The exterior approach applied to the inverse obstacle problem for the heat equation.

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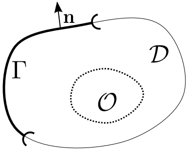
Contrôle, Problèmes Inverses et Applications
October 2014

The problem

- The data:
  - $\mathcal{D}$ open set of $\mathbb{R}^d$ ($d \geq 2$), with Lipschitz boundary
  - $T > 0$ a final time
  - $\Gamma \subset \partial\mathcal{D}$, $|\Gamma| > 0$
  - (g_D, g_N): (possibly noisy) data $\in L^2(0,T;L^2(\Gamma))$



- Problem:

Find a fixed object $\mathcal{O}$, $\overline{\mathcal{O}} \subset \mathcal{D}$, $\Omega := \mathcal{D} \setminus \overline{\mathcal{O}}$ connected, and $u \in V$, with

$$V := \left\{ v \in L^2(0,T;H^1(\Omega)),\ v' \in L^2(0,T;L^2(\Omega)) \right\}$$

s.t.

$$(\mathcal{P}) \begin{cases} u' - \Delta u = 0 & \text{in } (0,T) \times \Omega \\ u = g_D & \text{on } (0,T) \times \Gamma \\ \partial_n u = g_N & \text{on } (0,T) \times \Gamma \\ u = 0 & \text{on } (0,T) \times \partial\mathcal{O} \\ u(0,.) = 0 & \text{in } \Omega \end{cases}$$

Uniqueness result

- Based on the following uniqueness result:

Lemma

Assume $u \in V$ satisfies

$$\begin{cases} u' - \Delta u = 0 & \text{in } (0,T) \times \Omega \\ u = 0 & \text{on } (0,T) \times \Gamma \\ \partial_n u = 0 & \text{on } (0,T) \times \Gamma \end{cases}$$

then $u \equiv 0$ in $(0,T) \times \Omega$.

- Uniqueness result

Theorem

Let two domains $\mathcal{O}_1$, $\mathcal{O}_2$ and corresponding functions u_1, u_2 satisfy $(\mathcal{P})$ with the same data (g_D, g_N). Assume furthermore that for almost all $t \in (0,T)$, u_1 and u_2 are continuous w.r.t. x up to the boundary (i.e. $u_k \in L^2(0,T; C^0(\overline{\Omega}))$. Then $\mathcal{O}_1 = \mathcal{O}_2$ and $u_1 = u_2$.

State of art

• Shape derivative
- adjoint method: H. Harbrecht, J. Tausch, *On Shape Optimization with Parabolic State Equation*, preprint.
- integral equations: R. Chapko, R. Kress & J.-R. Yoon, *On the numerical solution of an inverse boundary value problem for the heat equation*, 1998.

• Several works for moving objects in fluids: C. Conca, P. Cumsille, J. Ortega & L. Rosier, *On the detection of a moving obstacle in an ideal fluid by a boundary measurement*, 1998.

Exterior approach

- Method developed for inverse obstacle problems with elliptic equations.
- The exterior approach is based on the following remark:

Property

- Let $(\omega_m)_{m\in\mathbb{N}}$ be a sequence of open sets such that $\mathcal{O} \subset \omega_{m+1} \subset \omega_m$. Then

$$\boxed{\omega_m \xrightarrow[m\to\infty]{d_H} \omega \supset \mathcal{O}}$$

- Suppose « $u = 0$ » on $\partial\omega$ for almost all t. Then $\omega = \mathcal{O}$.

- Question is: are we able to construct a sequence $(\omega_m)_{m\in\mathbb{N}}$ such that

- $\omega_{m+1} \subset \omega_m \rightsquigarrow$ OK

- $\omega_m \supset \mathcal{O}$

- by construction, $u = 0$ on $(0, T) \times \partial\omega$

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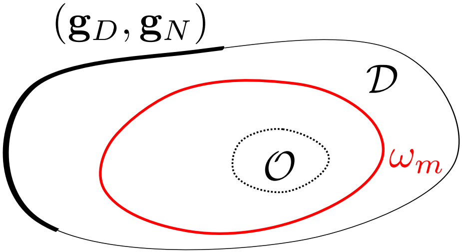
Construction of the sequence $(\omega_m)_{m\in\mathbb{N}}$

