Moving interfaces in Control and Inverse problems: Theory and numerical simulations.

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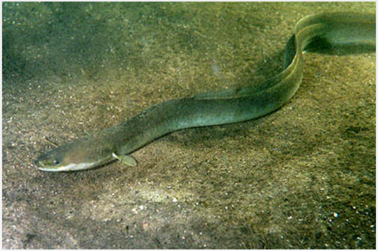
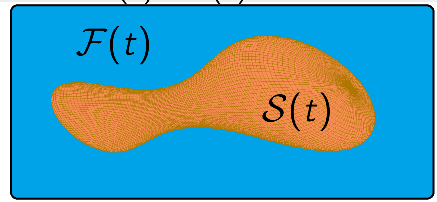
Plan

Some physical situations



Fluid-Solid model

$$\mathcal{O} = \mathcal{F}(t) \cup \overline{\mathcal{S}(t)} \subset \mathbb{R}^2 \text{ or } \mathbb{R}^3.$$



- The deformation of the solid induces an additional velocity in the fluid-solid interface. It translates into a **Dirichlet condition** for the fluid velocity, considered as viscous and incompressible.
- This condition influences the behavior of the environing fluid.
- The response of the fluid is a **force $\sigma \mathbf{n}$** which acts in the interface fluid/solid.
  It determines the **dynamics** of the solid, and thus its **position**.
- The deformation has to satisfy a set of **nonlinear constraints**.

Coupled system: Navier-Stokes and Newton laws

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u - \nu\Delta u + \nabla p = 0, \qquad x \in \mathcal{F}(t), \quad t \in (0,T),$$

$$\mathrm{div}\ u = 0, \qquad x \in \mathcal{F}(t), \quad t \in (0,T),$$

$$u = 0, \qquad x \in \partial\mathcal{O}, \quad t \in (0,T),$$

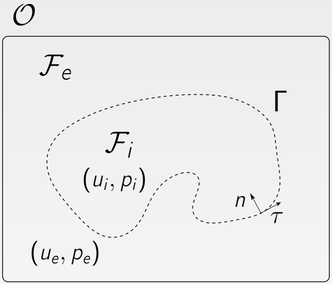
$$u = h'(t) + \omega(t)\wedge(x - h(t)) + w(x,t), \qquad x \in \partial\mathcal{S}(t), \quad t \in (0,T),$$

$$Mh''(t) = -\int_{\partial\mathcal{S}(t)} \sigma(u,p)n\mathrm{d}\Gamma, \qquad t \in (0,T),$$

$$(I\omega)'(t) = -\int_{\partial\mathcal{S}(t)} (x - h(t))\wedge\sigma(u,p)n\mathrm{d}\Gamma, \qquad t \in (0,T),$$

+ Initial conditions

The Immersed Boundary model



$$\begin{aligned}
\frac{\partial u}{\partial t} + (u\cdot\nabla)u - \nu\Delta u + \nabla p = f, & \qquad \text{in } \mathcal{O},\\
\mathrm{div}\ u = 0, & \qquad \text{in } \mathcal{O},\\
u = 0, & \qquad \text{on } \partial\mathcal{O},\\
u(\cdot,0) = u_0, & \qquad \text{in } \mathcal{O},\\
\Gamma(t) = X(\Gamma(0),t), &
\end{aligned}$$

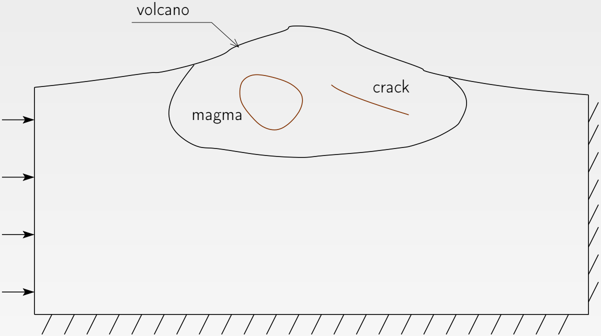
where $X(\cdot,t)$ is the Lagrangian mapping satisfying

$$\frac{\partial X}{\partial t}(y,t) = u(X(y,t),t), \quad X(y,0) = y, \quad y \in \mathcal{O},$$

and where the force f is defined on $\Gamma(t)$ through the expression

$$f(x,t) \quad = \quad \int_{\Gamma(0)} \tilde{f}(y,t)\delta\left(x - X(y,t)\right)\mathrm{d}\Gamma(y,0).$$

Cracks in materials



- In the whole solid: Elasticity model for the displacement (Lamé, viscoelastic solids, etc...)
- Across the crack: Traction force applied on the both sides
  $\rightarrow$ Following the **evolution** of a crack (opening, growth), or **identifying** an unknown crack step by step.

Problematics

- Handling variables/unknowns lying in **time-depending** domains.
- Handling non standard space-time **functional spaces**
- Dealing with **highly coupled** systems;
  In particular coupling related to the **geometry**.

  Several strategies:
  → Trying to uncouple the unknowns
  → Rewriting the systems in **non-depending time** domains
  → Or finding *appropriate* formulations
- Approaches for **theoretical** analysis are not necessarily convenient for performing **numerical** simulations...

Plan

A fluid-solid system

Let us consider a system involving a viscous incompressible fluid:

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u - \nu\Delta u + \nabla p = 0, \qquad x \in \mathcal{F}(t), \quad t \in (0,T),$$

$$\mathrm{div}\ u = 0, \qquad x \in \mathcal{F}(t), \quad t \in (0,T),$$

$$u = 0, \qquad x \in \partial\mathcal{O}, \quad t \in (0,T),$$

$$u = h'(t) + \omega(t) \wedge (x - h(t)), \qquad x \in \partial\mathcal{S}(t), \quad t \in (0,T),$$

$$Mh''(t) = -\int_{\partial\mathcal{S}(t)} \sigma(u,p)n\mathrm{d}\Gamma, \qquad t \in (0,T),$$

$$(I\omega)'(t) = -\int_{\partial\mathcal{S}(t)} (x-h(t)) \wedge \sigma(u,p)n\mathrm{d}\Gamma, \qquad t \in (0,T),$$

$$u(y,0) = u_0(y),\ y \in \mathcal{F}(0), \quad h'(0) = h_1 \in \mathbb{R}^3, \quad \omega(0) = \omega_0 \in \mathbb{R}^3.$$

$\mathcal{S}(t) = h(t) + \mathbf{R}_\omega(t)\mathcal{S}(0)$ and $\mathcal{F}(t) = \mathcal{O} \setminus \overline{\mathcal{S}(t)}$.

Rewriting in fixed domains

Given a reference geometric configuration, for instance the one at time $t = 0$: We want to deal with unknowns defined on $\mathcal{F}(0)$ instead of $\mathcal{F}(t)$.

The first step consists in defining in the fluid domain the change of variables which transforms $\mathcal{F}(0)$ into $\mathcal{F}(t)$:

$$\begin{array}{rccc} X(\cdot,t): & \mathcal{F}(0) & \longrightarrow & \mathcal{F}(t) \\ & y & \longmapsto & X(y,t) \end{array}$$

From that we can define **new variables** for the fluid state:

$$\tilde{u}(y,t) = u(X(y,t),t), \qquad \tilde{p}(y,t) = p(X(y,t),t).$$

The mapping $y \mapsto X(y,t)$ must be **invertible** for all time t, and we expect expect **regularity**, in space and time.

Some properties of the change of variables

Notation: $Y(\cdot, t)$ denotes the inverse of $X(\cdot, t)$.
The mapping $X(\cdot, t)$ shall be a C^1-diffeomorphism from $\mathcal{F}(0)$ onto $\mathcal{F}(t)$.
For a fluid-solid model, the mapping shall satisfy the conditions:

$$\left\{ \begin{array}{ll} \det \nabla X(\cdot, t) = 1 & \text{in } \mathcal{F}(0), \\ X(\cdot, t) = X_{\mathcal{S}}(\cdot, t) & \text{on } \mathcal{S}(0), \\ X(\cdot, t) = \mathrm{Id} & \text{on } \partial\mathcal{O}. \end{array} \right.$$

The original article: Inoue & Wakimoto (1977)

A. Inoue and M. Wakimoto, *On existence of the Navier-Stokes equation in a time dependent domain*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 24 (1977), pp. 303–319.

