

$$y_1(T) = 0 \text{ in } \Omega \Leftrightarrow w_2(T) = 0 \text{ in } \Omega.$$

Remark on the general dual system :

$$\begin{cases} -\partial_t \varphi = \Delta\varphi + A^*\varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(T,\cdot) = P^*\varphi_T & \text{in } \Omega. \end{cases} \quad \text{with} \quad P^* : \begin{array}{ccc} \mathbb{R}^p & \to & \mathbb{R}^p \times \mathbb{R}^{n-p} \\ \varphi_T & \mapsto & (\varphi_T, 0_{n-p})^*. \end{array}$$

PROPOSITION

1. *System (1) is* ***partially null controllable*** *on $(0,T)$, if and only if there exists $C_{obs} > 0$ such that for all $\varphi_T \in L^2(\Omega)^p$*

$$\|\varphi(0)\|^2_{L^2(\Omega)^n} \leqslant C_{obs} \int_0^T \|B^*\varphi\|^2_{L^2(\omega)^m}$$

with φ the solution to the dual system.

2. *System (1) is* ***partially approx. controllable*** *on $(0,T)$, if and only if for all $\varphi_T \in L^2(\Omega)^p$ the solution to the dual system satisfies*

$$B^*\varphi = 0 \text{ in } \omega \times (0,T) \Rightarrow \varphi = 0 \text{ in } Q_T.$$

Remark on the general dual system :

$$\begin{cases} -\partial_t \varphi = \Delta \varphi + A^* \varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(T, \cdot) = P^* \varphi_T & \text{in } \Omega. \end{cases} \quad \text{with} \quad P^* : \begin{array}{ccc} \mathbb{R}^p & \to & \mathbb{R}^p \times \mathbb{R}^{n-p} \\ \varphi_T & \mapsto & (\varphi_T, 0_{n-p})^*. \end{array}$$

PROPOSITION

1. *System (1) is* ***partially null controllable*** *on $(0, T)$, if and only if there exists $C_{obs} > 0$ such that for all $\varphi_T \in L^2(\Omega)^p$*

$$\|\varphi(0)\|^2_{L^2(\Omega)^n} \leqslant C_{obs} \int_0^T \|B^* \varphi\|^2_{L^2(\omega)^m}$$

   with φ the solution to the dual system.

2. *System (1) is* ***partially approx. controllable*** *on $(0, T)$, if and only if for all $\varphi_T \in L^2(\Omega)^p$ the solution to the dual system satisfies*

$$B^* \varphi = 0 \text{ in } \omega \times (0, T) \Rightarrow \varphi = 0 \text{ in } Q_T.$$

Remark on the general dual system :

$$\begin{cases} -\partial_t \varphi = \Delta\varphi + A^*\varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(T,\cdot) = P^*\varphi_T & \text{in } \Omega. \end{cases} \quad \text{with } P^* : \begin{array}{ccc} \mathbb{R}^p & \to & \mathbb{R}^p \times \mathbb{R}^{n-p} \\ \varphi_T & \mapsto & (\varphi_T, 0_{n-p})^*. \end{array}$$

Proposition

1. *System (1) is* ***partially null controllable*** *on* $(0,T)$*, if and only if there exists* $C_{obs} > 0$ *such that for all* $\varphi_T \in L^2(\Omega)^p$

$$\|\varphi(0)\|^2_{L^2(\Omega)^n} \leqslant C_{obs} \int_0^T \|B^*\varphi\|^2_{L^2(\omega)^m}$$

with φ *the solution to the dual system.*

2. *System (1) is* ***partially approx. controllable*** *on* $(0,T)$*, if and only if for all* $\varphi_T \in L^2(\Omega)^p$ *the solution to the dual system satisfies*

$$B^*\varphi = 0 \text{ in } \omega \times (0,T) \;\Rightarrow \varphi = 0 \text{ in } Q_T.$$

Proof in the case n=3,m=1, p=1, P=(1,0,0)

