Feedback stabilization of a fluid-rigid body interaction system

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1er octobre 2014

Outline

Presentation of the problem

The 1-d case

The multi-d case

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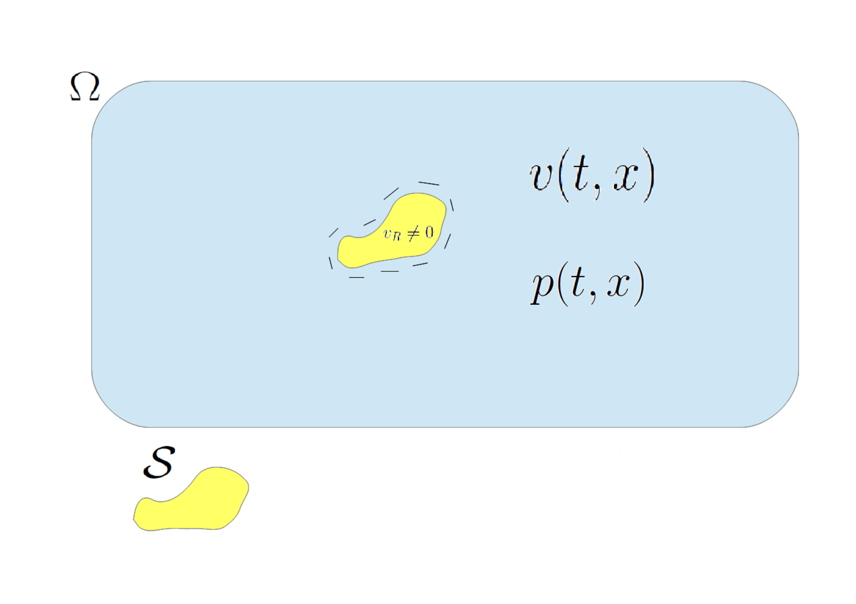
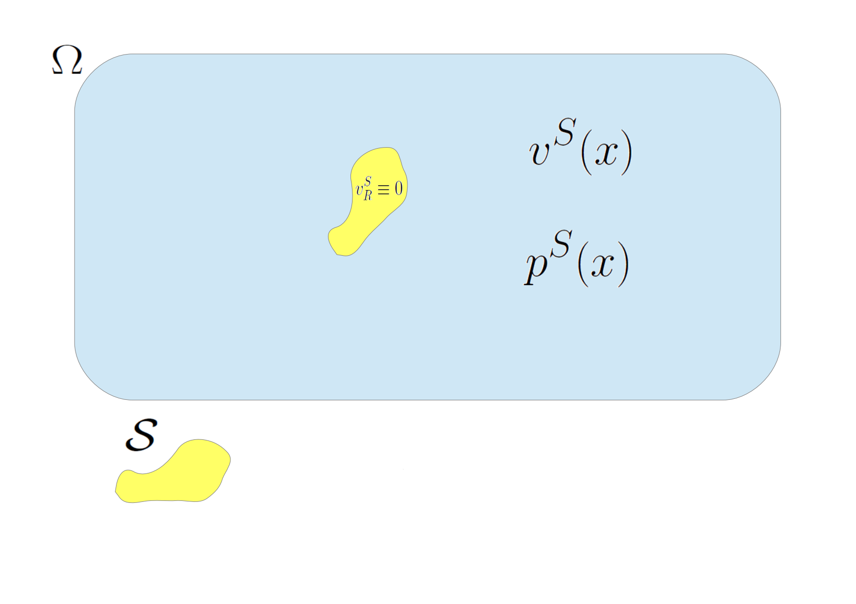
The multi-d case

2d fluid – rigid body interaction system



2d fluid – rigid body interaction system



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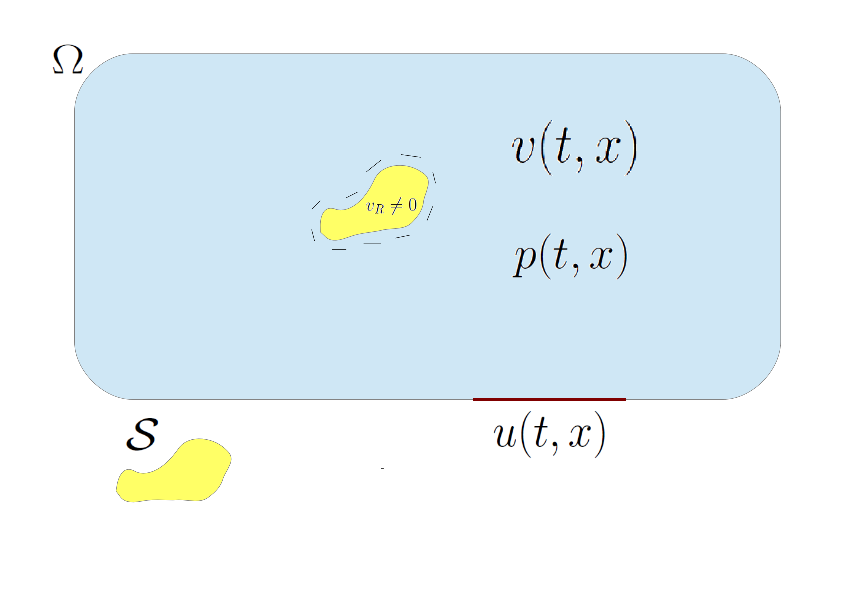

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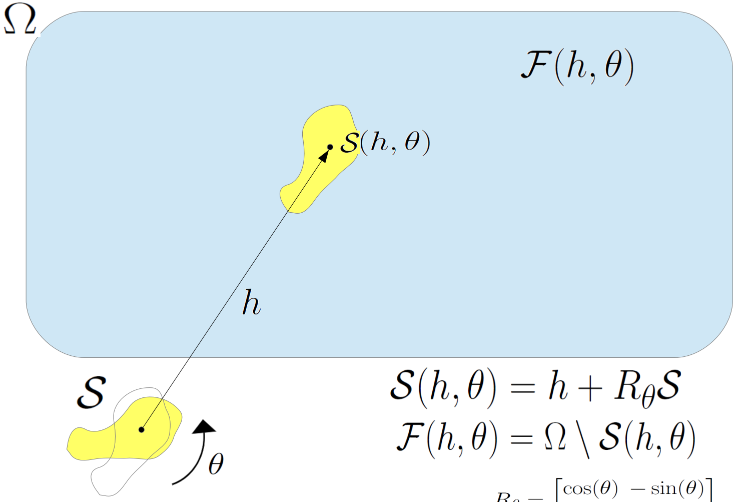
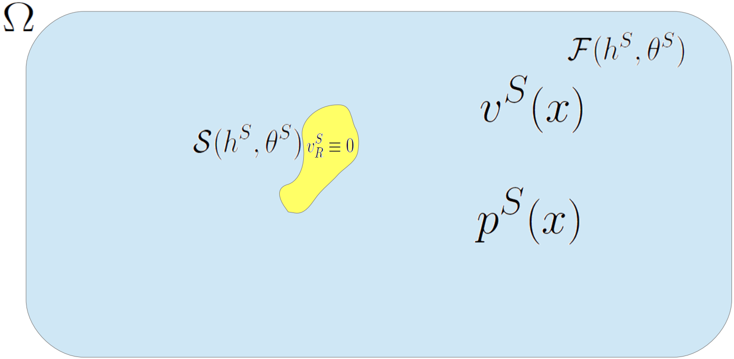


