Relaxation of an optimal design problem in Fracture Mechanic

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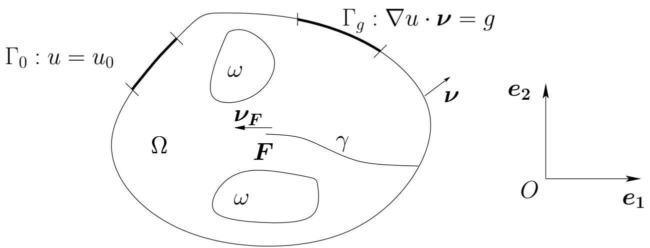

Figure: Crack domain Ω in $\mathbb{R}^2$

$$a_{\mathcal{X}_\omega}(\boldsymbol{x}) = \alpha \mathcal{X}_\omega(\boldsymbol{x}) + \beta(1 - \mathcal{X}_\omega(\boldsymbol{x})), \quad \boldsymbol{x} = (x_1, x_2) \in \Omega \tag{1}$$

$$\begin{cases} -\,div(a_{\mathcal{X}_\omega}(\boldsymbol{x})\nabla u) = 0 & \Omega, \\ u = u_0 & \Gamma_0 \subset \partial\Omega, \\ \beta\ \nabla u \cdot \boldsymbol{\nu} = g & \Gamma_g \subset \partial\Omega \end{cases} \tag{2}$$

Remark

If $g \in H^{1/2}(\partial\Omega)$ *and* $u_0 \in L^2(\partial\Omega) \Longrightarrow u \in H^1(\Omega)$.

1

1

Definition ("Energy" of the system)

$$E(u,\gamma) = \frac{1}{2}\int_\Omega a\chi_\omega |\nabla u|^2 dx - \int_{\Gamma_g} gud\sigma \tag{3}$$

Definition (Energy release rate)

*The energy release rate $\mathcal{T}$ is defined as minus the variation of E with respect the variation of **F** (in the direction e_1). Formally*

$$\mathcal{T} = -\lim_{\eta\to 0} \frac{E(u^\eta,\gamma^\eta) - E(u)}{|\gamma^\eta - \gamma|} \tag{4}$$

*where (u^η,γ^η) denotes an extension of **F**.*

*$\mathcal{T}$ - measure of the singularity of u at the point **F***

Criterion (Growth criterion of the crack, Griffith-1921)

*If $\mathcal{T}(u) \geq \mathcal{T}_c$ then **F** grows.*

$\mathcal{T}_c$ denotes an experimental value

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In order to prevent (or at least reduce) the growth of the crack, the idea is to act on the system in order to reduce the rate.

In this work, we minimize the rate with respect the distribution of α and β along the structure Ω

$$(P): \qquad \inf_{\mathcal{X}_\omega \in \mathcal{X}_{L,\mathcal{D}}} \mathcal{T}(u, \mathcal{X}_\omega)$$

where, for any $L \in (0,1)$ and a suitable compact set $\mathcal{D}$ included in $\overline{\Omega}$ such that $\boldsymbol{F} \in \mathcal{D}$,

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Very few contributions about the active control of crack growth

- P. Destuynder, *An approach to crack propagation control in structural dynamics*, C.R.Acad. Sci. Paris, Série II **306**, 953-956 (1988).
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The limit $\mathcal{T}$, finite and non negative, may be rigorously expressed in terms of u only. We introduce a velocity field

$$\boldsymbol{\psi} = (\psi_1, \psi_2) \in \boldsymbol{W} \equiv \{\boldsymbol{\psi} \in (W^{1,\infty}(\Omega, \mathbb{R}))^2, \boldsymbol{\psi} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega, \boldsymbol{\psi} = 0 \text{ on } \Gamma_g\}, \quad (5)$$

where $\boldsymbol{\nu}$ designates the unit outward normal to Ω. Let $\eta \in \mathbb{R}_+^\star$ and the transformation $\mathcal{F}^\eta : \boldsymbol{x} \to \boldsymbol{x} + \eta\boldsymbol{\psi}(\boldsymbol{x})$, so that $\mathcal{F}^\eta(\boldsymbol{F}) = \boldsymbol{F}^\eta$ and $\mathcal{F}^\eta(\gamma) = \gamma^\eta$; we first recall the following definition.

Definition (Mathematical definition of the energy release rate)

Let u be the solution of (2). The derivative of the functional $-E(u, \gamma)$ with respect to a variation of γ (more precisely of $\boldsymbol{F}$) in the direction ψ is defined as the Fréchet derivative in $\boldsymbol{W}$ at 0 of the application $\eta \to -E(u, (Id + \eta\boldsymbol{\psi})(\gamma))$, i.e.

$$-E(u, (Id + \eta\boldsymbol{\psi})(\gamma)) = -E(u, \gamma) - \eta \frac{\partial E(u, \gamma)}{\partial \gamma} \cdot \boldsymbol{\psi} + O(\eta^2). \quad (6)$$

In the sequel, we denote this derivative by $\mathcal{T}_{\boldsymbol{\psi}}(u, \mathcal{X}_\omega)$. ■

The energy release rate: the expression

Lemma

The first derivative of $-E$ with respect to γ in the direction $\boldsymbol{\psi} = (\psi_1, \psi_2) \in W$ is given by

$$\begin{aligned} \mathcal{T}_{\boldsymbol{\psi}}(u, \mathcal{X}_\omega) &= \int_\Omega a_{\mathcal{X}_\omega}(\boldsymbol{x}) \nabla u \cdot (\nabla \boldsymbol{\psi} \cdot \nabla u) dx - \frac{1}{2} \int_\Omega a_{\mathcal{X}_\omega}(\boldsymbol{x}) |\nabla u|^2 div(\boldsymbol{\psi}) dx \\ &= \int_\Omega a_{\mathcal{X}_\omega}(\boldsymbol{x}) (A_{\boldsymbol{\psi}}(\boldsymbol{x}) \nabla u, \nabla u) dx \end{aligned} \tag{7}$$

with

$$\begin{aligned} A_{\boldsymbol{\psi}}(\boldsymbol{x}) &= \nabla \boldsymbol{\psi} - \frac{1}{2} div(\boldsymbol{\psi}) I_2 = \nabla \boldsymbol{\psi} - \frac{1}{2} Tr(\nabla \boldsymbol{\psi}) I_2 \\ &= \frac{1}{2} \begin{pmatrix} \psi_{1,1} - \psi_{2,2} & 2\psi_{1,2} \\ 2\psi_{2,1} & \psi_{2,2} - \psi_{1,1} \end{pmatrix}. \end{aligned} \tag{8}$$

and where u is solution of (2).

Remark

Assuming that γ is rectilinear near $\boldsymbol{F}$ and moves along e_1, we can take $\psi_2 = 0$ so that

$$A_{\boldsymbol{\psi}}(\boldsymbol{x}) = \frac{1}{2} \begin{pmatrix} \psi_{1,1} & 2\psi_{1,2} \\ 0 & -\psi_{1,1} \end{pmatrix}. \tag{9}$$

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Since $\mathcal{T}_\psi$ is a shape derivative (with respect to $\boldsymbol{F}$), $\mathcal{T}_\psi$ should depend on the function $\psi \in \boldsymbol{W}$ only in a neighborhood of the crack tip $\boldsymbol{F}$. This invariance is true for all $\psi \in \boldsymbol{W}$ in the homogeneous case for which $\alpha = \beta$; in our situation, this invariance remains true if the material is homogeneous on the support of the function ψ, localized around $\boldsymbol{F}$. We therefore impose that $supp(\psi) \in \mathcal{D}$, where $\mathcal{D}$ is the set appearing in the definition of the admissible class $\mathcal{X}_{L,\mathcal{D}}$. We then take

$$\psi \in \boldsymbol{W}_{\mathcal{D}} = \{\psi \in \boldsymbol{W},\ supp(\psi) \subset \mathcal{D}\}.$$

This material assumption then permits to link the derivative $\mathcal{T}_\psi$, which is a mathematical quantity defined on Ω, to the thermo-dynamic strength $\mathcal{T}$ (locally defined on $\boldsymbol{F}$).