The previous cascade system is partially null controllable if and only if there exists $C > 0$ such that for all $\varphi_2^T \in L^2(\Omega)$ the solution to the dual system

$$\begin{cases} -\partial_t \varphi_1 = \Delta \varphi_1 + \alpha_1 \varphi_1 & \text{in } Q_T, \\ -\partial_t \varphi_2 = \Delta \varphi_2 + c_{12}\varphi_1 + c_{22}\varphi_2 + c_{32}\varphi_3 & \text{in } Q_T, \\ -\partial_t \varphi_3 = \Delta \varphi_3 + c_{13}\varphi_1 + c_{23}\varphi_2 + c_{33}\varphi_3 & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(T) = (0, \varphi_2^T, 0)^* & \text{in } \Omega \end{cases}$$

satisfies the observability inequality

$$\int_\Omega \varphi(0)^2 \leqslant C \int_{\omega \times (0,T)} \varphi_1^2.$$

We have $\varphi_1(T) = 0 \text{ in } \Omega \Rightarrow \varphi_1 = 0 \text{ in } Q_T$
and $\varphi_2^T \neq 0 \text{ in } \Omega \Rightarrow (\varphi_2(0), \varphi_3(0))^* \neq 0 \text{ in } \Omega.$
Thus the observability inequality is not satisfy.

Proof in the case n=3,m=1, p=1, P=(1,0,0)

The previous cascade system is partially null controllable if and only if there exists $C > 0$ such that for all $\varphi_2^T \in L^2(\Omega)$ the solution to the dual system

$$\begin{cases} -\partial_t\varphi_1 = \Delta\varphi_1 + \alpha_1\varphi_1 & \text{in } Q_T, \\ -\partial_t\varphi_2 = \Delta\varphi_2 + c_{12}\varphi_1 + c_{22}\varphi_2 + c_{32}\varphi_3 & \text{in } Q_T, \\ -\partial_t\varphi_3 = \Delta\varphi_3 + c_{13}\varphi_1 + c_{23}\varphi_2 + c_{33}\varphi_3 & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(T) = (0, \varphi_2^T, 0)^* & \text{in } \Omega \end{cases}$$

satisfies the observability inequality

$$\int_\Omega \varphi(0)^2 \leqslant C \int_{\omega\times(0,T)} \varphi_1^2.$$

We have $\varphi_1(T) = 0 \text{ in } \Omega \Rightarrow \varphi_1 = 0 \text{ in } Q_T$
and $\varphi_2^T \not\equiv 0 \text{ in } \Omega \Rightarrow (\varphi_2(0), \varphi_3(0))^* \not\equiv 0 \text{ in } \Omega.$
Thus the observability inequality is not satisfy.

Proof in the case n=3,m=1, p=1, P=(1,0,0)

(ii) s=2: We take $M(t) := (B|AB|M_3(t))$ with $M_3(T) = e_2$ or e_3.
(iii) s=3: We take $M := K = (B|AB|A^2B)$.

$\square$

Remark : We prove also the partial approximate controllability.

Time dependent matrices

Let us suppose that $A \in \mathcal{C}^{n-1}([0,T];\mathcal{L}(\mathbb{R}^n))$ and $B \in \mathcal{C}^{n}([0,T];\mathcal{L}(\mathbb{R}^m;\mathbb{R}^n))$ and define

$$\begin{cases} B_0(t) := B(t), \\ B_i(t) := A(t)B_{i-1}(t) - \partial_t B_{i-1}(t), \; 1 \leqslant i \leqslant n-1. \end{cases}$$

We can define for all $t \in [0,T]$

$$[A|B](t) := (B_0(t)|B_1(t)|...|B_{l-1}(t)).$$

Theorem (Ammar-Khodja et al 09)

1. *If there exist $t_0 \in [0, T]$ such that*

$$\mathrm{rank}[A|B](t_0) = n,$$

then system (1) is null controllable on (0,T).

2. *System (1) is null controllable on every interval (T_0, T_1) with $T_0 < T_1 \leqslant T$ if and only if there exists a dense subset E of $(0, T)$ such that $\mathrm{rank}[A|B](t) = n$ for every $t \in E$.*

Theorem

If

$$\mathrm{rank}\, P[A|B](T) = p,$$

then system (1) is partially null controllable on (0,T).