$$\mathcal{S}(h,\theta) = h + R_\theta \mathcal{S}$$

$$\mathcal{F}(h,\theta) = \Omega \setminus \mathcal{S}(h,\theta)$$

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

The stationary system

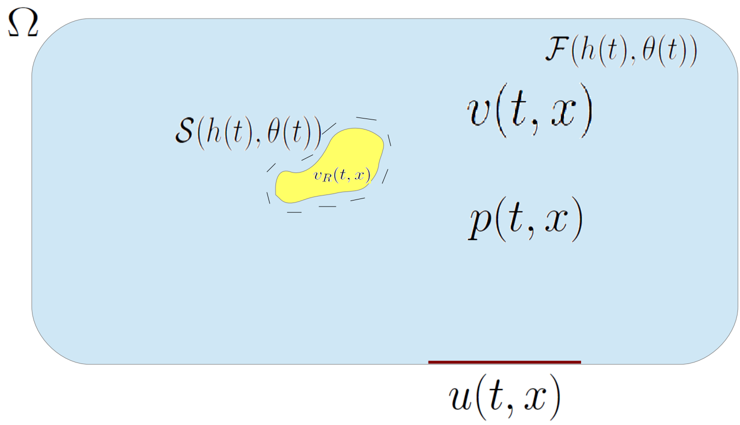


The stationary system

$$\begin{cases} (v^S \cdot \nabla)v^S - \nu\Delta v^S + \nabla p^S = f^S \quad \text{in } \mathcal{F}(h^S,\theta^S), \\ \operatorname{div} v^S = 0 \quad \text{in } \mathcal{F}(h^S,\theta^S), \\ v^S(x) = 0, \quad x \in \partial\mathcal{S}(h^S,\theta^S), \\ v^S(x) = b^S(x), \quad x \in \partial\Omega, \\ -\displaystyle\int_{\partial\mathcal{S}(h^S,\theta^S)} \mathbb{T}(v^S,p^S)n \, d\Gamma + f_M^S = 0, \\ -\displaystyle\int_{\partial\mathcal{S}(h^S,\theta^S)} (x-h^S)^\perp \cdot \mathbb{T}(v^S,p^S)n \, d\Gamma + f_I^S = 0 \end{cases}$$

$$\mathbb{T}(v,p) \overset{\text{def}}{=} 2\nu D(v) - p\,\mathrm{Id}, \quad \text{with} \quad D(v) \overset{\text{def}}{=} \frac{1}{2}\left((\nabla v) + {}^t(\nabla v)\right),$$

The controlled evolution system



$$v_R(t,x) = h'(t) + \theta'(t)(x - h(t))^{\perp}$$

The controlled evolution system

$$
\left\{
\begin{array}{l}
\partial_t v + (v \cdot \nabla) v - \nu \Delta v + \nabla p = f^S \quad \text{in } \mathcal{F}(h(t), \theta(t)), \\[2ex]
\operatorname{div} v = 0 \quad \text{in } \mathcal{F}(h(t), \theta(t)), \\[2ex]
v(t,x) = h'(t) + \theta'(t)(x - h(t))^{\perp}, \quad x \in \partial\mathcal{S}(h(t), \theta(t)), \\[2ex]
v(t,x) = b^S(x) + u(t,x), \quad x \in \partial\Omega, \\[2ex]
M h'' = - \displaystyle\int_{\partial\mathcal{S}(h(t),\theta(t))} \mathbb{T}(v,p) n \; d\Gamma + f_M^S, \\[2ex]
I \theta'' = - \displaystyle\int_{\partial\mathcal{S}(h(t),\theta(t))} (x - h)^{\perp} \cdot \mathbb{T}(v,p) n \; d\Gamma + f_I^S \\[2ex]
h(0) = h^0, \quad \theta(0) = \theta^0, \quad h'(0) = h^1, \quad \theta'(0) = \theta^1, \\
v(0,x) = v^0(x) \quad x \in \mathcal{F}(h^0, \theta^0).
\end{array}
\right.
$$

The stabilization problem

Q1. Can we find a feedback control of the form

$$u(t) = F(v(t), h(t), h'(t), \theta(t), \theta'(t)),$$

such that for

$$(v^0, h^0, h^1, \theta^0, \theta^1) \quad \text{“close to”} \quad (v^S, h^S, 0, \theta^S, 0)$$

the following exponential decay holds :

$$\|v(t)-v^S\| + |h'(t)| + |\theta'(t)| + |h(t)-h^S| + |\theta(t)-\theta^S| \leqslant C_0 e^{-\sigma t} \text{ ?}$$

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Q2. Is it possible to find a finite dimensional feedback law, i.e

$$u(t,x) = \sum_{j=1}^{N_\sigma} F_j(v(t), h(t), h'(t), \theta(t), \theta'(t)) v_j(x) \ ?$$

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Q3. The rate $\sigma > 0$ may be chosen arbitrarily large ?

Comments

- Case of "finite dimensional" deformable structure :

$$v(t,x) = Mr'(t), \quad t > 0, x \in \Gamma_s$$
$$r''(t) + Qr = -\int_{\Gamma_s} M^T \mathbb{T}(v,p)n \, d\Gamma$$

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- Control on the body :

$$v(t,x) = h'(t) + \theta'(t)(x - h(t))^{\perp} + u(t,x), \; x \in \partial\mathcal{S}(h(t),\theta(t)),$$

$$v(t,x) = b^S(x), \quad x \in \partial\Omega,$$

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$$Mh'' = -\int_{\partial\mathcal{S}(h(t),\theta(t))} \mathbb{T}(v,p)n \; d\Gamma + f_M^S + g_M(t),$$

$$I\theta'' = -\int_{\partial\mathcal{S}(h(t),\theta(t))} (x-h)^{\perp} \cdot \mathbb{T}(v,p)n \; d\Gamma + f_I^S + g_M(t),$$

References

Null controllability of Fluid-rigid body interaction syst.

- Imanuvilov and Takahashi (2007), 2d, ball.
- Boulakia and Osses (2008), 2d, symmetry assumption, H^3 initial datum.
- Boulakia and Guerrero (2013), 3d, H^2 i. d..
- Raymond and Vanninathan (2010, 2009), Lequeurre (2013), finite dimensional approximation of deformable solid.

Stabilization of Fluid-deformable body interaction syst.

- Raymond (2010), coupling with a controlled damped beam.
- Court (2014), control through the deformation of the moving solid.

In all above references the target state is zero ($v^S = 0$).
It is stable!

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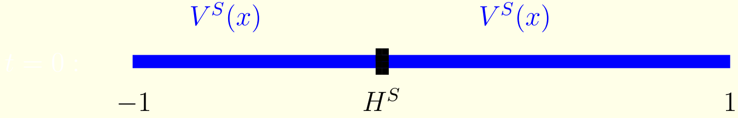
Stabilization of a 1d fluid – particle system

Feedback stabilization of a simplified 1d fluid–particle system, M. B. and Takéo Takahashi, *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 31, n. 2, pp. 36–389, 2014.

Simplified 1d fluid – particle system



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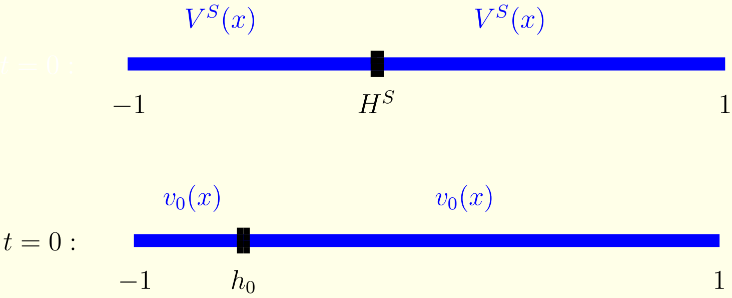

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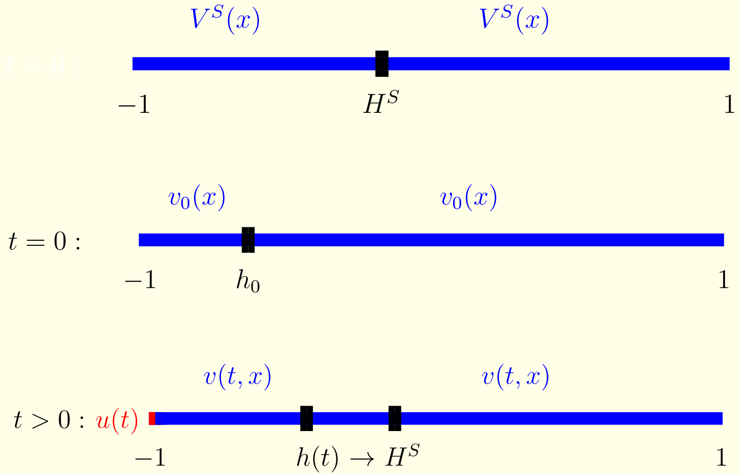
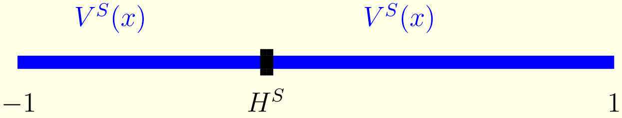