In this article the change of unknowns is this one:

$$\tilde{u}(y,t) = \nabla Y(X(y,t),t)u(X(y,t),t), \qquad \tilde{p}(y,t) = p(X(y,t),t).$$

With the property $\det \nabla X(y,t) = 1$, we keep for the new velocity the **free-divergence** condition:

$$\mathrm{div}\ \tilde{u}(y,t) \quad = \quad 0.$$

Treatment of non-linearities

For example, the term for the pressure becomes

$$\begin{aligned} p(x,t) &= \tilde{p}(Y(x,t),t), \quad x \in \mathcal{F}(t) \\ \nabla p(x,t) &= \nabla Y(x,t)^T \nabla \tilde{p}(Y(x,t),t), \quad x \in \mathcal{F}(t) \\ \nabla p(X(y,t),t) &= \nabla Y(X(y,t),t)^T \nabla \tilde{p}(y,t), \quad y \in \mathcal{F}(0) \end{aligned}$$

and the linearization process consists in writing

$$\nabla Y(X)^T \nabla \tilde{p} = \nabla \tilde{p} + \left(\nabla Y(X)^T - \mathrm{I}_{\mathbb{R}^3} \right) \nabla \tilde{p}.$$

$\rightarrow$ Necessity of Lipschitz estimates...

Rq: Much more complicated nonlinearities for $\tilde{u}$.

Strong solutions

Searching for strong solutions leads to this kind of regularity:

$$
\begin{aligned}
& u \circ X \in \mathrm{L}^2(0,T;\mathbf{H}^2(\mathcal{F}(0))) \cap \mathrm{H}^1(0,T;\mathbf{L}^2(\mathcal{F}(0))), \\
& p \circ X \in \mathrm{L}^2(0,T;\mathbf{H}^1(\mathcal{F}(0))), \\
& h' \in \mathrm{H}^1(0,T;\mathbb{R}^d), \quad \omega \in \mathrm{H}^1(0,T;\mathbb{R}^{1 \text{ or } 3}).
\end{aligned}
$$

Global strategy:

- Rewriting system in fixed domains (with chg. of var.)
- Linearization and study with the semi-group theory
- Solving the NL system with a fixed point method, with a **contracting** mapping whose the definition comes from the study of the linearized system.

The team from Nancy (France)

- T. Takahashi, *Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain*, Adv. Differential Equations, 2003.
- T. Takahashi, M. Tucsnak, *Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid*, JMFM 2004.
- P. Cumsille, T. Takahashi, *Wellposedness for the system modelling the motion of a rigid body of arbitrary form in an incompressible viscous fluid*, Cz. Math. Journal, 2008.
- J. San Martín, J.-F. Scheid, T. Takahashi, M. Tucsnak, *An Initial and Boundary Value Problem Modeling Fish-like Swimming*, ARMA 2008.

Controllability results in the rigid case

→ O. Imanuvilov, T. Takahashi, *Exact controllability of a fluid-rigid body system*, JMPA, 2007.

→ M. Boulakia, S. Guerrero, *Local null controllability of a fluid-solid interaction problem in dimension 3*, JEMS, 2011.

- Use of a **distributed control** on a sub-domain $\omega \subset\subset \mathcal{F}$ of the fluid.
- Use of a Carleman estimate for the linearized problem.
- The nonlinear problem treated by a fixed-point method.
- Local null controllability for the velocities (fluid+solid), exact controllability for the position of the solid.

Controllability: Towards the deformable case

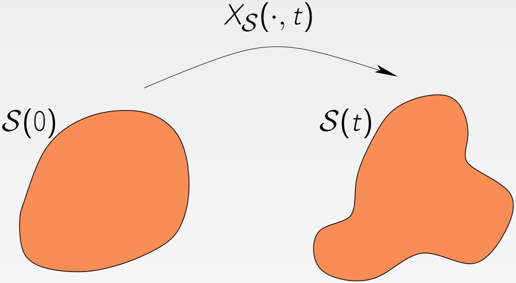
- J. San Martín, T. Takahashi, M. Tucsnak, *A Control Theoretic Approach to the Swimming of Microscopic Organisms*, Quart. Appl. Math. 2007.
- O. Glass, L. Rosier, *On the control of the motion of a boat*, M3AS 2011.

$$\begin{aligned} \mathcal{S}(t) &= h(t) + \mathbf{R}_{\omega}(t)\mathcal{S}(0) \\ u_{\mathcal{S}}(x,t) &= h'(t) + \omega(t) \wedge (x - h(t)) + w(x,t) \end{aligned}$$

- T. Chambrion & A. Munnier:
  - *Locomotion and control of a self-propelled shape-changing body in a fluid*, J. Nonlinear Sci. 2011.
  - *Generic Controllability of 3D Swimmers in a Perfect Fluid*, SIAM J. Control Optim. 2012.

The deformable case

Lagrangian description for the solid: $\mathcal{S}(t) = X_{\mathcal{S}}(\mathcal{S}(0), t)$.



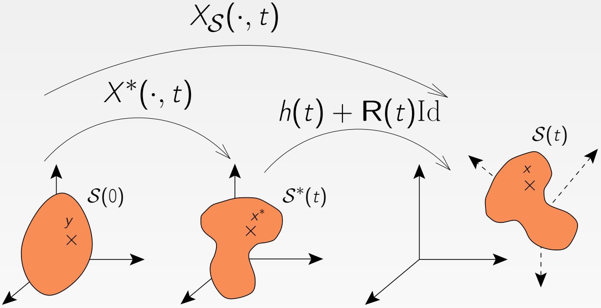
Mass conservation principle gives:

$$\rho_{\mathcal{S}}(X_{\mathcal{S}}(y,t),t) \quad = \quad \frac{\rho_{\mathcal{S}}(y,0)}{\det\left(\nabla X_{\mathcal{S}}(y,t)\right)}, \quad y \in \mathcal{S}(0).$$

The deformable case: Decomposition of movement

The Lagrangian mapping of the solid can be decomposed as follows:

$$X_{\mathcal{S}}(y,t) \quad = \quad h(t) + \mathbf{R}_{\omega}(t)X^*(y,t), \quad y \in \mathcal{S}(0).$$



The mapping $X^*(\cdot,t)$ is the deformation of the solid in **its own frame of reference**. We can consider it as a **control function**.

Constraints on the deformation

The mapping X^* has to satisfy a set of - nonlinear - constraints:

- Conservation of the **whole volume** of the solid:

$$\int_{\partial\mathcal{S}(0)} \frac{\partial X^*}{\partial t} \cdot \left(\mathrm{cof}\nabla X^*\right) n \mathrm{d}\Gamma \ = \ 0.$$

- Conservation of the linear momentum:

$$\int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0) X^*(y,t) \mathrm{d}y \ = \ 0.$$

- Conservation of the angular momentum:

$$\int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0) X^*(y,t) \wedge \frac{\partial X^*}{\partial t}(y,t) \mathrm{d}y \ = \ 0.$$

$+$ Regularity constraints (in relation with the functional framework).

Plan

Goals and motivations

- Goal: Describing how a body can deform itself in order to influence the whole/partial behavior of the coupled system.
- The underlying motivation is the **swim** of a deformable body inside a fluid, without any other help than its interaction with the environing fluid.
- Results: Controllability results of trajectories, **position**, velocities, etc...
- The methods and difficulties are different according to the model/scale...

Low Reynolds number

- The fluid equations are reduced to the Stokes system:

$$\left\{ \begin{array}{rl} -\nu\Delta u + \nabla p = 0, & \text{in } \mathcal{F}(t), \\ \mathrm{div}\ u = 0, & \text{in } \mathcal{F}(t). \end{array} \right.$$

  → J. Lohéac, J.-F. Scheid, M. Tucsnak, *Controllability and time optimal control for low Reynolds numbers swimmers*, Acta Appl. Math., 123(1):175–200, 2013.
  Controllability is obtained for deformation described by a set of 4 types of prescribed elementary deformations.

- Many other results exist at low Reynolds number:
  F. Alouges & L. Giraldi, A. DeSimone...
- T. Chambrion & A. Munnier for perfect fluids.

High Reynolds number

→ O. Glass, L. Rosier, *On the control of the motion of a boat*, M3AS 2011.

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0, \qquad x \in \mathcal{F}(t), \quad t \in (0,T),$$
$$\mathrm{div}\ u = 0, \qquad x \in \mathcal{F}(t), \quad t \in (0,T),$$

$$\lim_{|x| \to \infty} u = 0, \qquad t \in (0,T),$$

$$u \cdot n = (h'(t) + \omega(t) \wedge (x - h(t))) \cdot n + w(x,t), \qquad x \in \partial\mathcal{S}(t), \quad t \in (0,T),$$

$$Mh''(t) = \int_{\partial\mathcal{S}(t)} pn\mathrm{d}\Gamma, \qquad t \in (0,T),$$

$$(I\omega)'(t) = \int_{\partial\mathcal{S}(t)} (x - h(t)) \wedge pn\mathrm{d}\Gamma, \qquad t \in (0,T),$$

+ Initial conditions.