- 2-step algorithm :



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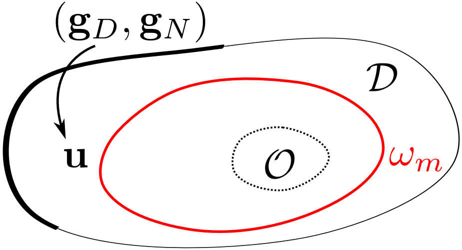

- from (g_D, g_N), obtain u in $(0, T) \times \mathcal{D} \setminus \overline{\omega_m}$

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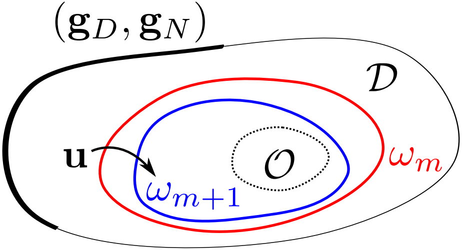

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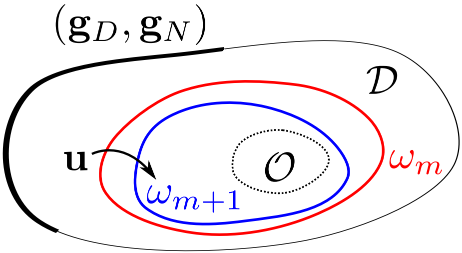

- from (g_D, g_N), obtain u in $(0, T) \times \mathcal{D} \setminus \overline{\omega_m}$ $\rightsquigarrow$ quasi-reversibility method

- from u, obtain a better exterior approximation of $\mathcal{O}$ $\rightsquigarrow$ level-set method.

Level-set method - construction

- We assume that we know $\begin{cases} \omega_m \supset \mathcal{O} \\ u \text{ in } (0,T) \times \mathcal{D} \setminus \overline{\omega_m} \end{cases}$
- Goal: construct ω_{m+1}, $\mathcal{O} \subset \omega_{m+1} \subset \omega_m$.

- Define $\tilde{u}$ and f:

- $$\left\{ \begin{array}{l} \tilde{u}(x) = \left(\int_0^T u(\eta, x)^2 \, d\eta \right)^{1/2} \quad \text{in } \Omega \\ \tilde{u}_{|\mathcal{O}} \in H_0^1(\mathcal{O})_- \end{array} \right. \Bigg| \Rightarrow \tilde{u} \in H^1(\mathcal{D}).$$

- $f \geq \Delta \tilde{u}$ (in the sense of $H^{-1}(\mathcal{D})$)

- w_{ω_m} unique solution of $\begin{cases} \Delta w = f \text{ in } \omega_m \\ w - \tilde{u} \in H_0^1(\omega_m) \end{cases}$

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Theorem

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Proof:

1) $\left.\begin{array}{l} \circ\, w_{\omega_m} \leq 0 \text{ a.e. in } \omega_{m+1} \\ \circ\, \omega \subset \omega_{m+1} \end{array}\right\} \Rightarrow w_{\omega_m} \leq 0$ a.e. in ω.

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1. Introduction
   - Inverse obstacle problem for the heat equation
   - Exterior approach
2. Level-set method
   - Construction
   - Convergence
3. Quasi-reversibility method
4. Numerical results
   - Data completion problems
   - Obstacle problems
5. Conclusion

Data completion problem

- We assume that we know $\begin{cases} \omega \supset \mathcal{O} \\ (g_D, g_N) \text{ in } (0,T)\times\Gamma, \Gamma \subset \partial\mathcal{D} \end{cases}$
- From now on, we define $\theta := \mathcal{D} \setminus \overline{\omega}$.
- Goal: retrieve $u \in V$ such that

$$(\mathcal{P}_c)\begin{cases} u' & = & \Delta u & \text{in } (0,T)\times\theta \\ u & = & g_D & \text{on } (0,T)\times\Gamma \\ \partial_n u & = & g_N & \text{on } (0,T)\times\Gamma \\ \Big(u(0,.) & = & 0 & \text{in } \theta\Big) \end{cases}$$

$\rightsquigarrow$ Ill-posed problem, admits at most one solution u. Highly sensitive to noise on data. No hope to retrieve exactly u.