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The energy release rate: the invariance

Lemma ((Local) Energy release rate)

Let $C(\boldsymbol{F}, r)$ be the circle of center $\boldsymbol{F}$ and radius $r > 0$, $\boldsymbol{\nu_c} = (\nu_{c,1}, \nu_{c,2})$ its outward normal and

$$\mathcal{T}_r(u, \mathcal{X}_\omega) = \frac{1}{2}\int_{C(\boldsymbol{F},r)} a_{\mathcal{X}_\omega}(\boldsymbol{x}) u_{,j} u_{,j} \psi_k \nu_{c,k} d\sigma - \int_{C(\boldsymbol{F},r)} a_{\mathcal{X}_\omega}(\boldsymbol{x}) u_{,j} u_{,k} \psi_k \nu_{c,j} d\sigma,$$

where u is solution of (2). The thermo-dynamic strength $\mathcal{T}$ is linked to $\mathcal{T}_{\boldsymbol{\psi}}$ as follows:

$$\mathcal{T}_{\boldsymbol{\psi}}(u, \mathcal{X}_\omega) = \lim_{r \to 0} \mathcal{T}_r(u, \mathcal{X}_\omega)\ (\boldsymbol{\psi} \cdot \boldsymbol{\nu})_{|\boldsymbol{F}} \equiv \mathcal{T}(u, \mathcal{X}_\omega)\ \boldsymbol{\psi}(\boldsymbol{F}) \cdot \boldsymbol{\nu_F}, \quad \forall \boldsymbol{\psi} \in \boldsymbol{W}_{\mathcal{D}}, \tag{10}$$

where $\boldsymbol{\nu_F} = (\nu_{F,1}, \nu_{F,2}) = (\pm 1, 0)$ denotes the orientation of the crack γ at the point $\boldsymbol{F}$. ■

It follows from (10) that the energy release rate $\mathcal{T}_{\boldsymbol{\psi}}$ is related to the strength $\mathcal{T}$ by

$$\mathcal{T}(u, \mathcal{X}_\omega) = \mathcal{T}_{\boldsymbol{\psi}}(u, \mathcal{X}_\omega), \quad \forall \boldsymbol{\psi} \in \boldsymbol{W}_{\mathcal{D}} \text{ such that } \boldsymbol{\psi}(\boldsymbol{F}) \cdot \boldsymbol{\nu_F} = \pm\psi_1(\boldsymbol{F}) = 1. \tag{11}$$

As a summary, if the conductivity is constant equal to β in $\mathcal{D}$, and if the function $\boldsymbol{\psi}$, which permits to define the virtual crack extension of $\boldsymbol{F}$, belongs to $\boldsymbol{W}_{\mathcal{D}}$ and satisfies $\psi_1(\boldsymbol{F}) = \pm 1$, then the energy release rate $\mathcal{T}$ may be related to the mathematical quantity $\mathcal{T}_{\boldsymbol{\psi}}$.

$$(P): \qquad \inf_{\mathcal{X}_\omega \in \mathcal{X}_{L,\mathcal{D}}} \mathcal{T}_{\psi}(u, \mathcal{X}_\omega) \tag{12}$$

- $L \in (0,1)$;
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The following formulation :

$$(RP) \quad \min_{s,t} \mathcal{I}_{\psi}(u,s) = \int_{\mathcal{D}} \beta(A_{\psi}\nabla u, \nabla u)dx \tag{13}$$

subject to the constraint

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- *(RP) is well-posed*
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Some steps of the proof from the Young measure approach

Suppose that $(\mathcal{X}_{\omega_n})_{(n>0)}$ is a minimizing sequence for (P) and let u_n be its corresponding sequence of solutions. Consider the two sequences of vectors

$$G_n(\boldsymbol{x}) = (\alpha\mathcal{X}_{\omega_n} + \beta(1-\mathcal{X}_{\omega_n}))\nabla u_n(\boldsymbol{x}), \quad H_n(\boldsymbol{x}) = \nabla u_n(\boldsymbol{x}). \tag{16}$$

Since both sequences are uniformly bounded in $(L^2(\Omega))^2$, we may associate with the pair (G_n, H_n) a family of parametrized measures $\nu = \{\nu_x\}_{x\in\Omega}$, a div-curl measure, supported in the union the two linear manifolds

$$\Lambda_\gamma = \{(\lambda,\rho)\in\mathbb{R}^2\times\mathbb{R}^2 : \rho = \gamma\lambda\}, \quad \gamma = \alpha, \beta \tag{17}$$

so that $supp(\nu_x) \subset \Lambda_\alpha \cup \Lambda_\beta$. As is usual, the measure ν_x may be written as

$$\nu_x = s(\boldsymbol{x})\nu_{x,\alpha} + (1-s(\boldsymbol{x}))\nu_{x,\beta} \tag{18}$$

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$$\lim_{n\to\infty} \mathcal{T}_\psi(u_n, \mathcal{X}_{\omega_n}) = \int_\Omega \left[\alpha s(\boldsymbol{x})A_\psi(\boldsymbol{x}) : \int_{\mathbb{R}^2} \lambda\lambda^T d\nu^{(1)}_{x,\alpha}(\lambda) + \beta(1-s(\boldsymbol{x}))A_\psi(\boldsymbol{x}) : \int_{\mathbb{R}^2} \lambda\lambda^T d\nu^{(1)}_{x,\beta}(\lambda)\right] dx \tag{19}$$

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We introduce the linear manifolds $\Lambda_\gamma = \{(\lambda,\rho)\in\mathbb{R}^2\times\mathbb{R}^2 : \rho = \gamma\lambda\}$ and

$$W(\boldsymbol{x},\rho,\lambda) = \begin{cases} \alpha A_{\boldsymbol{\psi}}(\boldsymbol{x}) : \lambda\lambda^T & \text{if} \quad (\rho,\lambda)\in\Lambda_\alpha\,, \\ \beta A_{\boldsymbol{\psi}}(\boldsymbol{x}) : \lambda\lambda^T & \text{if} \quad (\rho,\lambda)\in\Lambda_\beta\,, \\ +\infty & \text{else}, \end{cases} \tag{20}$$

and

$$V(\rho,\lambda) = \begin{cases} 1 & \text{if} \quad (\rho,\lambda)\in\Lambda_\alpha\,, \\ 0 & \text{if} \quad (\rho,\lambda)\in\Lambda_\beta\,, \\ +\infty & \text{else}. \end{cases} \tag{21}$$

Then we check that (P) is equivalent to the following new problem

$$(VP): \qquad \inf_{G,u}\int_\Omega W(\boldsymbol{x}, G(\boldsymbol{x}), \nabla u(\boldsymbol{x}))dx \tag{22}$$

subject to

$$\begin{cases} G\in L^2(\Omega;\mathbb{R}^2), \quad u\in H^1(\Omega;\mathbb{R}), \\ div\ G = 0 \quad \text{in} \quad H^{-1}(\Omega), \quad G(\boldsymbol{x}) = \beta\nabla u(\boldsymbol{x}) \quad \text{in} \quad \mathcal{D} \\ u = u_0 \quad \text{on} \quad \Gamma_0, \quad \beta\nabla u\cdot\boldsymbol{\nu} = g \quad \text{on} \quad \Gamma_g\subset\partial\Omega\backslash(\gamma\cup\Gamma_0), \\ \int_\Omega V(G(\boldsymbol{x}),\nabla u(\boldsymbol{x}))dx = L|\Omega|. \end{cases} \tag{23}$$

$$(RP): \qquad \min_{s,G,u} \int_{\Omega} CQW(\boldsymbol{x}, s(\boldsymbol{x}), G(\boldsymbol{x}), \nabla u(\boldsymbol{x}))dx \tag{24}$$

for u and G satisfying the previous constraints and

$$s \in S_{L,\mathcal{D}} \equiv \left\{ s \in L^{\infty}(\Omega, [0,1]), \|s\|_{L^1(\Omega)} = L|\Omega|, s = 0 \text{ in } \mathcal{D} \cup \partial\Omega \right\}. \tag{25}$$

