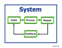
Nonlinear stabilization via a linear observability



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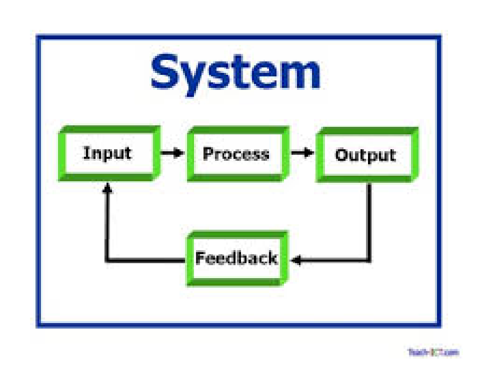

Collocated feedback stabilization

Outline

1. Linear stabilization
   - Introduction and main result
   - Applications to examples of linear PDEs

2. Nonlinear stabilization
   - Introduction and main results
   - Preliminary intermediate results
   - Comparison with an Euler scheme
   - Applications to examples of PDE's and dampings
     - Stabilization of the nonlinear damped wave equation
     - Stabilization of the nonlinear Euler-Bernoulli plate equation

3. Some references

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Outline

Linear stabilization

$$\begin{cases} \ddot{w}(t) + Aw(t) + BB^*\dot{w}(t) = 0, & t \geq 0, \\ w(0) = w^0, \dot{w}(0) = w^1, \end{cases} \qquad (1)$$

$A : \mathcal{D}(A) \subset H \to H$, self-adjoint, positive and boundedly invertible operator.

$$B : U \longrightarrow H_{-\frac{1}{2}},$$

$$H_{-\frac{1}{2}} = H^*_{\frac{1}{2}}, \qquad H_{\frac{1}{2}} = \mathcal{D}(A^{\frac{1}{2}})$$

$$B^* : H_{\frac{1}{2}} \longrightarrow U,$$

$$\ddot{w}(t) + Aw(t) = Bu(t), \qquad u(t) = -B^*\dot{w}(t)$$

$$E(t) = \frac{1}{2} \, \|(w(t), \dot{w}(t))\|^2_{H_{\frac{1}{2}} \times H} \, .$$

- $$E(T) - E(0) = -\int_0^T \|B^* \dot{w}(t)\|_U^2 \, dt$$

- The observability inequality:

$$E(0) \leq C_0 \int_0^T \|B^* \dot{w}(t)\|_U^2 \, dt$$

- Undamped second order diff. eq.

$$\ddot{\phi}(t) + A\phi(t) = 0, \quad t \geq 0, \tag{2}$$

$$\phi(0) = w^0, \dot{\phi}(0) = w^1. \tag{3}$$

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Assumption

Assume that for any $\gamma > 0$ we have

$$\sup_{\mathrm{Re}\lambda=\gamma} \left\| \lambda B^*(\lambda^2 I + A)^{-1} B \right\|_{\mathcal{L}(U)} < \infty. \tag{4}$$

Stability by observability

Theorem (A-Tucsnak)

The following assertions are equivalent.

- *The system (1) satisfies the exponential decay*

$$E(t) \leq Ce^{-\beta t}E(0), \quad \forall (w^0, w^1) \in H_{\frac{1}{2}} \times H.$$

- *There exist $T, C_0 > 0$ s.t. $\forall (\phi^0, \phi^1) \in \mathcal{D}(A) \times H_{\frac{1}{2}}$*

$$\int_0^T \left\|B^*\dot{\phi}(s)\right\|_U^2 ds \geq C_0 \left\|(\phi^0, \phi^1)\right\|_{H_{\frac{1}{2}} \times H}^2. \qquad (5)$$

Stability by observability

Theorem (A-Tucsnak)

The following assertions are equivalent.