Stationary states



For f^S, r^S, a^S and b^S given :

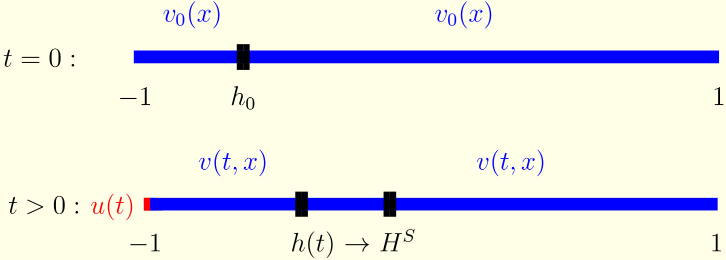
$$\begin{cases} -V^S_{yy} + V^S V^S_y = f^S \qquad (y \in (-1,1) \setminus \{H^s\}), \\ V^S(-1) = a^S, \quad V^S(H^S) = 0, \quad V^S(1) = b^S, \\ 0 = [V^S_y](H^S) + m r^S. \end{cases}$$

where $[f](x) \stackrel{\text{def}}{=} f(x^+) - f(x^-)$.

If $f^S = 0$, $r^S = 0$ then V^S is of the form :

- $V^S(y) = 2c \tan(c(y - H^S))$, $c \geqslant 0$
- $V^S(y) = -2c \tanh(c(y - H^S))$ $c \geqslant 0$

Simplified 1d fluid – particle system



$v = v(t,x)$ is the velocity of the fluid ; $h = h(t)$ is the position of the particle ; $u = u(t)$ is the control

$$
\begin{cases}
v_t - v_{xx} + vv_x = f^S \qquad (t \geqslant 0, \quad x \in (-1,1) \setminus \{h(t)\}), \\
v(t, h(t)) = \dot{h}(t), \\
m\ddot{h}(t) = [v_x](t, h(t)) + mr^S \qquad (t \geqslant 0), \\
v(t,-1) = a^S + u(t), \quad v(t,1) = b^S, \qquad (t \geqslant 0), \\
h(0) = h_0, \quad \dot{h}(0) = \ell_0, \quad v(0,x) = v_0(x) \quad x \in (-1,1) \setminus \{h_0\}.
\end{cases}
$$

Feedback Stabilization result

Theorem

Assume $f^S \in W^{2,\infty}((-1,1))$ *and* $V^S \in W^{1,\infty}((-1,1))$. *There exists* $\sigma_0 > 0$ *such that for all* $\sigma \in (0, \sigma_0)$, *there exist* $\mu > 0$ *and* $(\widehat{\varphi}, \widehat{g}, \widehat{k}) \in L^2(-1,1) \times \mathbb{R}^2$, *such that, if*

$$\|v_0 - V^S\|_{L^2((-1,1))} + |h_1| + |h_0 - H^S| \leqslant \mu$$

then there exists a unique solution

$$v \in L^2_{loc}(\mathbb{R}_+, H^1(-1,1)) \cap C(\mathbb{R}_+, L^2(-1,1)), \quad h \in H^1_{loc}(\mathbb{R}_+),$$

with

$$u(t) = \int_{-1}^{1} \big(v(t, X(h(t),y)) - V^S(y)\big)\, \widehat{\varphi} \mathrm{dy} + m\dot{h}(t)\widehat{g} + h(t)\widehat{k}.$$

Moreover, there is a constant $C > 0$ *such that*

$$\begin{aligned}\|v(t) - V^S\|_{L^2(-1,1)} &+ |\dot{h}(t)| + |h(t) - H^S| \\ &\leqslant Ce^{-\sigma t}\left(\|v_0 - V^S\|_{L^2((-1,1))} + |h_1| + |h_0 - H^S|\right).\end{aligned}$$

Feedback Stabilization result for an arbitrary rate σ

Theorem

Assume also that V^S satisfies one of the two conditions

$$V_y^S(H^S) \geqslant 0$$

or

$$\forall y \in (H^S, 1), \quad V_y^S(H^S) + \frac{V_y^S(y)}{2} + \frac{1}{4}(V^S(y))^2 + \left(\frac{\pi}{1-H^S}\right)^2 > 0.$$

Then, we can take $\sigma_0 = \infty$ in above Theorem.

Comments

1. We can construct a stationary state V^S such that system is not stabilizable for rates $\sigma < \sigma_0 \stackrel{\text{def}}{=} -V_y^S(H^S)$.

2. An analogue stabilization result is available with two controls, one on each side. In that case $\sigma_0 = \infty$.

3. An analogue stabilization result is available in the case of N particles.

References

Well-posedness.

- Vázquez and Zuazua (2003, 2006)

Controllability results.

- Doubova and Fernández-Cara (2005), $V^S \equiv 0$, use of two controls, one at each boundary.

- Liu, Takahashi and Tucsnak (2012), $V^S \equiv 0$, use of a single control.

General methodology for feedback stabilization

(Applied to N-S-E, Boussinesq, MHD, ...)

Step 1 : rewrite PDE system in the form

$$\mathbf{Y}' = A\mathbf{Y} + G(\mathbf{Y}, u) + Bu, \quad \mathbf{Y}(0) = \mathbf{Y}_0 \in H$$

Step 2 : find a finite rank operator $F : H \to U$ such that system

$$\mathbf{Y}' = A\mathbf{Y} + Bu, \quad \mathbf{Y}(0) = \mathbf{Y}_0 \in H$$

with

$$u = F\mathbf{Y}$$

satisfies

$$\|\mathbf{Y}(t)\|_H \leqslant Ce^{-\sigma t}\|\mathbf{Y}_0\|_H$$

Step 3 : for $\mathbf{Y}_0$ small enough, obtain a stable solution to

$$\mathbf{Y}' = \underbrace{A\mathbf{Y} + BF\mathbf{Y}}_{A_F\mathbf{Y}} + \underbrace{G(\mathbf{Y}, F\mathbf{Y})}_{N(\mathbf{Y})}, \quad \mathbf{Y}(0) = \mathbf{Y}_0$$

with a fixed-point procedure.

Step 1. Change of variables

$$X(h(t), y) \stackrel{\text{def}}{=} y + \eta(y)(h(t) - H^S),$$

with $\eta \in C^\infty((-1, 1))$ satisfying

- $\eta(H^S) = 1$, $\eta_y(H^S) = 0$,
- $\eta(1) = \eta(-1) = 0$.