High Reynolds number

→ O. Glass, L. Rosier, *On the control of the motion of a boat*, M3AS 2011.

with: $\mathcal{S}(t) = h(t) + \mathbf{R}_\omega(t)\mathcal{S}(0)$, $\mathcal{F}(t) = \mathbb{R}^2 \setminus \overline{\mathcal{S}(t)}$, $\partial\mathcal{S}(t) = \partial\mathcal{F}(t)$.

- Local controllability of the position/orientation and velocity of the boat, by a control acting on a part of the boundary of the boat.
- The **return method** of Coron is used for controlling the *potential* part of the flow.
- The *vorticity* part of the flow can be controlled by its initial value, for a convenient choice of the vorticity on the boundary of the boat.

Intermediate Reynolds number: The Navier-Stokes equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = 0, \qquad x \in \mathcal{F}(t), \quad t \in (0,T),$$

$$\mathrm{div}\ u = 0, \qquad x \in \mathcal{F}(t), \quad t \in (0,T),$$

$$u = 0, \qquad x \in \partial\mathcal{O}, \quad t \in (0,T),$$

$$u = h'(t) + \omega(t) \wedge (x - h(t)) + w(x,t), \qquad x \in \partial\mathcal{S}(t), \quad t \in (0,T),$$

$$Mh''(t) = -\int_{\partial\mathcal{S}(t)} \sigma(u,p) n \mathrm{d}\Gamma, \qquad t \in (0,T),$$

$$(I\omega)'(t) = -\int_{\partial\mathcal{S}(t)} (x - h(t)) \wedge \sigma(u,p) n \mathrm{d}\Gamma, \qquad t \in (0,T),$$

$$u(y,0) = u_0(y),\ y \in \mathcal{F}(0), \quad h'(0) = h_1 \in \mathbb{R}^3, \quad \omega(0) = \omega_0 \in \mathbb{R}^3.$$

$\mathcal{S}(t) = h(t) + \mathbf{R}_\omega(t) X^*(\mathcal{S}(0), t)$ and $\mathcal{F}(t) = \mathcal{O} \setminus \overline{\mathcal{S}(t)}$.

The main result

Theorem

If the initial conditions (u_0, h_1, ω_0) are small enough, then the fluid-solid system is stabilizable with an arbitrary exponential decay rate:
That means for all $\lambda > 0$ we can choose the deformation X^*, satisfying the nonlinear constraints, so that there exists $C > 0$ - depending only on u_0, h_1 and ω_0 - such that the solution (u, p, h', ω) satisfies:

$$\|(u(\cdot,t), h'(t), \omega(t))\|_{\mathbf{H}^1(\mathcal{F}(t))\times\mathbb{R}^3\times\mathbb{R}^3} \leq C\mathrm{e}^{-\lambda t}.$$

The linearized system

After rewriting in fixed domains and linearizing:

$$\begin{aligned}\frac{\partial u}{\partial t} - \nu\Delta u + \nabla p &= 0, \quad \text{in } (0,\infty)\times\mathcal{F}(0),\\ \text{div } u &= 0, \quad \text{in } (0,\infty)\times\mathcal{F}(0),\end{aligned}$$

$$\begin{aligned}u = 0, &\qquad \text{sur } \partial\mathcal{O}\times(0,\infty),\\ u = h'(t) + \omega(t)\wedge y + \zeta(y,t), &\qquad y \in \partial\mathcal{S}(0), \quad t\in(0,\infty),\end{aligned}$$

$$\begin{aligned}Mh''(t) &= -\int_{\partial\mathcal{S}} \sigma(u,p)n\mathrm{d}\Gamma, \quad t\in(0,\infty),\\ I_0\omega'(t) &= -\int_{\partial\mathcal{S}} y\wedge\sigma(u,p)n\mathrm{d}\Gamma, \quad t\in(0,\infty),\end{aligned}$$

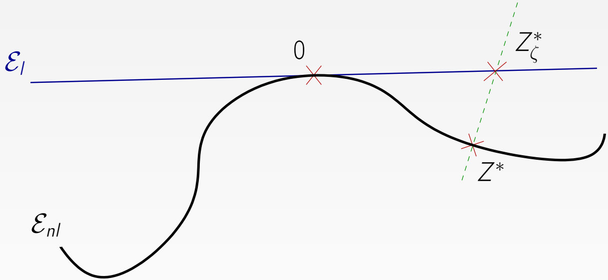
where: $\zeta = \frac{\partial X^*}{\partial t}_{|\partial\mathcal{S}(0)}$. The control ζ can be chosen under a **feedback** form, in order to **shift the spectrum** of the operator.

Nonlinear constraints on the control

We define: $Z^* = X^* - \mathrm{Id}_{\mathcal{S}}$,

$$\begin{aligned} \mathcal{E}_{nl} &= \{\text{displacements } Z^* \text{ such that } \mathfrak{F}(Z^*) = 0\}, \\ \mathcal{E}_{l} &= \left\{\text{displacements } Z^*_\zeta \text{ such that } \mathcal{D}_0\mathfrak{F}(Z^*_\zeta) = 0\right\}. \end{aligned}$$

The displacement $Z^*_\zeta = X^*_\zeta - \mathrm{Id}_{\mathcal{S}}$ is projected on $\mathcal{E}_{nl}$, representing displacements satisfying the NL constraints:



Ideas of the proof for stabilizing the full NL system

The stabilization of the NL system is treated by a fixed point method:

$$\textbf{small data} \Rightarrow \begin{cases} \text{small unknowns} \\ \textit{small} \text{ change of variables} \end{cases}$$
$$\Rightarrow \text{The control input for the NL is close to the one chosen for the linearized system}$$
$$\Rightarrow \text{Stabilization of the full NL system.}$$

For more details:

→ S.C, *Stabilization of a fluid-solid system, by the deformation of the self-propelled solid. Part I: The linearized system*, Evolution Equations and Control Theory, 2014.

→ S.C, *Stabilization of a fluid-solid system, by the deformation of the self-propelled solid. Part II: The nonlinear system*, Evolution Equations and Control Theory, 2014.

Plan

1. Introduction
2. Theoretical analysis of this class of problems
3. Control and Stabilization of a Fluid-Solid system
4. Numerics: Fictitious domain approaches
   - Overview of some methods
   - Illustration 1: Fluid-solid simulations
   - Illustration 2: Cracks inside volcanoes
   - Realistic simulations

Principles and interests

Fictitious domain methods:

- Principle: Considering boundaries **independent** of the mesh, which do not match to the mesh.
- Interest: No re-meshing is required.
- Objectives: When the boundaries are lead to move or to be changed, **updating the less things we need**.
- Price to pay: Working locally, special treatments, while guaranteeing convergence, etc...

Some examples: 1/ ALE formulations

→ San Martín, Smaranda, Takahashi, *Convergence of a finite element/ALE method for the Stokes equations in a domain depending on time*, JCAM 2009.

- The Eulerian formulation of the system is conserved, but **the mesh is deformed** in order to follow the motion of the boundary.
- Necessity of a "good" mesh moving algorithm.
- Appropriate when deformations are small: **Need of re-meshing** in any case, when distortions of the mesh become too large.
- The implementation is quite *technical*...

Some examples: 2/ The eXtended Finite Element Method

→ Moës, Dolbow, Belytschko 1999:
A finite element method for crack growth without remeshing, Internat. J. Numer. Methods Engrg.

- Main ideas:
  - a crack independent of a global mesh
  - a Finite Element Method for which the basis functions are **enriched** near the boundary of the crack, by **singular** functions:

$$\alpha_i \frac{\varphi_i(x)}{\sqrt{\mathrm{d}(x,\Gamma)}} \times \textbf{Heaviside}_{\Gamma}(x)$$

- **Robustness** with respect to the **geometry**: We expect the same behavior whatever the way the edges of the mesh are cut by the crack.

The Fictitious Domain approach we use - Main ideas

- Inspired by XFEM, the main **difference** is that we do not consider singular functions, only **Heaviside functions**:

$$H(x) = \begin{cases} 1 \text{ if } x \in \Omega \ (\textit{computational domain}) \\ 0 \text{ if } x \notin \Omega \end{cases}$$

  $\rightarrow$ It is more an impoverishment, not an enrichment of the standard basis functions.
- This **impoverishment** is a **simplification** of the way we consider boundaries.
- The price to pay is a lack of robustness w.r.t the geometry, and lack of convergence for dual variables...
  $\rightarrow$ Use of stabilization techniques.
- **Advantage**: Easiness of the implementation.