- A regularization method is required, for example:

- Kohn-Vogelius functional minimization
- quasi-reversibility method.

Quasi-reversibility (QR) method

• Introduced by Robert Lattes and Jacques-Louis Lions *The method of quasi-reversibility: applications to partial differential equations* (1969).

• Main idea: replace the ill-posed problem by a family of well-posed variational ones, of higher order, depending of a small regularization parameter $\varepsilon > 0$.

• Main issue: applied to problem $(\mathcal{P}_c)$, the «standard» QR method leads to the resolution of the variational problem

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$\Rightarrow$ numerical resolution needs spatial H^2 approximation
$\Rightarrow$ convergence towards exact solution u only if u is smooth enough
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First mixed formulation of QR for heat equation

- We define $V_{g_D} := \{v \in V,\ v = g_D \text{ on } (0,T)\times\Gamma\}$,

$V_0 := \{v \in V,\ v = 0 \text{ on } (0,T)\times\Gamma\}$,

$\tilde{V}_0 := \{v \in V,\ v = 0 \text{ on } (0,T)\times\Gamma_c,\ v(0,.) = v(T,.) = 0\}$.

- QR problem: find $(u_\varepsilon, \lambda_\varepsilon) \in V_{g_D} \times \tilde{V}_0$ such that for all $(v,\mu) \in V_0 \times \tilde{V}_0$,

$$\left\{\begin{array}{l} \displaystyle\int_0^T\int_\theta \left(-v\,\lambda_\varepsilon' + \nabla v\cdot\nabla\lambda_\varepsilon\right)\,dx\,dt + \varepsilon\int_0^T\int_\theta \left(u_\varepsilon'\,v' + \nabla u_\varepsilon\cdot\nabla v\right)\,dx\,dt = 0 \\ \displaystyle\int_0^T\int_\theta \left(u_\varepsilon'\,\mu + \nabla u_\varepsilon\cdot\nabla\mu\right)\,dx\,dt - \int_0^T\int_\theta \left(\lambda_\varepsilon'\,\mu' + \nabla\lambda_\varepsilon\cdot\nabla\mu\right) = \int_0^T\int_\Gamma g_N\,\mu\,ds\,dt. \end{array}\right.$$

Theorem

This problem admits a unique solution $(u_\varepsilon, \lambda_\varepsilon)$. Furthermore, we have

$$u_\varepsilon \xrightarrow[\varepsilon\to 0]{V} u, \quad \lambda_\varepsilon \xrightarrow[\varepsilon\to 0]{V} 0.$$

- Advantage: problem posed in V, H^1 in both space and time.
- Drawbacks: λ has no physical meaning (approximation of 0). For the moment, no method to set the parameter ε if noisy data.

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Second mixed formulation of QR - Introduction of second unknown

- By definition, $u \in V := \{ v \in L^2(0,T;H^1(\theta)),\ v' \in L^2(0,T;L^2(\theta)) \}$.

Define $\mathbf{p} := \nabla u \in L^2(0,T;L^2(\theta)^d)$. We have $\nabla \cdot \mathbf{p} = \Delta u = u' \in L^2(0,T;L^2(\theta))$, and $\mathbf{p} \cdot n = \partial_n u = g_N \in L^2(0,T;L^2(\Gamma))$.

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• Abstract point of view for QR: X, Y Hilbert spaces, $P : X \mapsto Y$, P linear continuous, $\mathcal{K}(P) = \{0\}$, $\mathcal{R}(P) \neq Y$, $\overline{\mathcal{R}(P)} = Y$. Problem: for $y \in Y$, find $x \in X$ s.t. $Px = y$.

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Second mixed formulation of QR - Convergence

Theorem

For any $\varepsilon > 0$, [QR] problem admits a unique solution $(u_\varepsilon, \mathbf{p}_\varepsilon)$.
Furthermore, if $(\mathcal{P}_c)$ admits a (necessarily) unique solution $u \in V$, then

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- Advantages: u and ∇u are approximated in their natural spaces, additional unknown $\mathbf{p}_\varepsilon$ is an approximation of the flux, involved Hilbert spaces have natural F.E. approximations (Lagrange, Raviart-Thomas).