We set

$$V(t, y) \stackrel{\text{def}}{=} v(t, X(h(t), y)) \quad \text{and} \quad V = W + V^S$$

To simplify we assume for the following :

$$H^S = 0.$$

Step 1. System after change of variables

$$\begin{cases} W_t - W_{yy} + h\left(\eta' V_y^S - \eta f^S\right)_y + (V^S W)_y - \eta \ell V_y^S = G\,(W, \ell, h) \\ \qquad\qquad\qquad\qquad\qquad\qquad (t \geqslant 0, \quad y \in (-1,1) \setminus \{0\}), \\ W(t,0) = \ell(t) \qquad (t \geqslant 0), \\ \dot{\ell}(t) = \frac{1}{m}[W_y](t,0) \qquad (t \geqslant 0), \\ \dot{h}(t) = \ell(t) \qquad (t \geqslant 0), \\ W(t,-1) = u(t), \quad W(t,1) = 0 \qquad (t \geqslant 0), \\ h(0) = h_0, \quad \ell(0) = \ell_0, \\ W(0,y) = V_0(y) - V^S(y) \quad y \in (-1,1) \setminus \{0\}. \end{cases}$$

Step 1. Abstract setting

$$H \stackrel{\text{def}}{=} L^2((-1,1)) \times \mathbb{R}^2$$

$$\mathcal{D}(A) \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} W \\ \ell \\ h \end{bmatrix} \in H \,\middle|\, \begin{array}{l} W \in H^2((-1,1) \setminus \{0\}) \cap H_0^1((-1,1)), \\ W(0) = \ell \end{array} \right\}.$$

$$A \begin{bmatrix} W \\ \ell \\ h \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} W_{yy} - h\left(\eta' V_y^S - \eta f^S\right)_y - (V^S W)_y + \eta \ell V_y^S \\ \dfrac{1}{m}[W_y](0) \\ \ell \end{bmatrix}$$

$$B : \mathbb{R} \to [\mathcal{D}(A^*)]', \qquad B^* \begin{bmatrix} \varphi \\ g \\ k \end{bmatrix} = \varphi_y(-1)$$

$$\begin{bmatrix} W \\ \ell \\ h \end{bmatrix}' = A \begin{bmatrix} W \\ \ell \\ h \end{bmatrix} + G\left(\begin{bmatrix} W \\ \ell \\ h \end{bmatrix} \right) + Bu,$$

Step 2. Stabilizability of the homogeneous linear system

Summary. The 1-d linearized fluid-particle system rewrites

$$\mathbf{Y}' = A\mathbf{Y} + Bu \in [\mathcal{D}(A^*)]', \quad \mathbf{Y}(0) = \mathbf{Y}_0 \in H \tag{1}$$

where

- $A : \mathcal{D}(A) \subset H \to H$ generates an analytic semigroup on H and has compact resolvent.
- $B : U \to [\mathcal{D}(A^*)]'$ and $(\lambda_0 - A)^{1-\epsilon} B : H \to H$ bounded.

Goal.
For $\sigma > 0$ find a feedback control of the form

$$u(t) = \sum_{j=1}^{N_\sigma} (\mathbf{Y}(t), \varepsilon_j)_H v_j$$

such that $\forall \mathbf{Y}_0 \in H$:

$$\|\mathbf{Y}(t)\|_H \leqslant C e^{-\sigma t} \|\mathbf{Y}_0\|_H$$

Step 2. Stabilizability of the homogeneous linear system

Theorem (Hautus/Fattorini criterion for stabilizability)

System (1) *is stabilizable by a finite dimensional feedback control of the form*

$$u(t) = \sum_{j=1}^{N_\sigma} (\mathbf{Y}(t), \Pi v_j)_H v_j$$

for an exponential rate of decrease $\sigma > 0$ *if and only if for all* $\lambda \in \mathbb{C}\ \Re\lambda \geqslant -\sigma$ *:*

$$\lambda \mathbf{Y} - A^* \mathbf{Y} = 0 \quad \text{and} \quad B^* \mathbf{Y} = 0 \quad \Longrightarrow \quad \mathbf{Y} = 0. \qquad (2)$$

Remark.

- N_σ must be $\geqslant$ to the maximum of geometric multiplicities of A's eigenvalues with real part greater that $-\sigma$.
- The set of admissible families $v = (v, \ldots, v_{N_\sigma})$ form an open and dense subset of U^{N_σ}.
- Π is calculated from a finite dimensional Riccati equation.

Ref.

Fattorini (1966), Triggiani (1975), B., Takahashi (2011, 2013).

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Step 2. Criterion for stabilizability

Assume

$$\lambda \in \mathbb{C}, \quad \Re\lambda \geqslant -\sigma,$$

and

$$\begin{cases} \lambda\varphi - \varphi_{yy} - V^S\varphi_y = 0, \quad y \in (-1,1)\setminus\{0\}, \\ \varphi(0) = g, \\ m\lambda g = [\varphi_y](0) + k + \displaystyle\int_{-1}^{1} \eta V_y^S \varphi \mathrm{dy}, \\ \lambda k = \displaystyle\int_{-1}^{1} \left(-\eta' V_y^S + \eta f^S\right)_y \varphi \mathrm{dy}, \\ \varphi(-1) = \varphi(1) = 0, \end{cases}$$

and

$$\varphi_y(-1) = 0.$$

Do we have $g = k = 0$ and $\varphi \equiv 0$ in $(-1,1)$?

Step 2. Criterion for stabilizability

Standart uniqueness result yield $\varphi \equiv 0$ in $(-1,0)$ and then $g = \varphi(0) = 0$ and

$$\begin{cases} \lambda\varphi - \varphi_{yy} - V^S\varphi_y = 0, \quad y \in (0,1), \\ \varphi(0) = \varphi(1) = 0, \\ \lambda\varphi_y(0) = \int_0^1 \left(\eta' V_y^S - \eta f^S\right)_y \varphi \mathrm{dy} - \int_0^1 \lambda\eta V_y^S \varphi \mathrm{dy} \end{cases}$$

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Mutliplying by ηV_y^S and integrating by parts, we obtain

$$0 = (\lambda - V_y^S(0))\varphi_y(0). \tag{3}$$

If $\lambda \neq V_y^S(0)$ then standart uniqueness result yields $\varphi \equiv 0$ in $(0, 1)$. But if $\lambda = V_y^S(0)$...

Step 2. Existence of an uncontrollable mode

Does there exists a non null solution to the following problem ?

$$\begin{cases} V_y^S(0)\varphi - \varphi_{yy} - V^S\varphi_y = 0, \quad y \in (0,1), \\ \varphi(0) = \varphi(1) = 0. \end{cases} \qquad (\star)$$

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- The answer is **negative** if

$$V_y^S(0) \geqslant 0$$

or

$$\forall y \in (0,1), \quad V_y^S(0) + \frac{V_y^S(y)}{2} + \frac{1}{4}(V^S(y))^2 + \pi^2 > 0.$$

In this case we set $\sigma_0 = \infty$.

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In this case we set $\sigma_0 = \infty$.

- There exists $V^S \in C^\infty([0,1])$ for which the answer is **positive**.

In this case we set $\sigma_0 = -V_y^S(0)$.

Step 3. Stabilization theorem for nonlinear system

Theorem

For all $\sigma \in (0, \sigma_0)$ *there exist* $\mu > 0$, $(\widehat{\varphi}, \widehat{g}, \widehat{k}) \in L^2(-1,1) \times \mathbb{R}^2$, *such that, if*

$$\|W_0\|_{L^2((-1,1))} + |\ell_0| + |h_0| \leqslant \mu$$

then system

$$\begin{cases} W_t - W_{yy} + h\left(\eta' V_y^S - \eta f^S\right)_y + (V^S W)_y - \eta \ell V_y^S = G(W, \ell, h) \\ W(t,0) = \ell(t) \qquad (t \geqslant 0), \\ m\dot{\ell}(t) = [W_y](t,0) \qquad (t \geqslant 0), \\ \dot{h}(t) = \ell(t) \qquad (t \geqslant 0), \\ W(t,-1) = \int_{-1}^{1} W(t)\widehat{\varphi}\,\mathrm{dy} + m\dot{h}(t)\widehat{g} + h(t)\widehat{k} \qquad (t \geqslant 0). \\ W(0) = W_0, \quad \ell(0) = \ell_0, \quad h(0) = h_0. \end{cases}$$

admits a unique solution
$W \in L^2_{\mathrm{loc}}(\mathbb{R}_+, H^1(-1,1)) \cap C(\mathbb{R}_+, L^2(-1,1))$, $h \in C^1(\mathbb{R}_+)$.
Moreover, there exists $C > 0$ *such that*

$$\|W(t)\|_{L^2(-1,1)} + |\ell(t)| + |h(t)| \leqslant Ce^{-\sigma t}\left(\|W_0\|_{L^2((-1,1))} + |\ell_0| + |h_0|\right).$$