The constrained quasi-convex density CQW is computed by solving the problem in measure :

$$\begin{aligned} & CQW(\boldsymbol{x}, s(\boldsymbol{x}), G(\boldsymbol{x}), \nabla u(\boldsymbol{x})) \\ & = \inf_{\nu} \left\{ \alpha s(\boldsymbol{x}) A_{\psi}(\boldsymbol{x}) : \int_{\mathbb{R}^2} \lambda\lambda^T d\nu^{(1)}_{x,\alpha}(\lambda) + \beta(1 - s(\boldsymbol{x})) A_{\psi}(\boldsymbol{x}) : \int_{\mathbb{R}^2} \lambda\lambda^T d\nu^{(1)}_{x,\beta}(\lambda) \right\} \end{aligned} \tag{26}$$

for any measure ν subject to

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Computation of *CQW*

- Step 2: Computation of CPW, a lower bound of CQW

$$CPW(\boldsymbol{x}, s(\boldsymbol{x}), G(\boldsymbol{x}), \nabla u(\boldsymbol{x}))$$
$$= \inf_{\nu} \left\{ \alpha s(\boldsymbol{x}) A_{\psi}(\boldsymbol{x}) : \int_{\mathbb{R}^2} \lambda\lambda^T d\nu^{(1)}_{x,\alpha}(\lambda) + \beta(1 - s(\boldsymbol{x})) A_{\psi}(\boldsymbol{x}) : \int_{\mathbb{R}^2} \lambda\lambda^T d\nu^{(1)}_{x,\beta}(\lambda) \right\} \quad (28)$$

for any measure ν subject to

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$\Longrightarrow$ Mathematical programming problem for the moments $\int_{\mathbb{R}^2} \lambda\lambda^T d\nu^{(1)}_{\gamma}$

- Step 3: See if the optimal measure for *CPW* is a div-curl measure

(Sufficient condition)

Suppose that ρ_i, λ_i, $i = 1, 2$ are four vectors in $\mathbb{R}^2$ such that

$$(\rho_2 - \rho_1) \cdot (\lambda_2 - \lambda_1) = 0. \quad (30)$$

Then the probability measure

$$\mu = s\delta_{(\rho_1,\lambda_1)} + (1 - s)\delta_{(\rho_2,\lambda_2)} \quad (31)$$

is a div-curl Young measure for all $s \in [0, 1]$. ∎

Computation of *CQW*

- Step 2: Computation of CPW, a lower bound of CQW

$$CPW(\boldsymbol{x}, s(\boldsymbol{x}), G(\boldsymbol{x}), \nabla u(\boldsymbol{x})) = \inf_{\nu}\left\{\alpha s(\boldsymbol{x})A_{\psi}(\boldsymbol{x}) : \int_{\mathbb{R}^2} \lambda\lambda^T d\nu^{(1)}_{x,\alpha}(\lambda) + \beta(1-s(\boldsymbol{x}))A_{\psi}(\boldsymbol{x}) : \int_{\mathbb{R}^2} \lambda\lambda^T d\nu^{(1)}_{x,\beta}(\lambda)\right\} \quad (28)$$

for any measure ν subject to

$$\begin{cases} \nu = \{\nu_x\}_{x\in\Omega}, \quad \nu_x = s(\boldsymbol{x})\nu_{\boldsymbol{x},\alpha} + (1-s(\boldsymbol{x}))\nu_{\boldsymbol{x},\beta}, \ \mathrm{supp}(\nu_{\boldsymbol{x},\gamma}) \subset \Lambda_\gamma, \\ \nu \quad \text{is measure satisfying the commutation property}, \\ G(\boldsymbol{x}) = \int_{\mathbb{R}^2} \rho d\nu_x(\lambda,\rho), \quad div\ G = 0 \text{ weakly in } \Omega, \quad \nabla u(\boldsymbol{x}) = \int_{\mathbb{R}^2} \lambda d\nu_x(\lambda,\rho). \end{cases} \quad (29)$$

$\Longrightarrow$ Mathematical programming problem for the moments $\int_{\mathbb{R}^2} \lambda\lambda^T d\nu^{(1)}_\gamma$

- Step 3: See if the optimal measure for *CPW* is a div-curl measure

Lemma (Sufficient condition)

Suppose that $\rho_i, \lambda_i, i = 1, 2$ are four vectors in $\mathbb{R}^2$ such that

$$(\rho_2 - \rho_1)\cdot(\lambda_2 - \lambda_1) = 0. \quad (30)$$

Then the probability measure

$$\mu = s\delta_{(\rho_1,\lambda_1)} + (1-s)\delta_{(\rho_2,\lambda_2)} \quad (31)$$

is a div-curl Young measure for all $s \in [0, 1]$. ∎

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Step 2 : Constrained Quasi-Convexification- Lower bound

Concerning the first moment of ν, we may write

$$(\lambda, \rho) = \int_{\Lambda} (x, y) d\nu(x, y) = s \int_{\mathbb{R}^2} (x, \alpha x) d\nu_{\alpha}^{(1)}(x) + (1 - s) \int_{\mathbb{R}^2} (x, \beta x) d\nu_{\beta}^{(1)}(x) \quad (32)$$

where $\nu_{\gamma}^{(1)}$ is the projection of ν_{γ} onto the first copy of $\mathbb{R}^2$ of the product $\mathbb{R}^2 \times \mathbb{R}^2$. By introducing

$$\lambda_{\gamma} = \int_{\mathbb{R}^2} x d\nu_{\gamma}^{(1)}(x), \quad (33)$$

we have $\lambda = s\lambda_{\alpha} + (1 - s)\lambda_{\beta}$, $\rho = s\alpha\lambda_{\alpha} + (1 - s)\beta\lambda_{\beta}$, and then

$$\lambda_{\alpha} = \frac{1}{s(\beta - \alpha)}(\beta\lambda - \rho), \quad \lambda_{\beta} = \frac{1}{(1 - s)(\beta - \alpha)}(\rho - \alpha\lambda). \quad (34)$$

Moreover, the commutation with the inner product yields the relation

$$\lambda^T \rho = \int_{\Lambda} x^T y d\nu(x, y) = \alpha s \int_{\mathbb{R}^2} x^T x d\nu_{\alpha}^{(1)}(x) + \beta(1 - s) \int_{\mathbb{R}^2} x^T x d\nu_{\beta}^{(1)}(x). \quad (35)$$

To find a lower bound of *CQW*, we retain just the relevant property expressed in the commutation (35), so that we regard feasible measures ν as Young measures which satisfy this commutation property, but are not necessarily a div-curl Young measure. We introduce

$$\boldsymbol{X_{\gamma} = \int_{\mathbb{R}^2} x x^T d\nu_{\gamma}^{(1)}(x), \quad \gamma = \alpha, \beta} \quad (36)$$

a convex combination of symmetric rank-one matrices. It is well-known that

$$\boldsymbol{X_{\gamma} \geq \lambda_{\gamma}\lambda_{\gamma}^T, \quad \gamma = \alpha, \beta} \quad (37)$$

in the usual sense of symmetric matrices, i.e. that $X_{\gamma} - \lambda_{\gamma}\lambda_{\gamma}^T$ is semi-definite positive. The relation (35) becomes

$$\lambda^T \rho = \lambda \cdot \rho = \alpha s Tr(X_{\alpha}) + \beta(1 - s) Tr(X_{\beta}). \quad (38)$$