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Case of noisy data

• Noisy data (g_D^δ, g_N^δ) s.t. $\|g_k^\delta - g_k\|_{L^2(0,T;L^2(\Gamma))} \leq \delta$, $k = D$ ou N. Such data can directly be used in the mixed formulation.

• Admissible strategy: for any choice $\varepsilon := \varepsilon(\delta)$ such that

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Iterated quasi-reversibility

• Another option: set $\varepsilon > 0$, define $x_\varepsilon^{(0)}$ to be the solution of [QR] problem, and $x_\varepsilon^{(M+1)}$ the unique element of X s.t.

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Suppose it exists a unique $x \in X$ such that $Px = y$. Then, for any choice of parameter of regularization $\varepsilon > 0$, we have

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Let $y^\delta \in Y$ s.t. $\|y^\delta - y\|_Y \leq \delta$, and $x_{\varepsilon,\delta}^{(M)}$ the corresponding QR sequence. For $r > 1$, define $M(\delta) := \inf\left\{M \in \mathbb{N}, \|Px_{\varepsilon,\delta}^{(M)} - y^\delta\|_Y \leq r\,\delta\right\}$. Then

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- Initialization: $\omega_0 \supset \mathcal{O}$.

- Iterations

- obtain approximation $\mathbf{u}$ outside ω_m $\begin{cases} \text{QR: first mixed formulation} \\ \text{QR: second mixed formulation} \end{cases}$

- construct $\omega_{m+1} \rightsquigarrow$ Poisson equation.

- Parameters:

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  - f: source of Poisson problem
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One-dimensional results

- Domain of study: $(0, T) \times \theta := (0,1) \times (1,2)$, $\Gamma := \{1\}$.
- $\Delta_x \sim \frac{1}{100}$, $\Delta_t \sim \frac{1}{100} \Rightarrow$ matrix $\sim 40400 \times 40400$.
- Exact solutions: $u_1(t,x) := 2\,t + x^2$, $u_2(t,x) := \sin(x)\,\exp(-t)$.
- Noise on both Dirichlet and Neumann data, amplitude 5%. $\varepsilon := 10^{-4}$.

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u_1 u_2

Exact solution, noisy data, and reconstructed solution on $(0,1) \times \{1\}$.

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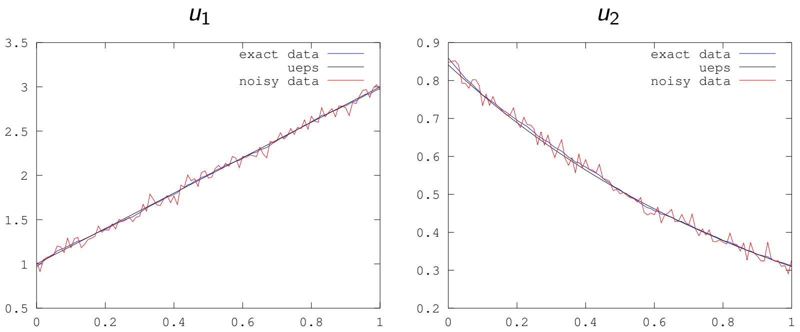

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Two-dimensional results

- $\theta := \mathcal{D} \setminus \overline{\mathcal{O}}_i$, $\Gamma := \partial\mathcal{D}$, with, for $\eta \in (0, 2\pi)$,

$$\partial\mathcal{D} := (1 + 0.01\ \sin(3\,\eta)) \begin{bmatrix} \cos(\eta) \\ \sin(\eta) \end{bmatrix},\ \partial\mathcal{O}_1 := \begin{bmatrix} 0.2 \\ 0 \end{bmatrix} + (0.4 + 0.1\cos(\eta) - 0.02\sin(3\,\eta)) \begin{bmatrix} \cos(\eta) \\ \sin(\eta) \end{bmatrix},$$

$$\partial\mathcal{O}_2 := (0.5 + 0.2\ \cos(2\,\eta) + 0.1\sin(\eta)) \begin{bmatrix} \cos(\eta) \\ \sin(\eta) \end{bmatrix}.$$