Outline

Presentation of the problem

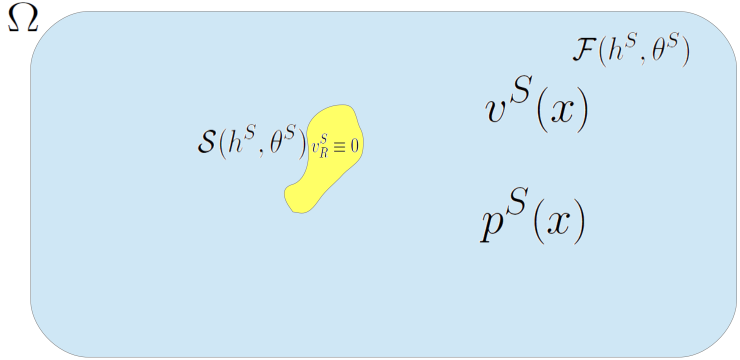
The 1-d case

The multi-d case

Stabilization of a Fluid-Rigid body Interaction System

Feedback Stabilization of a Fluid-Rigid body Interaction System, M. B. and Takéo Takahashi, *Advances in Differential Equations*, vol. 19, n. 11–12, pp. 1137–1184, 2014.

The stationary system

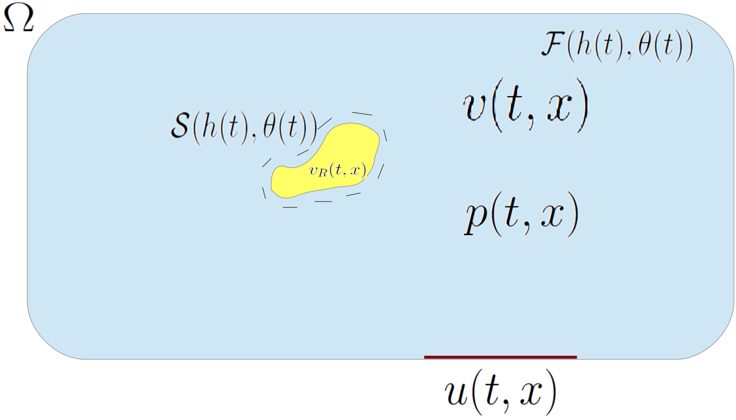


The stationary system

$$
\begin{cases}
(v^S \cdot \nabla)v^S - \nu \Delta v^S + \nabla p^S = f^S \quad \text{in } \mathcal{F}(h^S, \theta^S), \\[2ex]
\operatorname{div} v^S = 0 \quad \text{in } \mathcal{F}(h^S, \theta^S), \\[2ex]
v^S(x) = 0, \quad x \in \partial\mathcal{S}(h^S, \theta^S), \\[2ex]
v^S(x) = b^S(x), \quad x \in \partial\Omega, \\[2ex]
-\displaystyle\int_{\partial\mathcal{S}(h^S,\theta^S)} \mathbb{T}(v^S, p^S) n \, d\Gamma + f_M^S = 0, \\[2ex]
-\displaystyle\int_{\partial\mathcal{S}(h^S,\theta^S)} (x - h^S)^\perp \cdot \mathbb{T}(v^S, p^S) n \, d\Gamma + f_I^S = 0
\end{cases}
$$

$$
\mathbb{T}(v,p) \stackrel{\text{def}}{=} 2\nu D(v) - p\,\mathrm{Id}, \quad \text{with} \quad D(v) \stackrel{\text{def}}{=} \frac{1}{2}\big((\nabla v) + {}^t(\nabla v)\big),
$$

The controlled evolution system



$$v_R(t, x) = h'(t) + \theta'(t)(x - h(t))^{\perp}$$

The controlled evolution system

$$
\left\{
\begin{array}{l}
\partial_t v + (v \cdot \nabla) v - \nu \Delta v + \nabla p = f^S \quad \text{in } \mathcal{F}(h(t), \theta(t)), \\
\\
\operatorname{div} v = 0 \quad \text{in } \mathcal{F}(h(t), \theta(t)), \\
\\
v(t,x) = h'(t) + \theta'(t)(x - h(t))^{\perp}, \quad x \in \partial \mathcal{S}(h(t), \theta(t)), \\
\\
v(t,x) = b^S(x) + u(t,x), \quad x \in \partial \Omega, \\
\\
M h'' = -\displaystyle\int_{\partial \mathcal{S}(h(t),\theta(t))} \mathbb{T}(v,p) n \; d\Gamma + f^S_M, \\
\\
I \theta'' = -\displaystyle\int_{\partial \mathcal{S}(h(t),\theta(t))} (x - h)^{\perp} \cdot \mathbb{T}(v,p) n \; d\Gamma + f^S_I \\
\\
h(0) = h^0, \quad \theta(0) = \theta^0, \quad h'(0) = h^1, \quad \theta'(0) = \theta^1, \\
v(0,x) = v^0(x) \quad x \in \mathcal{F}(h^0, \theta^0).
\end{array}
\right.
$$

The stabilization result

- **Expression of the feedback control.**

$$u(t,x) = \sum_{j=1}^{N_\sigma} F_j(v(t), h(t), h'(t), \theta(t), \theta'(t)) v_j(x) \qquad (4)$$

$$t > 0,\ x \in \partial\Omega,$$

where

$$F_j(v,h,\ell,\theta,r) = M R_{\theta^S-\theta}\ell \cdot \xi_j + I r \zeta_j + (h - h^S) \cdot a_j + (\theta - \theta^S) b_j$$
$$+ \int_{\mathcal{F}(h^S,\theta^S)} \big(\mathrm{Cof}(\nabla X(h,\theta,y))^* v(t, X(h,\theta,y)) - v^S(y)\big) \cdot \varphi_j(y) \mathrm{d}y$$

  - $X(h,\theta,\cdot) : \Omega \to \Omega$ is a smooth change of variables such that

$$X(h,\theta,\mathcal{F}(h^S,\theta^S)) = \mathcal{F}(h,\theta)$$

  - $N_\sigma \in \mathbb{N}$, $v_j \in \mathbf{L}^2(\partial\Omega)$,
  - $(\varphi_j, \xi_j, \zeta_j, a_j, b_j) \in \mathbf{L}^2(\mathcal{F}(h^S,\theta^S)) \times \mathbb{R}^6$.

- **Notation.**

$$v(t,x) = \begin{cases} v(t,x) & \text{in } \mathcal{F}(h(t),\theta(t)) \\ h'(t) + \theta'(t)(x-h(t)) & \text{in } \mathcal{S}(h(t),\theta(t)) \end{cases}$$

$$v^0(x) = \begin{cases} v^0(x) & \text{in } \mathcal{F}(h^0,\theta^0) \\ h^1 + \theta^1(x-h^0) & \text{in } \mathcal{S}(h^0,\theta^0) \end{cases}$$

Theorem (2d feedback stabilization)

Assume $f^S \in \boldsymbol{W}^{2,\infty}$ *and* $(v^S, p^S) \in \boldsymbol{W}^{2,\infty} \times W^{1,\infty}$ *and*

$$h^0 = h^S \quad \text{and} \quad \theta^0 = \theta^S.$$

For all $\sigma > 0$*, there exist a feedback control of the form* (4) *such that if* $v^0 \in \mathbf{L}^2(\Omega)$ *satisfies* $\|v^0 - v^S\|_{\mathbf{L}^2(\Omega)} \leqslant c_0$*, then there exists a closed-loop solution* (v, p, h, θ) *satisfying :*

$$\|v(t) - v^S\|_{\mathbf{L}^2(\Omega)} + |h(t) - h^S| + |\theta(t) - \theta^S| \leqslant Ce^{-\sigma t}\|v_0 - v^S\|_{\mathbf{L}^2(\Omega)}.$$

Comments

1. Initial perturbations of initial positions are not allowed in the feedback control case (i.e. $h^0 = h^S$, $\theta^0 = \theta^S$).