Illustration 1: A Stokes problem with boundary conditions

We consider the following Stokes problem:

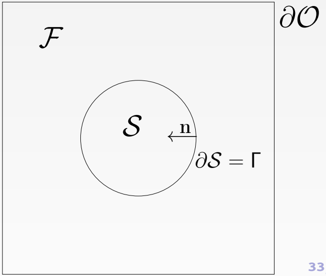
$$\begin{cases} -\nu\Delta \mathbf{u} + \nabla p & = & \mathbf{f} & \text{in } \mathcal{F}, \\ \operatorname{div} \mathbf{u} & = & 0 & \text{in } \mathcal{F}, \\ \mathbf{u} & = & 0 & \text{on } \partial\mathcal{O}, \\ \mathbf{u} & = & \mathbf{g} & \text{on } \Gamma, \end{cases}$$

where $\mathbf{f} \in \mathbf{L}^2(\mathcal{F})$, $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$.

$\boldsymbol{\lambda}$: multiplier associated with the Dirichlet condition on Γ.

Goal: Obtaining an optimal approximation of $\sigma(\mathbf{u}, p)\mathbf{n}$, for boundaries independent of the mesh.

$\sigma(\mathbf{u}, p) = \nu\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) - p\mathrm{Id}.$



Principles

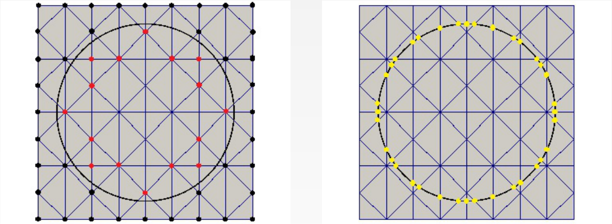
- The fluid-solid interface is represented by a **level-set** function.
- The basis functions are **cut** near the level-set:

$$\mathbf{V} = \left\{ \mathrm{v} \in \mathbf{H}^1(\mathcal{F}) \mid \mathrm{v} = 0 \text{ on } \partial\mathcal{O} \right\}, \qquad Q = \mathrm{L}^2_0(\mathcal{F}), \qquad \mathbf{W} = \left(\mathbf{H}^{1/2}(\Gamma)\right)',$$

$$\tilde{\mathbf{V}}^h \subset \mathbf{H}^1(\mathcal{O}), \qquad \tilde{Q}^h \subset \mathrm{L}^2_0(\mathcal{O}), \qquad \tilde{\mathbf{W}}^h \subset \mathbf{L}^2(\mathcal{O}),$$

$$\mathbf{V}^h = \tilde{\mathbf{V}}^h_{|\mathcal{F}}, \qquad Q^h = \tilde{Q}^h_{|\mathcal{F}}, \qquad \mathbf{W}^h = \tilde{\mathbf{W}}^h_{|\Gamma}.$$

- Selection of degrees of freedom:



→ See XFEM, by Moës, Dolbow and Belytschko for cracked domains in 1999.

A mixed formulation

An augmented Lagrangian technique, à la **Barbosa-Hughes** (1991-1992), is carried out in order to **stabilize** the convergence for the multiplier $\boldsymbol{\lambda}$:

$$L(\mathbf{u}, p, \boldsymbol{\lambda}) = L_0(\mathbf{u}, p, \boldsymbol{\lambda}) - \frac{\gamma}{2} \int_\Gamma \left| \boldsymbol{\lambda} - \sigma(\mathbf{u}, p)\mathbf{n} \right|^2 \mathrm{d}\Gamma,$$

where:

$$\begin{aligned} L_0(\mathbf{u}, p, \boldsymbol{\lambda}) = {} & \nu \int_\mathcal{F} |D(\mathbf{u})|^2 \mathrm{d}\mathcal{F} - \int_\mathcal{F} p \mathrm{div}\ \mathbf{u} \mathrm{d}\mathcal{F} \\ & - \int_\mathcal{F} \mathbf{f} \cdot \mathbf{u} \mathrm{d}\Gamma - \int_\Gamma \boldsymbol{\lambda} \cdot (\mathbf{u} - \mathbf{g}) \mathrm{d}\Gamma. \end{aligned}$$

We choose $\gamma = \gamma_0 * h$ and $\gamma_0 > 0$ has to be chosen **judiciously**.

A mixed formulation

The extended mixed formulation is then:

Find $(\mathbf{u}, p, \boldsymbol{\lambda}) \in \mathbf{V} \times Q \times \mathbf{W}$ such that

$$\begin{cases} \mathcal{A}((\mathbf{u}, p, \boldsymbol{\lambda}); \mathbf{v}) = \mathcal{L}(\mathbf{v}) & \forall \mathbf{v} \in \mathbf{V}, \\ \mathcal{B}((\mathbf{u}, p, \boldsymbol{\lambda}); q) = 0 & \forall q \in Q, \\ \mathcal{C}((\mathbf{u}, p, \boldsymbol{\lambda}); \boldsymbol{\mu}) = \mathcal{G}(\boldsymbol{\mu}), & \forall \boldsymbol{\mu} \in \mathbf{W}, \end{cases}$$

where:

$$\mathcal{A}((\mathbf{u}, p, \boldsymbol{\lambda}); \mathbf{v}) = 2\nu \int_{\mathcal{F}} D(\mathbf{u}) : D(\mathbf{v}) \mathrm{d}\mathcal{F} - \int_{\mathcal{F}} p \operatorname{div} \mathbf{v} \mathrm{d}\mathcal{F} - \int_{\Gamma} \boldsymbol{\lambda} \cdot \mathbf{v} \mathrm{d}\Gamma$$
$$-4\nu^2\gamma \int_{\Gamma} (D(\mathbf{u})\mathbf{n}) \cdot (D(\mathbf{v})\mathbf{n}) \, \mathrm{d}\Gamma + 2\nu\gamma \int_{\Gamma} p \, (D(\mathbf{v})\mathbf{n} \cdot \mathbf{n}) \, \mathrm{d}\Gamma + 2\nu\gamma \int_{\Gamma} \boldsymbol{\lambda} \cdot (D(\mathbf{v})\mathbf{n}) \, \mathrm{d}\Gamma,$$
$$\mathcal{B}((\mathbf{u}, p, \boldsymbol{\lambda}); q) = -\int_{\mathcal{F}} q \operatorname{div} \mathbf{u} \mathrm{d}\mathcal{F} + 2\nu\gamma \int_{\Gamma} q \, (D(\mathbf{u})\mathbf{n} \cdot \mathbf{n}) \, \mathrm{d}\Gamma - \gamma \int_{\Gamma} pq \mathrm{d}\Gamma - \gamma \int_{\Gamma} q \boldsymbol{\lambda} \cdot \mathbf{n} \mathrm{d}\Gamma,$$
$$\mathcal{C}((\mathbf{u}, p, \boldsymbol{\lambda}); \boldsymbol{\mu}) = -\int_{\Gamma} \boldsymbol{\mu} \cdot \mathbf{u} \mathrm{d}\Gamma + 2\nu\gamma \int_{\Gamma} \boldsymbol{\mu} \cdot (D(\mathbf{u})\mathbf{n}) \mathrm{d}\Gamma - \gamma \int_{\Gamma} p(\boldsymbol{\mu} \cdot \mathbf{n}) \mathrm{d}\Gamma - \gamma \int_{\Gamma} \boldsymbol{\lambda} \cdot \boldsymbol{\mu} \mathrm{d}\Gamma \, .$$

Results

→ See also J. Haslinger and Y. Renard, for the Poisson problem, 2009.

- **Theoretical** convergence:
  For $\gamma_0 > 0$ tiny enough, an *inf-sup* condition is automatically satisfied for the triplet $(\mathbf{u}, p, \boldsymbol{\lambda})$.
  → optimal rate of convergence for $\boldsymbol{\lambda} = \sigma(\mathbf{u}, p)\mathbf{n}$.

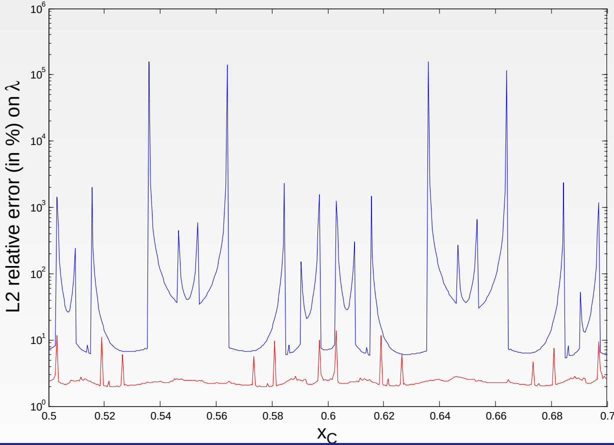
- Rates of convergence verified **numerically**.

- Good behavior with respect to the **geometry**:

  Let us analyze the errors on the approximation of $\boldsymbol{\lambda}$ for different geometric configurations, by considering **different manners of cutting** the fluid domain by the level-set.