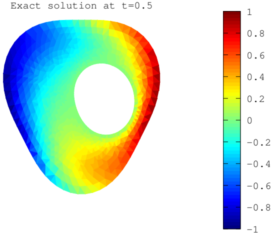
- «Exact solution» obtained by resolution of the direct problem

$$\begin{cases} u' - \Delta u = 0 \text{ in } (0,1) \times \theta \\ u = g_D \text{ on } (0,1) \times \partial\mathcal{D} \\ u = 0 \text{ on } (0,1) \times \partial\mathcal{O}_i \\ u(0,.) = 0 \text{ in } \theta \end{cases} \qquad \text{with } g_D := 4\ \cos(\theta(x,y) - 4\,\pi\,t)\,(1-t)t$$

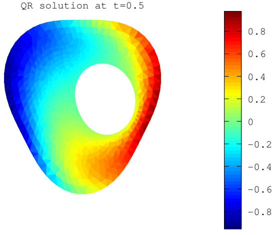
- $\varepsilon = 10^{-4}$, $\delta = 5\%$ (only on Dirichlet data), space F.E. $P_1 \times R.T_0$.

Two-dimensional results - Case $\mathcal{O}_1$

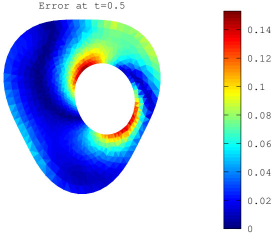
Exact solution



QR solution

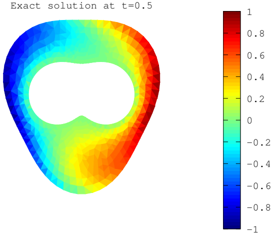


Error

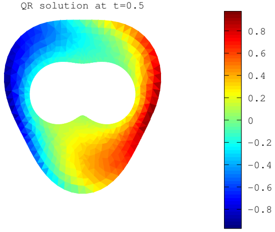


Two-dimensional results - Case $\mathcal{O}_2$

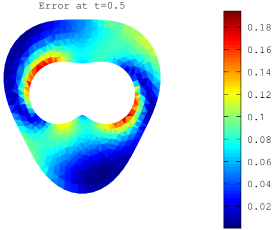
Exact solution



QR solution



Error



Obstacle problems

- Domain of study: $(0,1)\times\mathcal{D}\setminus\overline{\mathcal{O}}$, with $\mathcal{D}$ as previously and various $\mathcal{O}$, $\Gamma := \partial\mathcal{D}$.

- Data computed as previously:

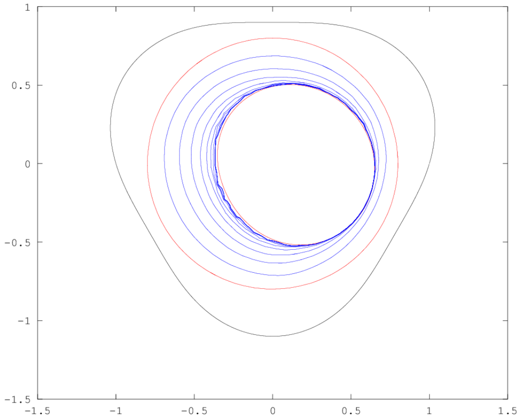
$$\left\{\begin{array}{l} u' - \Delta u = 0 \text{ in } (0,1)\times\mathcal{D}\setminus\overline{\mathcal{O}} \\ u = g_D \text{ on } (0,1)\times\partial\mathcal{D} \\ u = 0 \text{ on } (0,1)\times\partial\mathcal{O} \\ u(0,.) = 0 \text{ in } \mathcal{D}\setminus\overline{\mathcal{O}} \end{array}\right. \qquad \text{with } g_D := 4\cos(\theta(x,y) - 4\,\pi\,t)\,(1-t)t.$$

- $\varepsilon = 10^{-4}$, no noise on data, $f \sim 10$.

- We add in the QR formulation the initial condition $u(0,.) = 0$.

Testing the level-set method

• In that case, we use the exact solution as boundary condition for the Poisson problem.