2. An analogue stabilization result for strong solutions ($v^0 \in \mathbf{H}^1(\Omega)$ + c.c.) is available for perturbed initial positions, and even in 3d, but with a **dynamical control**.

3. The uniqueness of the controlled solution is not proved.

$\rightsquigarrow$ c.f. Glass and Sueur (2012).

Step1. Change of variables

- To simplify we assume for the following :

$$h^S = 0, \quad \theta^S = 0, \quad \left(\text{then } \begin{cases} \mathcal{S}(h^S, \theta^S) = \mathcal{S} \\ \mathcal{F}(h^S, \theta^S) = \mathcal{F} \end{cases} \right).$$

- We define the change of variables

$$X(h(t), \theta(t), y) \stackrel{\text{def}}{=} y + \eta(y) \left[h(t) + (R_{\theta(t)} - I_2) y \right],$$

with $\eta \in C^\infty(\Omega)$ such that

$$X(h(t), \theta(t), y) = \begin{cases} y \ \text{ near } \partial\Omega \\ h(t) + R_{\theta(t)} y \ \text{ near } \partial\mathcal{S} \end{cases}$$

- We set

$$\tilde{v}(t, y) \stackrel{\text{def}}{=} \mathrm{Cof}(\nabla X(h(t), \theta(t), y))^* v(t, X(h(t), \theta(t), y)), \ \tilde{v} = w + v^S$$

$$\tilde{p}(t, y) \stackrel{\text{def}}{=} p(t, X(h(t), \theta(t), y)), \quad \tilde{p} = q + p^S$$

$$\ell(t) \stackrel{\text{def}}{=} R_{-\theta(t)} h'(t), \quad \omega(t) \stackrel{\text{def}}{=} \theta'(t)$$

- The new unknowns are : $w, q, \ell, \omega, h, \theta$.

Step1. System after change of variables

$$
\left\{
\begin{aligned}
&\partial_t w - \nu \Delta w + \Gamma^S(h,\theta,\ell,\omega) + (w\cdot\nabla)v^S + (v^S\cdot\nabla)w + \nabla q = \\
&\qquad\qquad G_1(w,h,\theta,\ell,\omega) + G_2(h,\theta,\partial_t w,q) \text{ in } (0,+\infty)\times\mathcal{F},\\
&\operatorname{div} w = 0 \quad \text{in } (0,+\infty)\times\mathcal{F},\\
&w = \ell + \omega y^\perp \quad \text{on } (0,+\infty)\times\partial\mathcal{S},\\
&w = u \quad \text{on } \ (0,+\infty)\times\partial\Omega,\\
&M\ell' = -\int_{\partial\mathcal{S}} \mathbb{T}(w,q)n \,\mathrm{d}\Gamma - \theta(f_M^S)^\perp + g_3(\theta,\omega,\ell), \quad t>0,\\
&I\omega' = -\int_{\partial\mathcal{S}} y^\perp\cdot\mathbb{T}(w,q)n\,\mathrm{d}\Gamma, \quad t>0,\\
&h' = \ell + g_4(\theta,\ell), \quad t>0,\\
&\theta' = \omega, \quad t>0,\\
&h(0)=h^0, \quad \theta(0)=\theta^0, \quad \ell(0)=\ell^0, \quad \omega(0)=\omega^0,\\
&w(0,y) = w^0(y) \quad y\in\mathcal{F}.
\end{aligned}
\right.
$$

Step1. System after change of variables

$$
\begin{cases}
\partial_t w - \nu\Delta w + \Gamma^S(h,\theta,\ell,\omega) + (w\cdot\nabla)v^S + (v^S\cdot\nabla)w + \nabla q = \\
\qquad G_1(w,h,\theta,\ell,\omega) + G_2(h,\theta,\partial_t w,q) \text{ in } (0,+\infty)\times\mathcal{F}, \\
\operatorname{div} w = 0 \quad \text{in } (0,+\infty)\times\mathcal{F}, \\
w = \ell + \omega y^\perp \quad \text{on } (0,+\infty)\times\partial\mathcal{S}, \\
w = u \quad \text{on } (0,+\infty)\times\partial\Omega, \\
M\ell' = -\displaystyle\int_{\partial\mathcal{S}} \mathbb{T}(w,q)n \,\mathrm{d}\Gamma - \theta(f_M^S)^\perp + g_3(\theta,\omega,\ell), \quad t>0, \\
I\omega' = -\displaystyle\int_{\partial\mathcal{S}} y^\perp\cdot\mathbb{T}(w,q)n \,\mathrm{d}\Gamma, \quad t>0, \\
h' = \ell + g_4(\theta,\ell), \quad t>0, \\
\theta' = \omega, \quad t>0, \\
h(0)=h^0, \quad \theta(0)=\theta^0, \quad \ell(0)=\ell^0, \quad \omega(0)=\omega^0, \\
w(0,y) = w^0(y) \quad y\in\mathcal{F}.
\end{cases}
$$

Step1. System after change of variables

$$
\left\{
\begin{aligned}
&\partial_t w - \nu \Delta w + \Gamma^S(h,\theta,\ell,\omega) + (w\cdot\nabla)v^S + (v^S\cdot\nabla)w + \nabla q = \\
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&\operatorname{div} w = 0 \quad \text{in } (0,+\infty)\times\mathcal{F},\\
&w = \ell + \omega y^{\perp} \quad \text{on } (0,+\infty)\times\partial\mathcal{S},\\
&w = u \quad \text{on } (0,+\infty)\times\partial\Omega,\\
&M\ell' = -\int_{\partial\mathcal{S}} \mathbb{T}(w,q)n \,\mathrm{d}\Gamma - \theta(f_M^S)^{\perp} + g_3(\theta,\omega,\ell), \quad t>0,\\
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&\theta' = \omega, \quad t>0,\\
&h(0)=h^0, \quad \theta(0)=\theta^0, \quad \ell(0)=\ell^0, \quad \omega(0)=\omega^0,\\
&w(0,y) = w^0(y) \quad y\in\mathcal{F}.
\end{aligned}
\right.
$$

Step1. System after change of variables

$$
\left\{
\begin{aligned}
&\partial_t w - \nu \Delta w + \Gamma^S(h,\theta,\ell,\omega) + (w\cdot\nabla)v^S + (v^S\cdot\nabla)w + \nabla q = \\
&\qquad\qquad G_1(w,h,\theta,\ell,\omega) + G_2(h,\theta,\partial_t w, q) \text{ in } (0,+\infty)\times\mathcal{F},\\
&\operatorname{div} w = 0 \quad \text{in } (0,+\infty)\times\mathcal{F},\\
&w = \ell + \omega y^\perp \quad \text{on } (0,+\infty)\times\partial\mathcal{S},\\
&w = u \quad \text{on } (0,+\infty)\times\partial\Omega,\\
&M\ell' = -\int_{\partial\mathcal{S}} \mathbb{T}(w,q)n \,\mathrm{d}\Gamma - \theta(f_M^S)^\perp + g_3(\theta,\omega,\ell), \quad t>0,\\
&I\omega' = -\int_{\partial\mathcal{S}} y^\perp\cdot\mathbb{T}(w,q)n \,\mathrm{d}\Gamma, \quad t>0,\\
&h' = \ell + g_4(\theta,\ell), \quad t>0,\\
&\theta' = \omega, \quad t>0,\\
&h(0)=h^0, \quad \theta(0)=\theta^0, \quad \ell(0)=\ell^0, \quad \omega(0)=\omega^0,\\
&w(0,y) = w^0(y) \quad y\in\mathcal{F}.
\end{aligned}
\right.
$$