Step 2 : A lower bound of the Constrained Quasi-Convexification

Similarly, the cost may be written in term of the variable X_γ as follows :

$$s\alpha A_\psi : X_\alpha + (1-s)\beta A_\psi : X_\beta = s\alpha Tr(A_\psi X_\alpha) + (1-s)\beta Tr(A_\psi X_\beta) \tag{39}$$

from the relation $A_\psi : X_\gamma = Tr(A_\psi X_\gamma)$, $\gamma = \alpha, \beta$. Consequently, in seeking a lower bound of the constrained quasiconvexification, we are led to consider the mathematical programming problem

$$\min_{X_\alpha, X_\beta} C(X_\alpha, X_\beta) = \alpha s Tr(A_\psi X_\alpha) + \beta(1-s) Tr(A_\psi X_\beta) \tag{40}$$

subject to the constraints

$$\lambda^T \rho = \lambda \cdot \rho = \alpha s Tr(X_\alpha) + \beta(1-s) Tr(X_\beta), \quad X_\gamma \geq \lambda_\gamma \lambda_\gamma^T. \tag{41}$$

We first realize that the set of vectors for which the constraints yield a non-empty set takes place if

$$\alpha s Tr(\lambda_\alpha \lambda_\alpha^T) + \beta(1-s) Tr(\lambda_\beta \lambda_\beta^T) \leq \lambda \cdot \rho \tag{42}$$

i.e. if

$$\begin{aligned} B(\rho, \lambda) &\equiv \lambda \cdot \rho - \alpha s |\lambda_\alpha|^2 - \beta(1-s)|\lambda_\beta|^2 \geq 0 \\ &= (\lambda_\beta - \lambda_\alpha) \cdot (\beta\lambda_\beta - \alpha\lambda_\alpha) \end{aligned} \tag{43}$$

using that $Tr(\lambda \rho^T) = \lambda \cdot \rho$.

Proposition (Non diagonal case)

For any $s \in L^\infty(\Omega)$ *and* $(\lambda, \rho) = (\nabla u, G)$ *satisfying all the constraints,*

$$m(s,\lambda,\rho) = \begin{cases} \frac{1}{2}\Big[-\sqrt{\psi_{1,1}^2+\psi_{1,2}^2}(\rho\cdot\lambda - \alpha s|\lambda_\alpha|^2 - \beta(1-s)|\lambda_\beta|^2) \\ \qquad + \psi_{1,1}(\alpha s\lambda_{\alpha,1}^2 + (1-s)\beta\lambda_{\beta,1}^2) - \psi_{1,1}(\alpha s\lambda_{\alpha,2}^2 + (1-s)\beta\lambda_{\beta,2}^2) \\ \qquad + 2\psi_{1,2}(\alpha s\lambda_{\alpha,1}\lambda_{\alpha,2} + (1-s)\beta\lambda_{\beta,1}\lambda_{\beta,2})\Big] & \text{if } \mathbf{B}(\rho,\lambda)\geq 0 \\ +\infty & \text{else} \end{cases} \tag{44}$$

is a lower bound for the constrained quasi-convexified CQW of W:

$$m(s,\lambda,\rho) \leq CQW(s,\lambda,\rho). \tag{45}$$

$\lambda_\gamma = \lambda_\gamma(s,\lambda,\rho)$, $\gamma = \alpha, \beta$ *are defined by (34).* ■

We note

$$A_\psi = \frac{1}{2}\begin{pmatrix} \psi_{1,1} & 2\psi_{1,2} \\ 0 & -\psi_{1,1} \end{pmatrix} \equiv \begin{pmatrix} a & 2b \\ 0 & -a \end{pmatrix} \tag{46}$$

and made the change of variables $\boldsymbol{Y_\gamma = X_\gamma - \lambda_\gamma \lambda_\gamma^T}$ so that the cost and the constraints are transformed into

$$\min_{Y_{\gamma,11}, Y_{\gamma,22}, Y_{\gamma,12}} \alpha s(a(Y_{\alpha,11} - Y_{\alpha,22}) + 2bY_{\alpha,12}) + \beta(1-s)((a(Y_{\beta,11} - Y_{\beta,22}) + 2bY_{\beta,12})) + A \tag{47}$$

and

$$\begin{cases} s\alpha(Y_{\alpha,11} + Y_{\alpha,22}) + (1-s)\beta(Y_{\beta,11} + Y_{\beta,22}) = \boldsymbol{B}, \\ Y_{\gamma,11} + Y_{\gamma,22} \geq 0, \quad Y_{\gamma,11}Y_{\gamma,22} \geq Y_{\gamma,12}^2 \quad \gamma = \alpha, \beta \end{cases} \tag{48}$$

where the constant A is defined by

$$A = \alpha s(a(\lambda_{\alpha,1}^2 - \lambda_{\alpha,2}^2) + 2b\lambda_{\alpha,1}\lambda_{\alpha,2}) + \beta(1-s)((a(\lambda_{\beta,1}^2 - \lambda_{\beta,2}^2) + 2b\lambda_{\beta,1}\lambda_{\beta,2})). \tag{49}$$

The minimum of the linear cost is reached on the boundary of the convex sets

$$\Gamma_\gamma = \left\{(Y_{\gamma,11}, Y_{\gamma,22}, Y_{\gamma,12}) \in \mathbb{R}^3, Y_{\gamma,11} \geq 0, Y_{\gamma,22} \geq 0, Y_{\gamma,11}Y_{\gamma,22} \geq Y_{\gamma,12}^2\right\}, \quad \gamma = \alpha, \beta \tag{50}$$

which implies $Y_{\gamma,11}Y_{\gamma,22} = Y_{\gamma,12}^2$. Therefore, we can introduce the new variables $Z_\gamma \equiv (Z_{\gamma,11}, Z_{\gamma,22})^T$ so that $Y_{\gamma,11} = Z_{\gamma,11}^2$, $Y_{\gamma,22} = Z_{\gamma,22}^2$ and $\epsilon_\gamma = \pm 1$ and then $Z_{\gamma,11}Z_{\gamma,22} = \epsilon_\gamma Y_{\gamma,12}$ reducing the problem to

$$\begin{aligned} \min_{Z_{\gamma,11}, Z_{\gamma,22}, \epsilon_\gamma} C(Z_\gamma, \epsilon_\gamma) &= \alpha s(a(Z_{\alpha,11}^2 - Z_{\alpha,22}^2) + 2b\epsilon_\alpha Z_{\alpha,11}Z_{\alpha,22}) \\ &+ \beta(1-s)((a(Z_{\beta,11}^2 - Z_{\beta,22}^2) + 2b\epsilon_\beta Z_{\beta,11}Z_{\beta,22})) + A \end{aligned} \tag{51}$$

under the constraint

$$s\alpha(Z_{\alpha,11}^2 + Z_{\alpha,22}^2) + (1-s)\beta(Z_{\beta,11}^2 + Z_{\beta,22}^2) = B. \tag{52}$$

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Introducing the Lagrangian L and the multiplier p

$$L(Z_\gamma, p) = C(Z_\gamma, \epsilon_\gamma) - p\Big(s\alpha(Z^2_{\alpha,11} + Z^2_{\alpha,22}) + (1-s)\beta(Z^2_{\beta,11} + Z^2_{\beta,22}) - \mathbf{B}\Big), \tag{53}$$

we arrive at the optimality conditions :

$$A_{\psi,\epsilon_\gamma} Z_\gamma = pZ_\gamma, \qquad A_{\psi,\epsilon_\gamma} = \begin{pmatrix} a & b\epsilon_\gamma \\ b\epsilon_\gamma & -a \end{pmatrix}. \tag{54}$$

The resolution of a spectral problem leads to

$$p = -\sqrt{a^2+b^2}, \quad Z_\gamma = a_\gamma\Big(b\epsilon_\gamma, -(a+\sqrt{a^2+b^2})\Big)^T \tag{55}$$

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Therefore, the cost, independent of ϵ_γ is obtained for the lowest eigenvalue (independent here of the sign of a) :

$$\min C(Z_\gamma, \epsilon_\gamma) = -\sqrt{a^2+b^2}\,\mathbf{B} + A \tag{58}$$

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$$(a^2_\alpha s\alpha + a^2_\beta(1-s)\beta)(b^2 + (a+\sqrt{a^2+b^2})^2) = \mathbf{B}. \tag{59}$$