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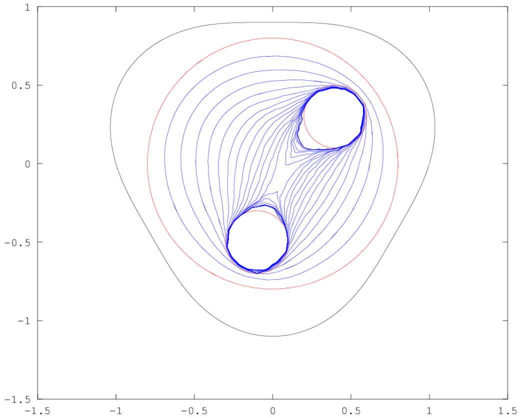
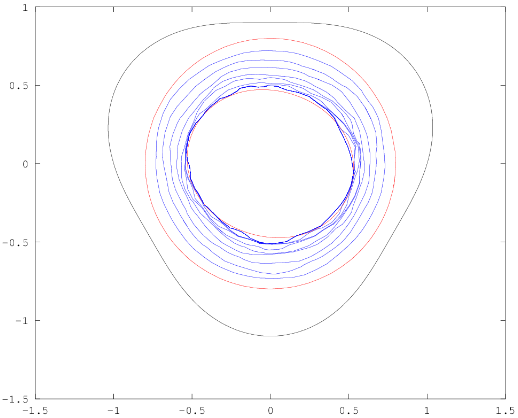


Testing the exterior approach

• In that case, we use the QR solution as boundary condition for the Poisson problem.



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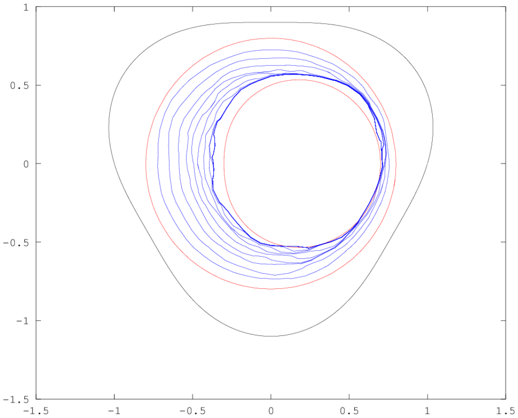

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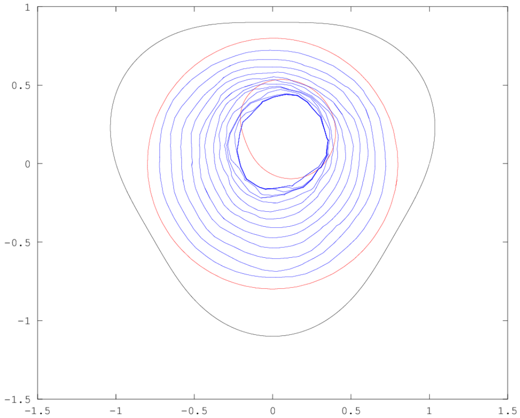

Conclusion

• Exterior approach is a method without optimization to reconstruct fixed Dirichlet obstacles, based on the coupling of

- the resolution of a Cauchy problem for parabolic equation: QR method
- a level-set method: topological changes.

• Future works:

- wave equation - fixed obstacles
- Stokes (and Navier-Stokes) equations
- B.C. on objects.

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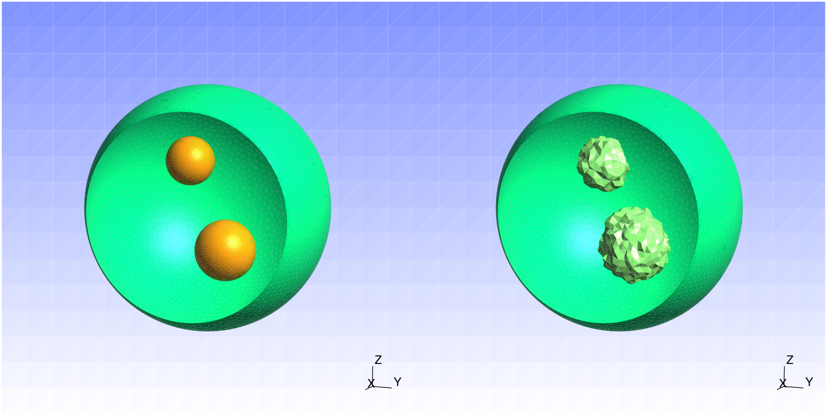
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