Step 1. Abstract setting

$$H \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} w \\ \ell \\ \omega \\ h \\ \theta \end{bmatrix} \in \mathbf{L}^2(\mathcal{F}) \times \mathbb{R}^6 \,\middle|\, \begin{array}{l} w \cdot n = (\ell + \omega y^\perp) \cdot n \ \text{ on } \partial\mathcal{S}, \\ w \cdot n = 0 \ \text{ on } \partial\Omega, \\ \operatorname{div} w = 0 \ \text{ in } \mathcal{F}, \end{array} \right\}.$$

$$\mathcal{D}(A) \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} w \\ \ell \\ \omega \\ h \\ \theta \end{bmatrix} \in H \,\middle|\, \begin{array}{l} w \in \mathbf{H}^2(\mathcal{F}) \\ w = \ell + \omega y^\perp \ \text{ on } \partial\mathcal{S}, \\ w = 0 \ \text{ on } \partial\Omega, \end{array} \right\}$$

$$A \begin{bmatrix} w \\ \ell \\ \omega \\ a \\ \theta \end{bmatrix} \stackrel{\text{def}}{=} P_H \begin{bmatrix} \nu\Delta w - \Gamma^S(h,\theta,\ell,\omega) - (w\cdot\nabla)v^S - (v^S\cdot\nabla)w \\ -M^{-1}\displaystyle\int_{\partial\mathcal{S}} 2\nu D(w)n \,\mathrm{d}\Gamma - M^{-1}\theta(f_M^S)^\perp \\ -I^{-1}\displaystyle\int_{\partial\mathcal{S}} y^\perp \cdot 2\nu D(w)n \,\mathrm{d}\Gamma \\ \ell \\ \omega \end{bmatrix}$$

A generates an analytic semigroup and has a compact resolvent.

Step1. Abstract setting

$$B : \mathbf{V}^0(\partial\Omega) \to [\mathcal{D}(A^*)]', \quad B^* \begin{bmatrix} \varphi \\ \xi \\ \zeta \\ a \\ b \end{bmatrix} = -\mathbb{T}(\varphi, \pi)n$$

$$\mathbf{X} \overset{\text{def}}{=} \begin{bmatrix} w \\ \ell \\ \omega \\ h \\ \theta \end{bmatrix}, \mathbf{G}(\mathbf{X}, \partial_t w, q) \overset{\text{def}}{=} P_H \begin{bmatrix} G_1(w, h, \theta, \ell, \omega) + G_2(h, \theta, \partial_t w, q) \\ g_3(\theta, \omega, \ell) \\ 0 \\ g_4(\theta, \ell) \\ 0 \end{bmatrix},$$

$$\begin{aligned} (I - P_H)\mathbf{X} &= (I - P_H)Du, \\ P_H\mathbf{X}' &= AP_H\mathbf{X} + Bu + \mathbf{G}(\mathbf{X}, \partial_t w, q) \ \text{ in } [\mathcal{D}(A^*)]', \\ P_H\mathbf{X}(0) &= P_H\mathbf{X}^0 \end{aligned}$$

- Final abstract formulation :

$$\mathbf{Y}' = A\mathbf{Y} + \mathbf{G}(\mathbf{Y}, \partial_t w, q) + Bu, \quad \mathbf{Y}(0) = \mathbf{Y}^0$$

Step 2. Criterion for stabilizability

Assume $\lambda \in \mathbb{C}$, $\Re\lambda \geqslant -\sigma$ and

$$\left\{\begin{array}{c}
\lambda\varphi - \nu\Delta\varphi + (\nabla v^S)^*\varphi - (v^S\cdot\nabla)\varphi + \nabla\pi = 0 \quad \text{in } \mathcal{F}, \quad \displaystyle\int_{\partial\Omega}\pi \mathrm{d}\Gamma = 0,\\
\operatorname{div}\varphi = 0 \quad \text{in } \mathcal{F},\\
\varphi = \xi + \zeta y^\perp \quad \text{on } \partial\mathcal{S},\\
\varphi = 0 \quad \text{and} \quad \mathbb{T}(\varphi,\pi)n = 0 \quad \text{on } \partial\Omega,\\
M\lambda\xi + \displaystyle\int_{\partial\mathcal{S}}\mathbb{T}(\varphi,\pi)n \ \mathrm{d}\Gamma + \begin{bmatrix}\int_{\mathcal{F}}\Gamma_4\cdot\varphi \ \mathrm{d}y\\ \int_{\mathcal{F}}\Gamma_5\cdot\varphi \ \mathrm{d}y\end{bmatrix} - a = 0,\\
I\lambda\zeta + \displaystyle\int_{\partial\mathcal{S}} y^\perp\cdot\mathbb{T}(\varphi,\pi)n \ \mathrm{d}\Gamma + \int_{\mathcal{F}}\Gamma_6\cdot\varphi \ \mathrm{d}y - b = 0,\\
\lambda a + \begin{bmatrix}\int_{\mathcal{F}}\Gamma_1\cdot\varphi \ \mathrm{d}y\\ \int_{\mathcal{F}}\Gamma_2\cdot\varphi \ \mathrm{d}y\end{bmatrix} = 0,\\
\lambda b + (f_M^S)^\perp\cdot\xi + \displaystyle\int_{\mathcal{F}}\Gamma_3\cdot\varphi \ \mathrm{d}y = 0.
\end{array}\right.$$

Do we have $(\varphi,\xi,\zeta,a,b) = (0,0,0,0,0)$?

Step 3. Construction of a fixed-point solution

$$
\left\{
\begin{aligned}
&\partial_t w - \nu \Delta w + \Gamma^S(h,\theta,\ell,\omega) + (w\cdot\nabla)v^S + (v^S\cdot\nabla)w + \nabla q = \\
&\qquad G_1(w,h,\theta,\ell,\omega) + G_2(h,\theta,\partial_t w, q) \text{ in } (0,+\infty)\times\mathcal{F},\\
&\operatorname{div} w = 0 \quad \text{in } (0,+\infty)\times\mathcal{F},\\
&w = \ell + \omega y^\perp \quad \text{on } (0,+\infty)\times\partial\mathcal{S},\\
&w = \mathfrak{F}_\sigma(w,\ell,\omega,h,\theta) \quad \text{on } (0,+\infty)\times\partial\Omega,\\
&M\ell' = -\int_{\partial\mathcal{S}} \mathbb{T}(w,q)n \,\mathrm{d}\Gamma - \theta(f_M^S)^\perp + g_3(\theta,\omega,\ell), \quad t>0,\\
&I\omega' = -\int_{\partial\mathcal{S}} y^\perp\cdot\mathbb{T}(w,q)n \,\mathrm{d}\Gamma, \quad t>0,\\
&h' = \ell + g_4(\theta,\ell), \quad t>0,\\
&\theta' = \omega, \quad t>0,\\
&h(0)=h^0,\ \theta(0)=\theta^0,\ \ell(0)=\ell^0,\ \omega(0)=\omega^0,\ w(0,y)=w^0(y)\ \ y\in\mathcal{F}.
\end{aligned}
\right.
$$

$$
\mathfrak{F}_\sigma(w,\ell,\omega,h,\theta) = \sum_{j=1}^{N_\sigma}\left(\int_{\mathcal{F}} w\cdot\varphi_j \mathrm{d}y + M\ell\cdot\xi_j + I\omega\zeta_j + h\cdot a_j + \theta b_j\right) v_j
$$