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According to the previous computation, the optimal second moment are of the form

$$X_\gamma = \lambda_\gamma \lambda_\gamma^T + a_\gamma^2 \begin{pmatrix} \psi_{1,2}^2 & -\psi_{1,2}(\psi_{1,1} + \sqrt{\psi_{1,1}^2 + \psi_{1,2}^2}) \\ -\psi_{1,2}(\psi_{1,1} + \sqrt{\psi_{1,1}^2 + \psi_{1,2}^2}) & (\psi_{1,1} + \sqrt{\psi_{1,1}^2 + \psi_{1,2}^2})^2 \end{pmatrix} \tag{60}$$

leading to the cost $-\sqrt{\psi_{1,1}^2 + \psi_{1,2}^2}\boldsymbol{B} + A$. But, on $\Omega/\mathcal{D}$, the radial function ψ is zero so that,

$$\boldsymbol{X_\gamma = \lambda_\gamma \lambda_\gamma^T, \quad x \in \Omega/\mathcal{D}} \tag{61}$$

i.e. in particular

$$X_{\gamma,ii} = \int_{\mathbb{R}} x_i^2 d\nu_\gamma^{(1,i)}(x_i) = \left(\int_{\mathbb{R}} x_i d\nu_\gamma^{1,i}(x_i)\right)^2 = (\lambda_{\gamma,i})^2, \quad i = 1, 2 \tag{62}$$

where $\nu_\gamma^{(1,i)}$ denotes the projection of $\nu^{(1)}$ onto the i-th copy of $\mathbb{R}^2$. From the strict convexity of the square function, this implies that $\nu_\gamma^{(1,i)} = \delta_{\lambda_{\gamma,i}}$, i.e.

$$\nu_\alpha^{(1,i)} = \delta_{\frac{\beta\lambda_i - \rho_i}{s(\beta-\alpha)}}, \quad \nu_\beta^{(1,i)} = \delta_{\frac{\rho_i - \alpha\lambda_i}{(1-s)(\beta-\alpha)}}. \tag{63}$$

Remark that this is compatible with the third equality $X_{\gamma,12} = \lambda_{\gamma,1}\lambda_{\gamma,2}^T$. This also implies (see for instance (59)) the equality in (42), i.e. that

$$\boldsymbol{B} = \lambda \cdot \rho - \alpha s|\lambda_\alpha|^2 - \beta(1-s)|\lambda_\beta|^2 = 0. \tag{64}$$

Consequently, the optimal value $m(s, \lambda, \rho)$ may be recovered by the following measure

$$\boldsymbol{\nu = s\delta_{(\alpha\lambda_\alpha, \lambda_\alpha)} + (1-s)\delta_{(\beta\lambda_\beta, \lambda_\beta)}} \tag{65}$$

which is a first order (div-curl) laminate, the div-curl condition $(\beta\lambda_\beta - \alpha\lambda_\alpha)\cdot(\lambda_\beta - \lambda_\alpha) = 0$ (analogous to a rank one condition for H^1 −gradient Young measure) being equivalent precisely to $\boldsymbol{B} = 0$.

Conclusion

Theorem

The variational problem

$$(RP): \qquad \min_{s,u,G} \int_\Omega m(s, \nabla u, G)dx \tag{66}$$

subject to

$$\begin{cases} s \in L^\infty(\Omega, [0,1]), s = 0 \text{ in } \mathcal{D} \cup \partial\Omega, \int_\Omega s(\boldsymbol{x})dx = L|\Omega|, \\ u \in H^1(\Omega), \quad u = u_0 \text{ on } \Gamma_0, \quad \beta\nabla u \cdot \boldsymbol{\nu} = g \text{ on } \Gamma_g, \\ G \in (L^2(\Omega))^2, \quad div\ G = 0 \text{ weakly in } \Omega, \quad G = \beta\nabla u \text{ weakly in } \mathcal{D} \end{cases} \tag{67}$$

where m is defined by (44) is a relaxation of (VP) in the sense that the minimum of (RP) exists and equals the minimum of (VP). Moreover, the underlying Young measure associated with (RP) can be found in the form of a first order laminate whose direction of lamination are given explicitly in terms of the optimal solution (u, G): precisely, the normal are orthogonal to $\lambda_\beta - \lambda_\alpha$.

Simplified conclusion

The above formulation may be simplified by taking into account that $\boldsymbol{B} = 0$. Precisely, we use (34) to express $\boldsymbol{B} = (\beta\lambda_\beta - \alpha\lambda_\alpha)\cdot(\lambda_\beta - \lambda_\alpha) = 0$ as follows

$$(\rho - \lambda^-(s)\lambda)\cdot(\rho - \lambda^+(s)\lambda) = 0 \tag{68}$$

in terms of the harmonic and arithmetic mean of α, β with weight s.

Theorem

The variational problem

$$(\overline{RP}): \qquad \min_{s,u,G} \int_\Omega F(s, \nabla u, G)dx \tag{69}$$

subject to

$$\begin{cases} s \in L^\infty(\Omega, [0,1]),\ s = 0 \textit{ in } \mathcal{D} \cup \partial\Omega,\ \int_\Omega s(\boldsymbol{x})dx = L|\Omega|, \\ u \in H^1(\Omega), \quad u = u_0 \textit{ on } \Gamma_0, \quad \beta\nabla u\cdot\boldsymbol{\nu} = g \textit{ on } \Gamma_g, \\ G \in (L^2(\Omega))^2, \quad div\ G = 0 \textit{ weakly in } \Omega, \quad G = \beta\nabla u \textit{ weakly in } \mathcal{D} \\ (G - \lambda^-(s)\nabla u)\cdot(G - \lambda^+(s)\nabla u) = 0 \textit{ in } L^2(\Omega), \end{cases} \tag{70}$$

where F, deduced from m, is defined

$$\begin{aligned} F(s,\lambda,\rho) = \frac{1}{2}\Big[&\psi_{1,1}(\alpha s\lambda_{\alpha,1}^2 + (1-s)\beta\lambda_{\beta,1}^2) - \psi_{1,1}(\alpha s\lambda_{\alpha,2}^2 + (1-s)\beta\lambda_{\beta,2}^2) \\ &+ 2\psi_{1,2}(\alpha s\lambda_{\alpha,1}\lambda_{\alpha,2} + (1-s)\beta\lambda_{\beta,1}\lambda_{\beta,2})\Big] \end{aligned} \tag{71}$$

is a relaxation of (VP) in the sense that the minimum of $(\overline{RP})$ exists and equals the minimum of (VP). ■

A transformation

Following [4] we remark that $\boldsymbol{B} = 0$ is equivalent to

$$\left|\rho - \frac{\lambda^+(s) + \lambda^-(s)}{2}\lambda\right|^2 = \left(\frac{\lambda^+(s) - \lambda^-(s)}{2}\right)^2 |\lambda|^2. \tag{72}$$

Therefore, by introducing the additional variable $t(\boldsymbol{x}) \in \mathbb{R}^2$ such that $|t| = 1$, we may write $\rho = G(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$ under the form (we use that $\lambda^-(s) \leq \lambda^+(s)$ for all $s \in (0, 1)$)

$$\rho = \underbrace{\frac{\lambda^+(s) + \lambda^-(s)}{2}}_{\equiv A(s)} \lambda + \underbrace{\frac{\lambda^+(s) - \lambda^-(s)}{2}}_{\equiv C(s)} |\lambda| t \equiv \phi(s, t, \lambda). \tag{73}$$

We have

$$A(s) = \frac{2\alpha\beta + s(1-s)(\beta-\alpha)^2}{2(\alpha(1-s)+\beta s)}, \qquad C(s) = \frac{s(1-s)(\beta-\alpha)^2}{2(\alpha(1-s)+\beta s)}. \tag{74}$$

The relation *div* $G = 0$ then permits to recover u as the solution of a *nonlinear* equation under a divergence form (having in mind that $\lambda = \nabla u$):

$$\begin{cases} div(A(s)\nabla u + C(s)|\nabla u| t) = 0, & \text{in } \Omega, \\ u = u_0, & \text{on } \Gamma_0, \\ \beta \nabla u \cdot \boldsymbol{\nu} = g, & \text{on } \Gamma_g. \end{cases} \tag{75}$$

We assume that this problem is well-posed in $H^1(\Omega)$.