Robustness with respect to the geometry

Domain: $[0,1] \times [0,1]$; $h = 0.025$; $(\mathbf{u}, p, \boldsymbol{\lambda}) \rightarrow$ P2/P1/P0.



Solid: Disk of radius $R = 0.21$.

x_C = abscissa of the center of the solid.

blue: without stabilization
red: with stabilization, $\gamma_0 = 0.05$.

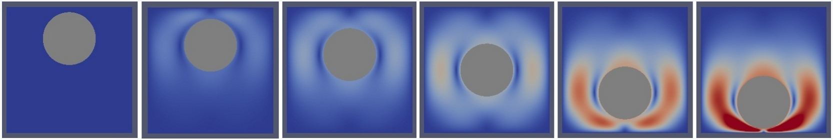
Free fall of a ball for a low Reynolds number

$\mathbf{h}$: position of the center of mass.

$$\begin{aligned} M\mathbf{h}_2''(t) &= -\alpha[\mathbf{h}(t)]_2\mathbf{h}_2'(t) - Mg, \\ \alpha[\mathbf{h}(t)] &= \int_{\mathcal{S}(\mathbf{h}(t))} \sigma(\hat{\mathbf{u}},\hat{p})\mathbf{n}\mathrm{d}\Gamma, \end{aligned}$$

where

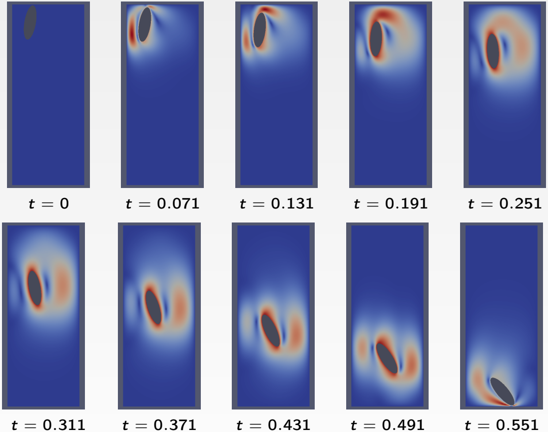
$$\left\{ \begin{array}{rcll} -\nu\Delta\hat{\mathbf{u}} + \nabla\hat{p} &=& 0 & \text{in } \mathcal{F}, \\ \mathrm{div}\ \hat{\mathbf{u}} &=& 0 & \text{in } \mathcal{F}, \\ \hat{\mathbf{u}} &=& 0 & \text{on } \partial\mathcal{O}, \\ \hat{\mathbf{u}} &=& (0,1)^T & \text{on } \Gamma. \end{array} \right.$$



Intensity of the velocity represented at $t = 0$, $t = 21$, $t = 31$, $t = 41$, $t = 51$, $t = 54$.

Extension to the incompressible Navier-Stokes equations

→ Simulation of the **free fall** of an ellipse in 2D
(implicit Euler scheme in time + Newton method):



Next steps...

→ **Next step**: making the solid deform itself, in order to simulate the swim for an intermediate Reynolds number (work in progress).

→ **Final step**: Choosing the deformation of the solid as a **control** function, in order to stabilize a flow, or to control a trajectory.

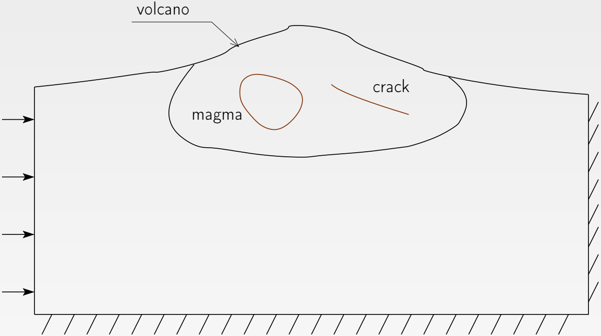
References:

- J. Baiges, R. Codina, F. Henke, S. Shahmiri and W. Wall, *A symmetric method for weakly imposing Dirichlet boundary conditions in embedded finite element meshes*, Int. J. Numer. Meth. Engng, 2012.
- S.C, M. Fournié and A. Lozinski, *A fictitious domain approach for the Stokes problem based on the extended finite element method*, Int. J. Numer. Meth. Fluids., 2013.
- A. Massing, M. Larson, A. Logg, M. E. Rognes, *A stabilized Nitsche fictitious domain method for the Stokes problem*, J. Sci. Comput., 2014.

Illustration 2: Cracks inside volcanoes

Labex ClerVolc: The context of Clermont-Ferrand.

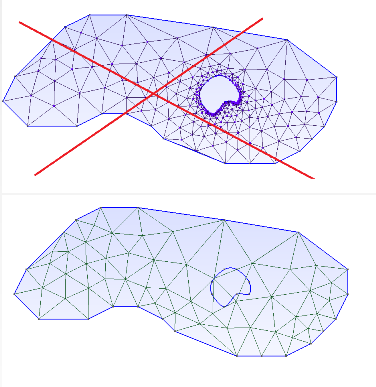
Numerical simulations of displacements inside volcanos



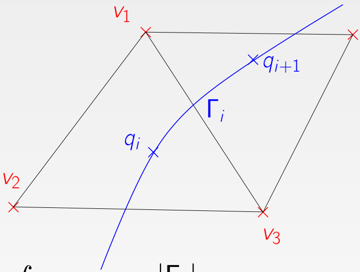
- **Direct problem**: Computing **discontinuous** displacements for non-homogeneous and anisotropic media, for various topographies.
- **Inverse problem**: Recovering information on inside cracks, from surface measurements: geometries, localization...
  $\rightarrow$ Better comprehension of volcanic activities.

From Dirichlet conditions to jump conditions

To take the crack into account, we consider a **non-matching** mesh:



For Dirichlet conditions (weakly):



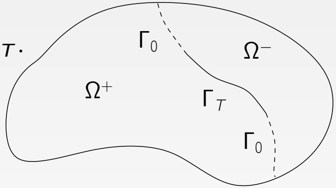
$$\int_{\Gamma_i} qv \simeq \frac{|\Gamma_i|}{2}(q_i v_i + q_{i+1} v_{i+1}),$$
$$v_i = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$\rightarrow$ For jump conditions, necessity of **duplicating** the degrees of freedom around the crack.

An elliptic boundary-value problem: The Lamé system

The displacement inside the volcano is assumed to satisfy:

$$\left\{\begin{array}{rl} -\mathrm{div}\ \sigma_{\mathcal{S}}(u) = f & \text{in } \Omega = \Omega^+ \cup \Omega^-, \\ u = u_{\partial\Omega} & \text{on } \partial\Omega, \\ (\sigma_{\mathcal{S}}(u)n)^{\pm} = pn^{\pm} & \text{on } \Gamma_T, \\ [u] = 0 & \text{across } \Gamma_0 = \Gamma \setminus \Gamma_T. \end{array}\right.$$



with $\sigma_{\mathcal{S}}(u) = 2\mu\varepsilon(u) + \lambda(\mathrm{div}\ u)\mathrm{I}_{\mathbb{R}^d}$,

$$\varepsilon(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right), \quad \lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}, \quad \mu = \frac{E}{2(1+\nu)},$$

and

$$[u] \;=\; u^+ - u^- \quad \text{(the jump across } \Gamma_0\text{)}.$$

Variational formulation

Weak solution = Saddle-point of the Lagrangian:

$$\mathcal{L}(u^+, u^-, \lambda) = \frac{1}{2}\int_{\Omega^+} \varepsilon(u^+) : \sigma_{\mathcal{S}}(u^+) + \frac{1}{2}\int_{\Omega^-} \varepsilon(u^-) : \sigma_{\mathcal{S}}(u^-)$$
$$- \int_{\Omega^+} u^+ \cdot f^+ - \int_{\Omega^-} u^- \cdot f^- - \int_{\Gamma_T} u^+ \cdot pn^+ - \int_{\Gamma_T} u^- \cdot pn^- + \int_{\Gamma_0} [u] \cdot \lambda$$

We are looking for (u^+, u^-, λ) such that for all $v^+, v^- \in \mathbf{H}^1_0(\Omega)$ and $\mu \in \mathbf{H}^{-1/2}(\Gamma_0)$:

$$\left\{ \begin{array}{rcl} \displaystyle\int_{\Omega^+} \varepsilon(u^+) : \sigma_{\mathcal{S}}(v^+) + \int_{\Gamma_0} v^+ \cdot \lambda & = & \displaystyle\int_{\Omega^+} f^+ \cdot v^+ + \int_{\Gamma_T} v^+ \cdot pn^+, \\ \displaystyle\int_{\Omega^-} \varepsilon(u^-) : \sigma_{\mathcal{S}}(v^-) - \int_{\Gamma_0} v^- \cdot \lambda & = & \displaystyle\int_{\Omega^-} f^- \cdot v^- + \int_{\Gamma_T} v^- \cdot pn^-, \\ \displaystyle\int_{\Gamma_0} u^+ \cdot \mu - \int_{\Gamma_0} u^- \cdot \mu & = & 0. \end{array} \right.$$