Step 3. Construction of a fixed-point solution

$$\text{Find a fixed-point to : } (\widehat{\omega}, \widehat{q}, \widehat{\ell}, \widehat{\omega}, \widehat{h}, \widehat{\theta}) \longmapsto (\omega, q, \ell, \omega, h, \theta)$$

$$\begin{cases}
\partial_t w - \nu\Delta w + \Gamma^S(h,\theta,\ell,\omega) + (w\cdot\nabla)v^S + (v^S\cdot\nabla)w + \nabla q = \\
\qquad G_1(\widehat{w},\widehat{h},\widehat{\theta},\widehat{\ell},\widehat{\omega}) + G_2(\widehat{h},\widehat{\theta},\partial_t\widehat{w},\widehat{q}) \text{ in } (0,+\infty)\times\mathcal{F}, \\
\operatorname{div} w = 0 \quad \text{in } (0,+\infty)\times\mathcal{F}, \\
w = \ell + \omega y^\perp \quad \text{on } (0,+\infty)\times\partial\mathcal{S}, \\
w = \mathfrak{F}_\sigma(w,\ell,\omega,h,\theta) \quad \text{on } (0,+\infty)\times\partial\Omega, \\
M\ell' = -\int_{\partial\mathcal{S}} \mathbb{T}(w,q)n \, \mathrm{d}\Gamma - \theta(f_M^S)^\perp + g_3(\widehat{\theta},\widehat{\omega},\widehat{\ell}), \quad t>0, \\
I\omega' = -\int_{\partial\mathcal{S}} y^\perp\cdot\mathbb{T}(w,q)n \, \mathrm{d}\Gamma, \quad t>0, \\
h' = \ell + g_4(\widehat{\theta},\widehat{\ell}), \quad t>0, \\
\theta' = \omega, \quad t>0, \\
h(0)=h^0,\ \theta(0)=\theta^0,\ \ell(0)=\ell^0,\ \omega(0)=\omega^0,\ w(0,y)=w^0(y) \ \ y\in\mathcal{F}.
\end{cases}$$

Step 3. Construction of a fixed-point solution

Difficulty.
We must look for "weak" solution...

$$\rightsquigarrow \text{otherwise} : w^0(x) = \mathfrak{F}_\sigma(w^0, \ell^0, \omega^0, h^0, \theta^0)(x),\ x \in \partial\Omega$$

But for w^0 only in $\mathbf{L}^2(\mathcal{F})$, $\partial_t w$, q are not integrable in time!

Step 3. Construction of a fixed-point solution

Difficulty.
We must look for "weak" solution...

$$\rightsquigarrow \text{otherwise} : w^0(x) = \mathfrak{F}_\sigma(w^0, \ell^0, \omega^0, h^0, \theta^0)(x),\ x \in \partial\Omega$$

But for w^0 only in $\mathbf{L}^2(\mathcal{F})$, $\partial_t w$, q are not integrable in time!

Solution.

- The singularity is localized at time $t = 0$:

$$t\partial_t w \text{ and } tp \quad \text{are square integrable in time}$$

- If $h^0 = h^S = 0$ and $\theta^0 = \theta^S = 0$ we have

$$\begin{aligned}\|G_2(h, \theta, \partial_t w, q)(t)\| &\leqslant C(|h(t)| + |\theta(t)|)(\|\partial_t w(t)\| + \|q(t)\|)\\ &\leqslant Ct(\|\partial_t w(t)\| + \|q(t)\|)\end{aligned}$$

Extensions and perspectives

1. Use of dynamical control to define **strong solutions** :

$$
\begin{aligned}
& u(t,x) = \sum_{j=1}^{N_\sigma} u_j(t) v_j(x) \quad t>0,\ x \in \partial\Omega, \\
& \bar{u} = (u_1, \ldots, u_{N_\sigma}) \\
& \bar{u}' + \Lambda \bar{u} = \sum_{j=1}^{N_\sigma} F_j(v(t), h(t), h'(t), \theta(t), \theta'(t)) e_j, \\
& \bar{u}(0) = 0.
\end{aligned}
$$

$\rightsquigarrow$ 3-d result and stabilization of positions i.e.

$$h^0 \neq h^S \quad \text{or} \quad \theta^0 \neq \theta^S$$

2. Use another change of variables to define **2-d weak solutions** for perturbed positions.

Extensions and perspectives

3 Control on/in the body : similar results are available for **Dirichlet control on** $\partial\mathcal{S}$ if $v^S \equiv 0$ and if the control acts on the whole boundary $\partial\mathcal{S}$.

Extensions and perspectives

Overdetermined system of the stabilizability criterion :

$$\left\{\begin{array}{c}
\lambda\varphi - \nu\Delta\varphi + (\nabla v^S)^*\varphi - (v^S\cdot\nabla)\varphi + \nabla\pi = 0 \quad \text{in } \mathcal{F}, \quad \displaystyle\int_{\partial\Omega}\pi \mathrm{d}\Gamma = 0,\\
\operatorname{div}\varphi = 0 \quad \text{in } \mathcal{F},\\
\varphi = \xi + \zeta y^{\perp} \quad \text{and} \quad \color{red}{\mathbb{T}(\varphi,\pi)n = 0} \quad \text{on } \partial\mathcal{S},\\
\varphi = 0 \qquad \text{on } \partial\Omega,\\
M\lambda\xi + \displaystyle\int_{\partial\mathcal{S}}\mathbb{T}(\varphi,\pi)n\ \mathrm{d}\Gamma + \begin{bmatrix} \int_{\mathcal{F}}\Gamma_4\cdot\varphi\ \mathrm{d}y \\ \int_{\mathcal{F}}\Gamma_5\cdot\varphi\ \mathrm{d}y \end{bmatrix} - a = 0,\\
I\lambda\zeta + \displaystyle\int_{\partial\mathcal{S}} y^{\perp}\cdot\mathbb{T}(\varphi,\pi)n\ \mathrm{d}\Gamma + \int_{\mathcal{F}}\Gamma_6\cdot\varphi\ \mathrm{d}y - b = 0,\\
\lambda a + \begin{bmatrix} \int_{\mathcal{F}}\Gamma_1\cdot\varphi\ \mathrm{d}y \\ \int_{\mathcal{F}}\Gamma_2\cdot\varphi\ \mathrm{d}y \end{bmatrix} = 0,\\
\lambda b + (f_M^S)^{\perp}\cdot\xi + \displaystyle\int_{\mathcal{F}}\Gamma_3\cdot\varphi\ \mathrm{d}y = 0.
\end{array}\right.$$

Extensions and perspectives

4 Control on/in the body : similar results are available for **Dirichlet control on $\partial\mathcal{S}$ and internal control in $\mathcal{S}$.**

Extensions and perspectives

Overdetermined system of the stabilizability criterion :

$$\left\{\begin{array}{c}
\lambda\varphi - \nu\Delta\varphi + (\nabla v^S)^*\varphi - (v^S\cdot\nabla)\varphi + \nabla\pi = 0 \quad \text{in } \mathcal{F}, \quad \displaystyle\int_{\partial\Omega}\pi \mathrm{d}\Gamma = 0, \\
\operatorname{div}\varphi = 0 \quad \text{in } \mathcal{F}, \\
\varphi = 0 \quad \text{and} \quad \mathbb{T}(\varphi,\pi)n = 0 \quad \text{on } \partial\mathcal{S}, \\
\varphi = 0 \qquad \text{on } \partial\Omega, \\
\displaystyle\int_{\partial\mathcal{S}}\mathbb{T}(\varphi,\pi)n \ \mathrm{d}\Gamma + \begin{bmatrix}\int_{\mathcal{F}}\Gamma_4\cdot\varphi \ \mathrm{d}y \\ \int_{\mathcal{F}}\Gamma_5\cdot\varphi \ \mathrm{d}y\end{bmatrix} - a = 0, \\
\displaystyle\int_{\partial\mathcal{S}} y^{\perp}\cdot\mathbb{T}(\varphi,\pi)n \ \mathrm{d}\Gamma + \int_{\mathcal{F}}\Gamma_6\cdot\varphi \ \mathrm{d}y - b = 0, \\
\lambda a + \begin{bmatrix}\int_{\mathcal{F}}\Gamma_1\cdot\varphi \ \mathrm{d}y \\ \int_{\mathcal{F}}\Gamma_2\cdot\varphi \ \mathrm{d}y\end{bmatrix} = 0, \\
\lambda b + \displaystyle\int_{\mathcal{F}}\Gamma_3\cdot\varphi \ \mathrm{d}y = 0.
\end{array}\right.$$

Extensions and perspectives

5 Case of “finite dimensional” deformable structure :

$$v(t,x) = Mr'(t), \quad t > 0, x \in \Gamma_s$$
$$r''(t) + Qr = -\int_{\Gamma_s} M^T \mathbb{T}(v,p)n \, d\Gamma$$

6 Case of deformable structure (beam, plate, Lamé).