[4] P. Pedregal, *Div-Curl Young measures and optimal design in any dimension*, Rev. Mat. Complut., (2007).

A transformation

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Theorem

Let F and ϕ be defined respectively by (71) and (73). The following formulation

$$(\underline{RP}): \qquad \min_{s,t} I(s,t) = \int_\Omega F(s,\nabla u,\phi(s,t,\nabla u))dx \tag{77}$$

subject to the constraints

$$\begin{cases} s\in L^\infty(\Omega,[0,1]),\ s=0 \textit{ in } \mathcal{D}\cup\partial\Omega,\ \int_\Omega s(\boldsymbol{x})dx = L|\Omega|, \\ t\in L^\infty(\Omega,\mathbb{R}^2), \quad |t|=1, \\ u\in H^1(\Omega), \quad u=u_0 \textit{ on } \Gamma_0, \quad \beta\nabla u\cdot\boldsymbol{\nu} = g \textit{ on } \Gamma_g, \\ div\ \phi(s,t,\nabla u) = 0 \textit{ weakly in } \Omega \end{cases} \tag{78}$$

is equivalent to the relaxation (RP). In particular, $(\underline{RP})$ is a full well-posed relaxation of (VP). ■

Remark

Since $s=0$ in $\mathcal{D}$, $\int_\Omega F(s,\nabla u,\phi(s,t,\nabla u))dx = \int_\Omega \beta(A_\psi\nabla u,\nabla u)dx$.

A transformation

$$\begin{cases} div(A(s)\nabla u + C(s)|\nabla u|t) = 0, & \text{in } \Omega, \\ u = u_0, & \text{on } \Gamma_0, \\ \beta\nabla u \cdot \boldsymbol{\nu} = g, & \text{on } \Gamma_g. \end{cases} \tag{76}$$

Theorem

Let F and ϕ be defined respectively by (71) and (73). The following formulation

$$(\underline{RP}): \qquad \min_{s,t} I(s,t) = \int_\Omega F(s, \nabla u, \phi(s,t,\nabla u))dx \tag{77}$$

subject to the constraints

$$\begin{cases} s \in L^\infty(\Omega, [0,1]), s = 0 \textit{ in } \mathcal{D} \cup \partial\Omega, \int_\Omega s(\boldsymbol{x})dx = L|\Omega|, \\ t \in L^\infty(\Omega, \mathbb{R}^2), \quad |t| = 1, \\ u \in H^1(\Omega), \quad u = u_0 \textit{ on } \Gamma_0, \quad \beta\nabla u \cdot \boldsymbol{\nu} = g \textit{ on } \Gamma_g, \\ div\ \phi(s,t,\nabla u) = 0 \textit{ weakly in } \Omega \end{cases} \tag{78}$$

is equivalent to the relaxation (RP). In particular, $(\underline{RP})$ is a full well-posed relaxation of (VP). ■

Remark

Since $s = 0$ in $\mathcal{D}$, $\int_\Omega F(s, \nabla u, \phi(s,t,\nabla u))dx = \int_\Omega \beta(A_\psi \nabla u, \nabla u)dx$.

The relaxation (through H-measure)

Theorem

The following formulation :

$$\textit{minimize in } (\theta, K^\star) \in \boldsymbol{RD} : \mathcal{T}_\psi(u,\theta) = \int_{\mathcal{D}} \beta(A_\psi \nabla u, \nabla u) dx \tag{79}$$

subject to the constraint

$$\begin{cases} \theta \in L^\infty(\Omega,[0,1]), \theta = 0 \textit{ in } \mathcal{D} \cup \partial\Omega, \int_\Omega \theta(\boldsymbol{x})dx = L|\Omega|, \\ u \in H^1(\Omega), \quad u = u_0 \textit{ on } \Gamma_0, \quad \beta \nabla u \cdot \boldsymbol{\nu} = g \textit{ on } \Gamma_g, \\ -div(K^\star \nabla u) = 0 \textit{ weakly in } \Omega \end{cases} \tag{80}$$

with

$$\boldsymbol{RD} = \{(\theta, K^\star) \in L^\infty(\Omega; [0,1] \times \mathcal{M}_N^s(\alpha,\beta)) : K^\star \in G_{\theta(x)} \; a.e. \; \boldsymbol{x} \in \Omega\} \tag{81}$$

is a full relaxation of (P). ■

Theorem

The first variation of I with respect to s and t in the direction δs and δt exist and are given respectively by

$$\frac{dI(s,t,u,p)}{ds}\cdot\delta s = \int_\Omega F_{,s}(s,\nabla u,\phi(s,t,\nabla u))\cdot\delta s\,dx + \int_\Omega\Big(A_{,s}(s)\nabla u\cdot\nabla p + B_{,s}(s)|\nabla u| t\cdot\nabla p\Big)\cdot\delta s\,dx \tag{82}$$

and

$$\frac{dI(s,t,u,p)}{dt}\cdot\delta t = \int_\Omega F_{,t}(s,\nabla u,\phi(s,t,\nabla u))\cdot\delta t\,dx + \int_\Omega B(s)|\nabla u|\delta t\cdot\nabla p\,dx \tag{83}$$

where $p\in H^1_{\Gamma_0}(\Omega)=\{v\in H^1(\Omega), v=0 \text{ on } \Gamma_0\}$ solves the adjoint problem

$$\int_\Omega F_{,u}(s,\nabla u,\phi(s,t,\nabla u))\cdot v\,dx + \int_\Omega\Big(A(s)\nabla v\cdot\nabla p + B(s)\frac{\nabla u\cdot\nabla v}{|\nabla u|}t\cdot\nabla p\Big)dx = 0, \tag{84}$$

for all v in $H^1_{\Gamma_0}(\Omega)$. $A_{,s}$ and $B_{,s}$ denote the partial derivative of A and B with respect to s and $F_{,t}$ the partial derivative of F with respect to t. ∎

At each iteration k, the solution u of the variational formulation

$$\int_{\Omega}\left(A(s^{(k)})\nabla u\cdot\nabla v+B(s^{(k)})|\nabla u|t^{(k)}\cdot\nabla v\right)dx=\int_{\Gamma_g} gv\,d\sigma,\quad \forall v\in H^1_{\Gamma_0}(\Omega) \tag{85}$$

(we use that $s=0$ on $\partial\Omega$ and that $A(0)=\beta$, $B(0)=0$)

is soved using the full Newton algorithm:

$$\begin{cases} u^0\in H^1(\Omega),\, u^0=u_0 \text{ on } \Gamma_0, \\ \displaystyle\int_{\Omega}\left(A(s^{(k)})\nabla u^{n+1}\cdot\nabla v+B(s^{(k)})\frac{\nabla u^{n+1}\cdot\nabla u^n}{|\nabla u^n|}\,t^{(k)}\cdot\nabla v\right)dx=\int_{\Gamma_g} gv\,d\sigma, \forall n>0, \forall v\in H^1_{\Gamma_0}(\Omega). \end{cases} \tag{86}$$

About the choice of ψ_1: Non diagonal case: $\psi_1 = \psi_1(x_1, x_2)$

$$\psi_1(\boldsymbol{x}) = \zeta(\mathrm{dist}(\boldsymbol{x}, \boldsymbol{F}))\nu_{F,1}, \quad \forall \boldsymbol{x} \in \Omega \tag{87}$$

defining the function $\zeta \in C^1(\mathbb{R}^+; [0, 1])$ as follows:

$$\zeta(r) = \begin{cases} 1 & r \leq r_1 \\ \dfrac{(r - r_2)^2(3r_1 - r_2 - 2r)}{(r_1 - r_2)^3} & r_1 \leq r \leq r_2 \\ 0 & r \geq r_2 \end{cases} \tag{88}$$

with $0 < r_1 < r_2 < \mathrm{dist}(\partial\Omega/\gamma, \boldsymbol{F}) = \inf_{\boldsymbol{x}\in\partial\Omega/\gamma} \mathrm{dist}(\boldsymbol{x}, \boldsymbol{F})$.