Abstract formulation

The mixed formulation is then:

Find $(u^+, u^-, \lambda) \in \mathbf{V}^+ \times \mathbf{V}^- \times \mathbf{W}$ such that

$$\left\{ \begin{array}{ll} a^+(u^+, v^+) + b^+(\lambda, v^+) = \mathcal{L}^+(v^+) & \forall v^+ \in \mathbf{V}^+, \\ a^-(u^-, v^-) - b^-(\lambda, v^-) = \mathcal{L}^-(v^-) & \forall v^- \in \mathbf{V}^-, \\ b^+(\mu, u^+) - b^-(\mu, u^+) = 0, & \forall \boldsymbol{\mu} \in \mathbf{W}, \end{array} \right.$$

with $\mathbf{W} = \mathbf{L}^2(\Gamma_0)$, $\mathbf{V}^+ = \left\{ v \in \mathbf{H}^1(\Omega^+) \mid v = 0 \text{ on } \partial\Omega \right\}$, $\mathbf{V}^- = \left\{ v \in \mathbf{H}^1(\Omega^-) \mid v = 0 \text{ on } \partial\Omega \right\}$,

$$\begin{array}{rcl} a^\pm(u^\pm, v^\pm) & = & \displaystyle\int_{\Omega^\pm} \varepsilon(u^\pm) : \sigma_{\mathcal{S}}(v^\pm), \\ b^\pm(\lambda, v^\pm) & = & \displaystyle\int_{\Gamma_0} v^\pm \cdot \lambda, \\ \mathcal{L}^\pm(v^\pm) & = & \displaystyle\int_{\Omega^\pm} f^\pm \cdot v^\pm + \int_{\Gamma_T} v^\pm \cdot pn^\pm. \end{array}$$

Discrete formulation: Matrix formulation

If $\{\phi_i^\pm\}$ and $\{\psi_i\}$ are the selected basis functions of the spaces $\tilde{\mathbf{V}}_h^\pm$ and $\tilde{\mathbf{W}}_h$ respectively, and

$$A^\pm = \left[\int_{\Omega^\pm} \sigma_{\mathcal{S}}(\varphi_i^\pm) : \varepsilon(\varphi_j^\pm)\right]_{ij}, \qquad B^\pm = \left[\int_{\Gamma_0} \varphi_i^\pm \cdot \psi_j\right]_{ij},$$

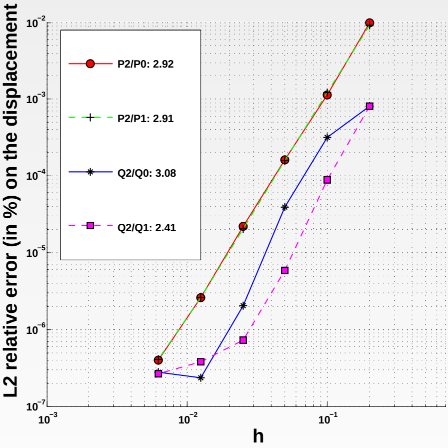
we solve:

$$\left(\begin{array}{c|c||c} A^+ & 0 & (B^+)^T \\ \hline 0 & A^- & -(B^-)^T \\ \hline\hline B^+ & -B^- & 0 \end{array}\right) \left(\begin{array}{c} u^+ \\ \hline u^- \\ \hline\hline \lambda \end{array}\right) = \left(\begin{array}{c} F^+ \\ \hline F^- \\ \hline\hline 0 \end{array}\right).$$

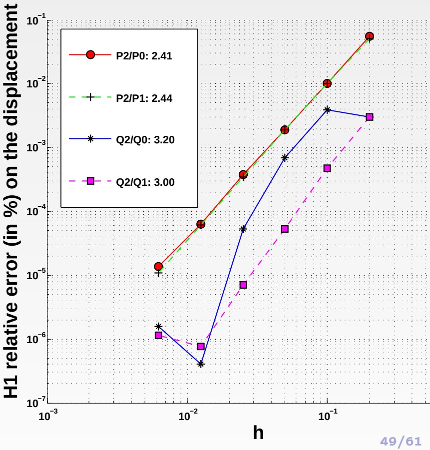
Remark: The integrations on the fictitious domains $\Omega^\pm$ and Γ_0 are made by the use of **Heaviside functions**.

Convergence with rates: Finite elements P2 and Q2

$L^2(\Omega)$-relative error (in %)

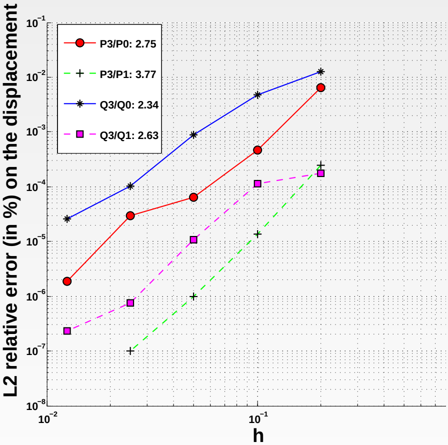


$H^1(\Omega)$-relative error (in %)

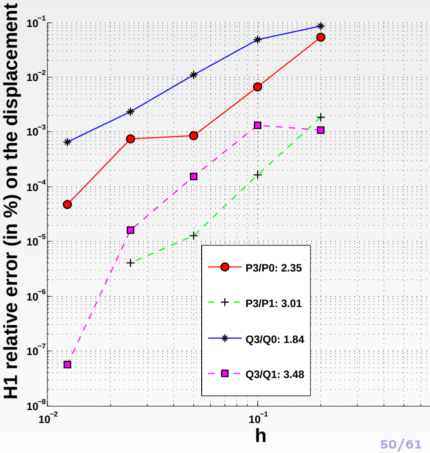


Convergence with rates: Finite elements P3 and Q3

$L^2(\Omega)$-relative error (in %)

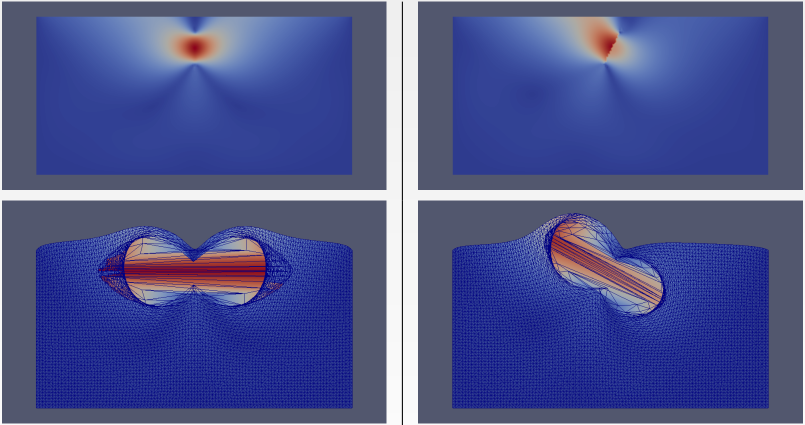


$H^1(\Omega)$-relative error (in %)



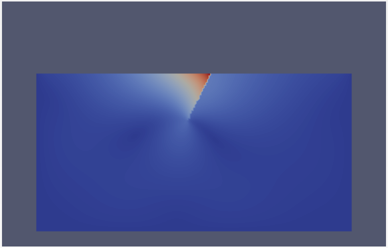
Physical tests: a crack inside the ground

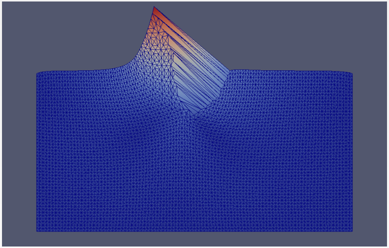
→ D. Pollard, P. Delaney, W. Duffield, E. Endo and A. Okamura, *Surface deformation in volcanic rift zones*, Tectonophysics, 1983.