Theorem (Ammar-Khodja et al 09)

1. *If there exist* $t_0 \in [0, T]$ *such that*

$$\mathrm{rank}[A|B](t_0) = n,$$

then system (1) is null controllable on (0,T).

2. *System (1) is null controllable on every interval* (T_0, T_1) *with* $T_0 < T_1 \leqslant T$ *if and only if there exists a dense subset* E *of* $(0, T)$ *such that* $\mathrm{rank}[A|B](t) = n$ *for every* $t \in E$.

Theorem

If

$$\mathrm{rank}\, P[A|B](T) = p,$$

then system (1) is partially null controllable on (0,T).

Plan

An example with $A = A(t)$

Let us consider the control problem

$$\begin{cases} \partial_t y_1 = \Delta y_1 + a y_2 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 & \text{in } Q_T. \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0, y_1(T) = 0 & \text{in } \Omega. \end{cases}$$

If $\mathbf{a} := \mathbf{a}(\mathbf{t})$, let us consider the change of variable $z_1 := y_1 + \mu y_2$

$$(\partial_t - \Delta) z_1 = (\partial_t \mu + a) y_2 + \mathbb{1}_\omega u \text{ in } Q_T.$$

If we choose μ solution to

$$\begin{cases} \partial_t \mu = -a \text{ in } [0, T], \\ \mu(T) = 0, \end{cases}$$

the system is not coupled. And thus it is partially null controllable.

Difficulty when $A = A(x)$

Let us consider the same control problem

$$\begin{cases} \partial_t y_1 = \Delta y_1 + a y_2 + \mathbb{1}_\omega u & \text{in } Q_T. \\ \partial_t y_2 = \Delta y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0, y_1(T) = 0 & \text{in } \Omega. \end{cases}$$

If now $\mathbf{a} := \mathbf{a}(\mathbf{x})$, let us consider the change of variable $z_1 := y_1 + \mu y_2$

$$(\partial_t - \Delta) z_1 = (a - \partial_t \mu + \Delta\mu + 2\nabla\mu \cdot \nabla) y_2 + \mathbb{1}_\omega u \text{ in } Q_T$$

What kind of μ can be chosen in this situation ???

[1] Benabdallah A.; Cristofol M.; Gaitan P.; De Teresa, Luz : Controllability to trajectories for some parabolic systems of three and two equations by one control force (2014).

[2] Coron J-M, Lissy P. : Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components (2014)

Difficulty when $A = A(x)$

Let us consider the same control problem

$$\begin{cases} \partial_t y_1 = \Delta y_1 + a y_2 + \mathbb{1}_\omega u & \text{in } Q_T. \\ \partial_t y_2 = \Delta y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0, y_1(T) = 0 & \text{in } \Omega. \end{cases}$$

If now $\mathbf{a} := \mathbf{a}(\mathbf{x})$, let us consider the change of variable $z_1 := y_1 + \mu y_2$

$$(\partial_t - \Delta) z_1 = (a - \partial_t \mu + \Delta\mu + 2\nabla\mu \cdot \nabla) y_2 + \mathbb{1}_\omega u \text{ in } Q_T$$

What kind of μ can be chosen in this situation ???

[1] Benabdallah A.; Cristofol M.; Gaitan P.; De Teresa, Luz : Controllability to trajectories for some parabolic systems of three and two equations by one control force (2014).

[2] Coron J-M, Lissy P. : Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components (2014)

Plan

Conclusions :

1. Necessary and sufficient conditions in the constant case.
2. Sufficient conditions in the time dependent case.
3. Same conditions concerning the (partial) approximate controllability.

Perspectives :

1. Example of non partial controllability in the space dependent case.
2. Numerical simulations and analysis.
3. Partial controllability of semilinear parabolic systems.

Conclusions :

1. Necessary and sufficient conditions in the constant case.
2. Sufficient conditions in the time dependent case.
3. Same conditions concerning the (partial) approximate controllability.

Perspectives :

1. Example of non partial controllability in the space dependent case.
2. Numerical simulations and analysis.
3. Partial controllability of semilinear parabolic systems.

Thank you for your attention !