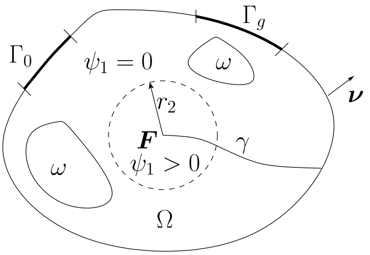


Figure: Choice of a radial function $\psi_1(\boldsymbol{x})$ leading to a non diagonal matrix $A_{\boldsymbol{\psi}}$.

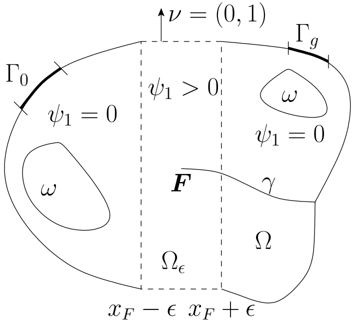
$$A_{\boldsymbol{\psi}}(\boldsymbol{x}) = \frac{1}{2}\begin{pmatrix} \psi_{1,1} & 2\psi_{1,2} \\ 0 & -\psi_{1,1} \end{pmatrix}. \tag{89}$$

About the choice of ψ_1: Diagonal case: $\psi_1 = \psi_1(x_1, x_2)$

$$\zeta(x_1) = \begin{cases} 0 & x_1 \le r_1 \\ \frac{(x_1-r_1)^2(2x_1+r_1-3r_2)}{r_1-r_2} & r_1 \le x_1 \le r_2, \\ 1 & r_2 \le x_1 \le r_3, \\ \frac{(x_1-r_4)^2(2x_1+r_4-3r_3)}{r_4-r_3} & r_3 \le x_1 \le r_4, \\ 0 & x_1 \ge r_4 \end{cases} \tag{90}$$

with

$$r_1 = x_F - \frac{2\epsilon}{3}, \quad r_2 = x_F - \frac{\epsilon}{3}, \quad r_3 = x_F + \frac{\epsilon}{3}, \quad r_4 = x_F + \frac{2\epsilon}{3}. \tag{91}$$



$$A_\psi(\boldsymbol{x}) = \frac{1}{2}\begin{pmatrix} \psi_{1,1} & 0 \\ 0 & -\psi_{1,1} \end{pmatrix}. \tag{92}$$

Figure: Choice of a function $\psi_1(\boldsymbol{x}) = \psi_1(x_1)\mathcal{X}_{\Omega_\epsilon}$ leading to a diagonal matrix A_ψ assuming the existence of a domain Ω_ϵ.

$$
\begin{aligned}
&\Omega = (0,1)^2, \quad \gamma = [1/2,1] \times \{a\}(a \in (0,1)), \quad \boldsymbol{F} = (1/2, a),\\
&\Gamma_0 = \Gamma_{0,1} \cup \Gamma_{0,2}, \quad u_0 = 0 \ on \ \Gamma_{0,1} = \{0\} \times [0,1], \quad u_0 = 1/2 \ on \ \Gamma_{0,2} = \{1\} \times [0.5, 0.8],\\
&\Gamma_g = \emptyset, \qquad (93)\\
&\mathcal{D} = \{\boldsymbol{x} \in \Omega, \|\boldsymbol{x} - \boldsymbol{F}\| \leq r_3\}, \quad r_3 = 0.05,\\
&r_1 = 0.015, \ r_2 = 0.045 < 0.3, \ \nu_{F,1} = -1
\end{aligned}
$$

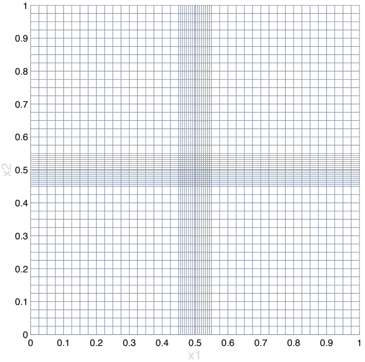


Figure: Example of quadrangulation of the unit square with a refinement on the support of the radial function ψ_1 (52 $\times$ 52 finite elements - 2916 degrees of freedom) around the point $\boldsymbol{F} = (1/2, 1/2)$.

$$
\begin{aligned}
&\Omega = (0,1)^2, \quad \gamma = [1/2, 1] \times \{a\}(a \in (0,1)), \quad \boldsymbol{F} = (1/2, a), \\
&\Gamma_0 = \Gamma_{0,1} \cup \Gamma_{0,2}, \quad u_0 = 0 \text{ on } \Gamma_{0,1} = \{0\} \times [0,1], \quad u_0 = 1/2 \text{ on } \Gamma_{0,2} = \{1\} \times [0.5, 0.8], \\
&\Gamma_g = \emptyset, \\
&\mathcal{D} = \{\boldsymbol{x} \in \Omega, \|\boldsymbol{x} - \boldsymbol{F}\| \leq r_3\}, \quad r_3 = 0.05, \\
&r_1 = 0.015, \; r_2 = 0.045 < 0.3, \; \nu_{F,1} = -1
\end{aligned}
\tag{93}
$$

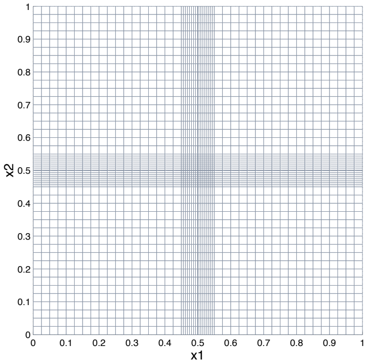


Figure: Example of quadrangulation of the unit square with a refinement on the support of the radial function ψ_1 (52 $\times$ 52 finite elements - 2916 degrees of freedom) around the point $\boldsymbol{F} = (1/2, 1/2)$.

Numerical experiments $(\alpha, \beta) = (1, 2)$

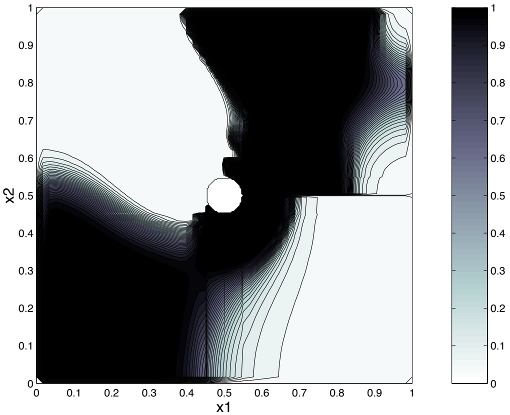


Figure: $(\alpha, \beta) = (1, 2)$- $L = 2/5$; $\boldsymbol{F} = (1/2, 1/2)$ - Iso-value of the density s^{opt} on the crack domain Ω with $s^{opt} = 0$ on $\partial\Omega$.

$$\|\boldsymbol{B}(\lambda, \rho)\|_{L^2(\Omega)} = \|(\rho - \lambda^-(s^{opt})\lambda) \cdot (\rho - \lambda^+(s^{opt})\lambda)\|_{L^2(\Omega)} \approx 1.32 \times 10^{-6}. \quad (94)$$

Moreover, we obtain

$$\|\rho - \lambda^+(s^{opt})\lambda\|_{L^2(\Omega)} \approx 3.13 \times 10^{-4}, \quad \|\rho - \lambda^-(s^{opt})\lambda\|_{L^2(\Omega)} \approx 4.21 \times 10^{-3}. \quad (95)$$

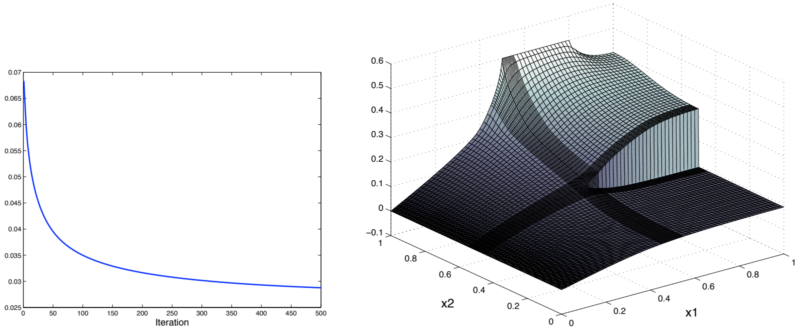


Figure: $(\alpha, \beta) = (1, 2)$- $L = 2/5$; $\boldsymbol{F} = (1/2, 1/2)$ - Evolution of the relaxed cost $I(s^{(k)}, t^{(k)})$ w.r.t the iteration (**Left**) and final solution u on Ω (**Right**).