Physical tests: a crack touching the surface

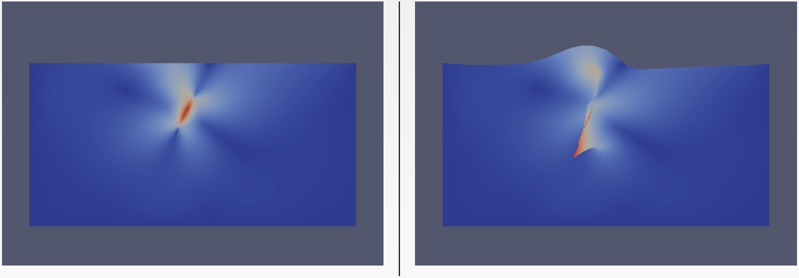
→ D. Pollard, P. Delaney, W. Duffield, E. Endo and A. Okamura, *Surface deformation in volcanic rift zones*, Tectonophysics, 1983.





Physical tests: tangential forces

Applying tangential forces: $(\sigma_{\mathcal{S}}(u)n)^{\pm} = pt^{\pm}$



Remarks and perspectives

- Remark: Optimization of the system resolution: solving faster
  $\rightarrow$ Direct solvers in the 2D case
  $\rightarrow$ Gradient algorithms in the 3D case

- **Inverse problem**: Geometric identification.
  From surface measurements, recovering information on inside cracks: their positions, shapes, estimation of the stress change, and displacements across them...
  $\rightarrow$ Importance of geometries (for the crack) which do not match to the mesh
  (no need of remeshing, only a **local** re-assembling).

No re-meshing

$$A^{\pm} = \left[\int_{\Omega^{\pm}} \sigma_{\mathcal{S}}(\varphi_i^{\pm}) : \varepsilon(\varphi_j^{\pm}) \right]_{ij}, \qquad A = \left[\int_{\Omega} \sigma_{\mathcal{S}}(\varphi_i) : \varepsilon(\varphi_j) \right]_{ij},$$

A: Global matrix, with φ which **do not see the crack** (standard finite elements, classical integration methods).

Reduction and Extension matrices (sparse binary matrices):

$$\tilde{A}^{+} = R^{+}A, \qquad \tilde{A}^{-} = R^{-}A,$$

with the properties: $R^{+}E^{+} = I$, $R^{+} = {E^{+}}^{T}$,
$R^{-}E^{-} = I$, $R^{-} = {E^{-}}^{T}$.

$\rightarrow$ Local re-assembling of the integration terms with Heaviside functions corresponding to the fracture

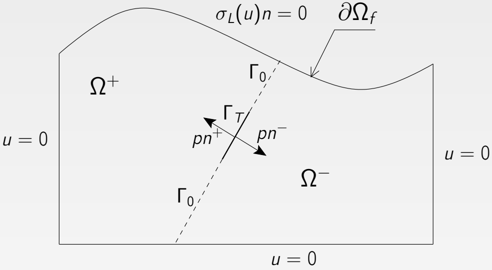
$\Rightarrow$ $\tilde{A}^{+}$ becomes A^{+}, and $\tilde{A}^{-}$ becomes A^{-}.

Theoretical inverse problem

$$J(\Omega) = \frac{1}{2}\int_{\partial\Omega_f} \left| u_{|\partial\Omega_f} - u_{obs} \right|^2$$

with: $\quad \Omega = \Omega^+ \cup \Omega^-$

(the presence of the crack is encoded into this splitting)



Sensitivity w.r.t the domain: $\quad \Omega_t = (\mathrm{Id} + t\theta)\Omega_0$

$$F(t) := J(\Omega_t) \quad \Rightarrow \quad F'(0) = DJ(\Omega_0).\theta$$

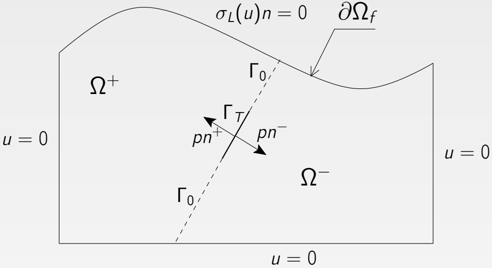
θ with support chosen around $\Gamma = \Gamma_0 \cup \Gamma_T$.

Theoretical inverse problem

$$J(\Omega) = \frac{1}{2}\int_{\partial\Omega_f} \left|u_{|\partial\Omega_f} - u_{obs}\right|^2$$

with: $\Omega = \Omega^+ \cup \Omega^-$

(the presence of the crack is encoded into this splitting)



$$DJ(\Omega).\theta = \int_{\Gamma_0} (\theta \cdot n) K_0(u,w) + \int_{\Gamma_T} (\theta \cdot n) K_T(w)$$

where w is solution of an **adjoint problem**.

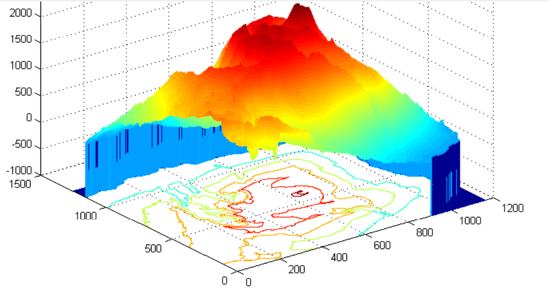
Difficulty: Parameterization of Γ_T

$\rightarrow$ finite degrees of freedom for θ

$\rightarrow$ discrete gradient algorithm for finding Ω_{opt}

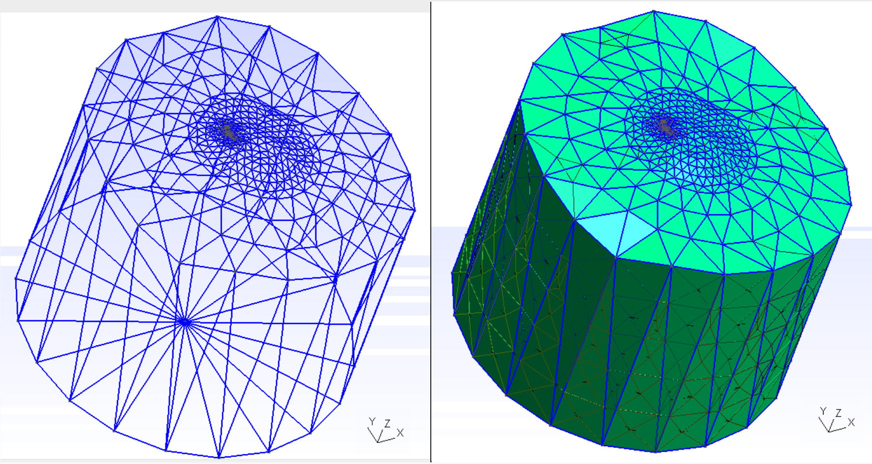
Realistic simulations

- Considering realistic topographies:
  the volcano *Piton de la Fournaise* (île de la Réunion).

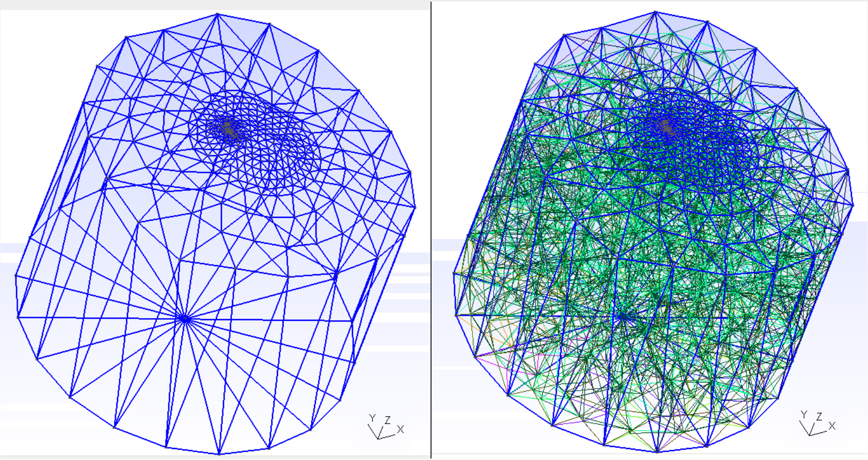


(Data provided by satellites pictures, produced by IGN: Institut Géographique National)

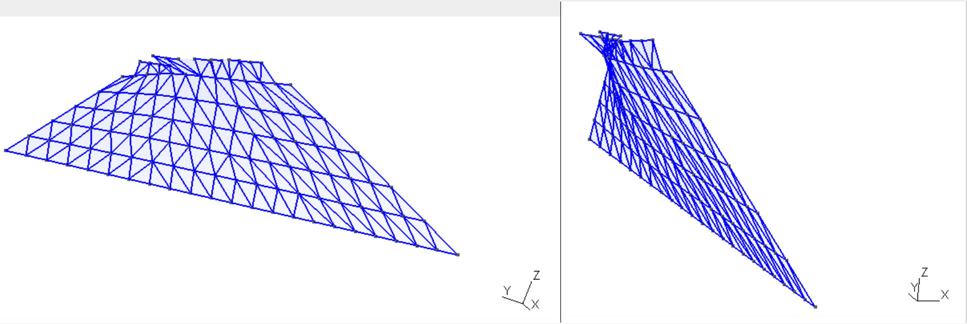
Computational domain



Computational domain

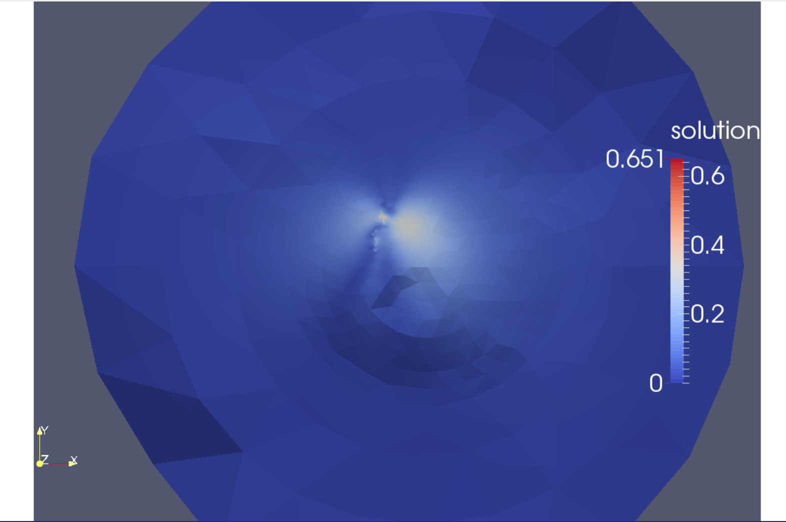


Realistic fractures (called *dikes*)



Triangles touching the surface → Openings due to traction forces

Results



Results

Warp by 1000:

Results

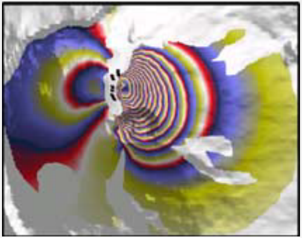
Warp by 2000:

Results

Warp by 3000:

Results

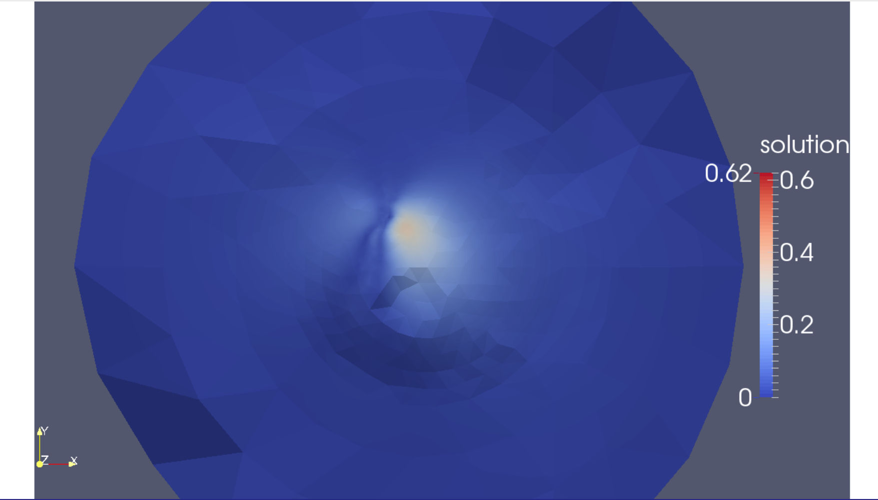
- Comparing quantitatively our results with experimental data obtained by **interferometry**:



Ref: Fukushima & Al.: *Finding realistic dike models from interferometric synthetic aperture radar data: The February 2000 eruption at Piton de la Fournaise*, J. Geophysical Research, 2005.

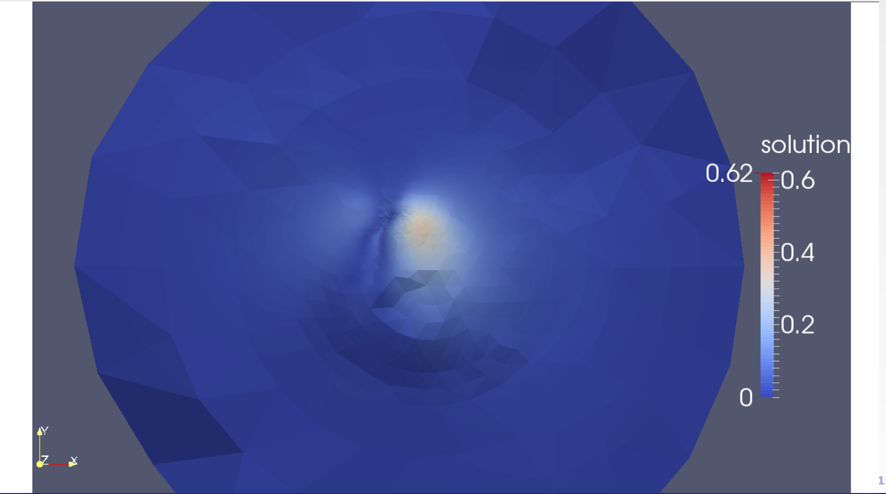
- Considering realistic elasticity coefficients obtained by **muon tomography**.

Results (without surface openings)



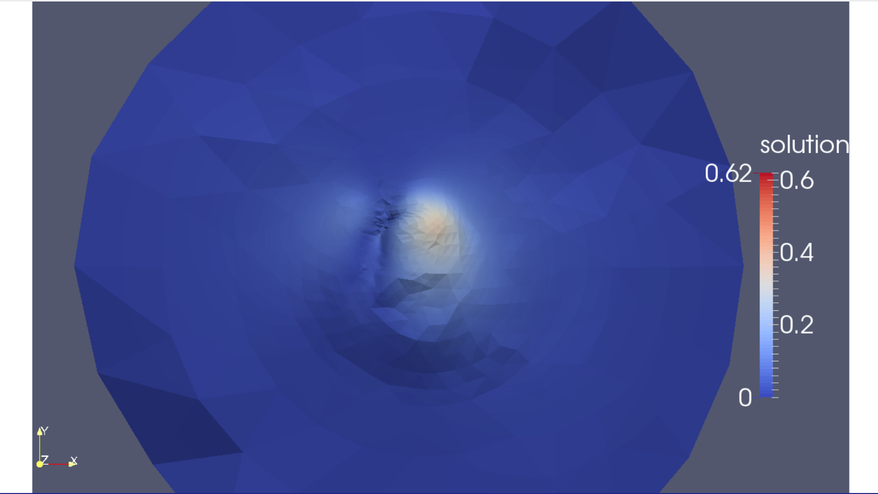
Results (without surface openings)

Warp by 1000:



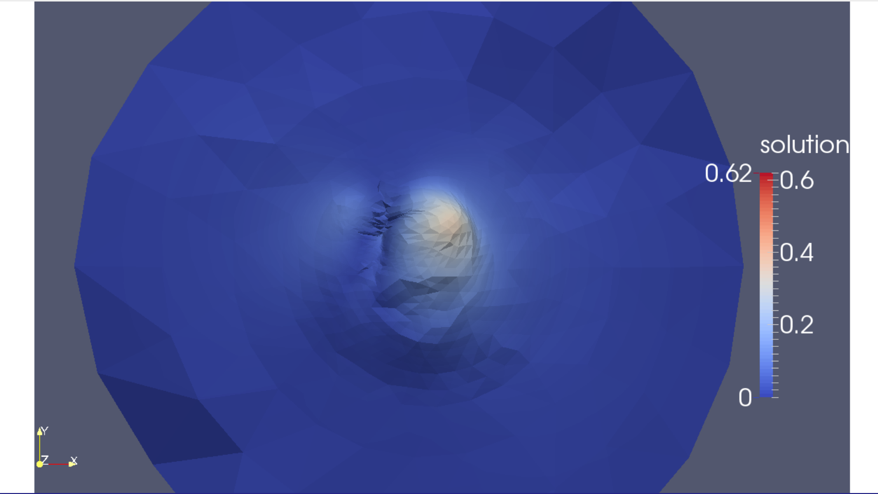
Results (without surface openings)

Warp by 2000:



Results (without surface openings)

Warp by 3000:



All the implementation/simulations are performed with Getfem++, a free finite element library developed in particular by Julien Pommier and Yves Renard:

Y. Renard, J. Pommier, Getfem++.
An open source generic C++ library for finite element methods,
http://home.gna.org/getfem/

THANK YOU FOR YOUR ATTENTION!