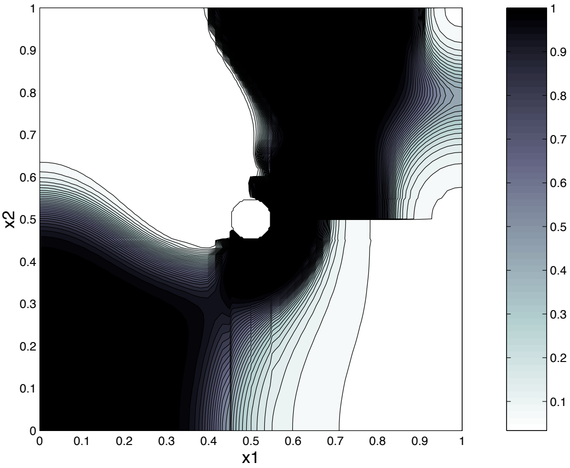


Figure: $(\alpha, \beta) = (1, 2)$- $L = 2/5$; $\boldsymbol{F} = (1/2, 1/2)$ - Iso-value of the density s on the crack domain with s free on $\partial\Omega$.

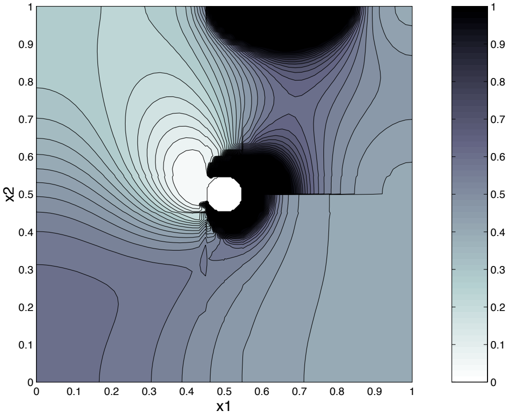


Figure: $(\alpha, \beta) = (1, 10)$- $L = 2/5$; $\boldsymbol{F} = (1/2, 1/2)$ - Iso-values of the density s on the crack domain.

$$\|\boldsymbol{B}(\lambda, \rho)\|_{L^2(\Omega)} = \|(\rho - \lambda^-(s^{opt})\lambda) \cdot (\rho - \lambda^+(s^{opt})\lambda)\|_{L^2(\Omega)} \approx 1.32 \times 10^{-5} \tag{96}$$

but

$$\|\rho - \lambda^+(s^{opt})\lambda\|_{L^2(\Omega)} \approx 8.21 \times 10^{-1}, \quad \|\rho - \lambda^-(s^{opt})\lambda\|_{L^2(\Omega)} \approx 4.09 \times 10^{-1}. \tag{97}$$

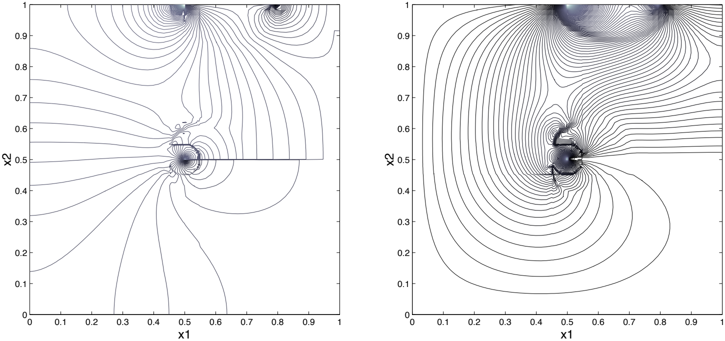


Figure: $(\alpha, \beta) = (1, 10)$- $L = 2/5$; $\boldsymbol{F} = (1/2, 1/2)$ - Iso-values of the components of the vector $\lambda_\beta - \lambda_\alpha$.

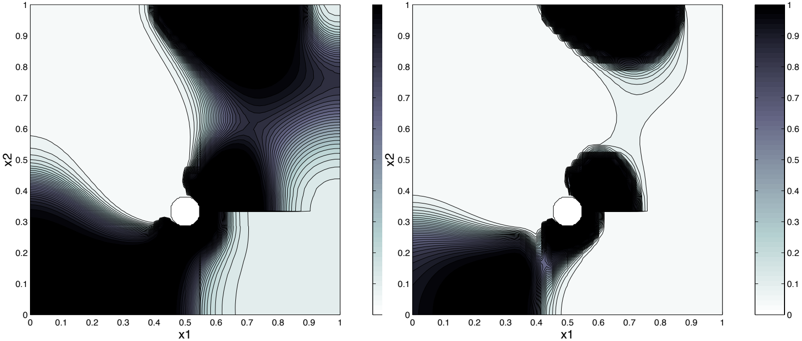


Figure: $(\alpha, \beta) = (1, 2)$- $\boldsymbol{F} = (1/2, 1/3)$ - Iso-values of the density s for $L = 2/5$ (**Left**) and $L = 1/5$ (**Right**).

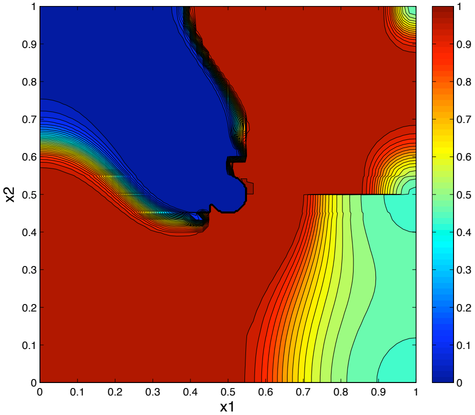


Figure: $(\alpha, \beta) = (1, 2)$- $\boldsymbol{F} = (1/2, 1/2)$ - Iso-value of the density s on the crack domain with s free on $\partial\Omega$

The optimal distribution corresponds to $L \approx 0.65$.

As a conclusion: minimization of the rate with respect to an extra boundary load

[5]

$$\inf_{h \in L^2(\partial\Omega)} g_\psi(u, G), \quad G = g\mathcal{X}_{\Gamma_g} + h\mathcal{X}_{\Gamma_h}, \Gamma_h \subset \partial\Omega/(\gamma \cup \Gamma_g) \tag{98}$$

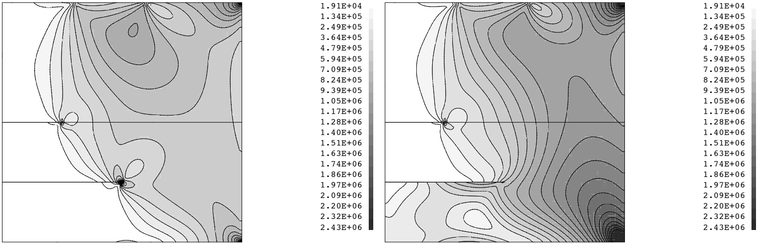


Figure: Iso-values of the Von Mises constraint in Ω - without extra-force : $g_\psi(\boldsymbol{u}, \boldsymbol{h}, 0) \approx 0.232N/m$; with extra force : $g_\psi(\boldsymbol{u}, \boldsymbol{h}^{opt}, \mathcal{X}_{\Gamma_h}) \approx 0.0556N/m$.

[5]

THANK YOU - MUCHAS GRACIAS - MERCI BEAUCOUP