- *The system* (1) *satisfies the exponential decay*

$$E(t) \leq Ce^{-\beta t}E(0), \quad \forall (w^0, w^1) \in H_{\frac{1}{2}} \times H.$$

- *There exist* $T, C_0 > 0$ *s.t.* $\forall (\phi^0, \phi^1) \in \mathcal{D}(A) \times H_{\frac{1}{2}}$

$$\int_0^T \left\| B^* \dot{\phi}(s) \right\|_U^2 ds \geq C_0 \, ||(\phi^0, \phi^1)||^2_{H_{\frac{1}{2}} \times H}. \qquad (5)$$

Example 1

$$\ddot{w}(t,x) - w_{xx}(t,x) + \dot{w}(t,\xi)\delta_{\xi} = 0,$$
$$w(t,0) = w_x(t,1) = 0, \quad t > 0,$$
$$w(0,x) = w^0(x), \dot{w}(0,x) = w^1(x), \quad x \in (0,1),$$

$\xi \in (0,1)$.

exponential stability $\Leftrightarrow \xi = \frac{p}{q}$, with p odd.

Example 2

$$\ddot{w}(t,x) - \triangle w(t,x) = 0, \quad t \geq 0,\ x \in \Omega, \tag{6}$$

$$w(t,x) = \frac{\partial(-\triangle)^{-1}\dot{w}}{\partial \nu}(t,x), \quad t \geq 0,\ x \in \Gamma_0, \tag{7}$$

$$w(t,x) = 0, \quad t \geq 0,\ x \in \Gamma_1, \tag{8}$$

$$w(0,x) = w^0(x), \dot{w}(0,x) = w^1(x), \quad t \geq 0,\ x \in \Omega, \tag{9}$$

where ν is the unit normal vector of Γ pointing towards the exterior of the open set Ω.

To write this problem in the abstract form, define

$$H = H^{-1}(\Omega),\ A = -\triangle, \quad D(A) = H_0^1(\Omega),$$
$$H_{\frac{1}{2}} = L^2(\Omega), \quad U = L^2(\Gamma_0).$$

The operator A can be extended to an operator
$A : L^2(\Omega) \longrightarrow (H^2(\Omega) \cap H_0^1(\Omega))'$.
The operators B and B^* are defined as

$$Bv = -ADv,\ \forall v \in L^2(\Gamma_0), \quad B^*g = -D^*Ag = \frac{\partial[(-\triangle)^{-1}g]}{\partial\nu}|_{\Gamma_0},$$

$\forall\, g \in L^2(\Omega)$, where D is the Dirichlet map defined by

$$D : L^2(\Gamma_0) \longrightarrow H^{\frac{1}{2}}(\Omega) \subset H^{\frac{1}{2}-2\sigma}(\Omega) = D(A^{\frac{1}{4}-\sigma}), \quad \text{for all } \sigma > 0,$$

$$h = Dg \Longleftrightarrow \begin{cases} \triangle h = 0 & in\ \Omega, \\ h = g & in\ \Gamma_0, \\ h = 0 & in\ \Gamma_1. \end{cases}$$

The problem (6)-(9) becomes

$$\ddot{w} + Aw(t) + BB^* \dot{w}(t) = 0, \quad t \geq 0, \tag{10}$$

$$w(0) = w^0, \dot{w}(0) = w^1. \tag{11}$$

In the Hilbert space $\mathcal{H} = L^2(\Omega) \times H^{-1}(\Omega)$ the operator $\mathcal{A}_d$ is given by

$$\mathcal{A}_d : \mathcal{D}(\mathcal{A}_d) \to \mathcal{H},$$

$$\mathcal{A}_d \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} =$$

$$\begin{pmatrix} u_2 \\ Au_1 - AD\dfrac{\partial[(-\triangle)^{-1} u_2]}{\partial \nu}|\Gamma_0 \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}_d) =$$

$$\left\{(u_1, u_2) \in H^1(\Omega)\times L^2(\Omega) : u_1 = \frac{\partial[(-\triangle)^{-1}u_2]}{\partial\nu} \text{ in } \Gamma_0,\ u_1 = 0 \text{ in } \Gamma_1\right\}.$$

The operators A and B defined above satisfy the assumption (4).

Consider the uncontrolled evolution problem

$$\ddot{\phi} - \triangle\phi = 0, \quad (t,x) \in (0,+\infty) \times \Omega, \tag{12}$$

$$\phi(t,x) = 0, \quad (t,x) \in (0,+\infty) \times \Gamma \tag{13}$$

$$\phi(0,x) = w^0(x), \quad \dot{\phi}(0,x) = w^1(x), \quad x \in \Omega. \tag{14}$$

Proposition

There exist $T_0 > 0$, $C_{T_0} > 0$ such that for all $T > T_0$ the solution ϕ of (12)-(14) satisfies

$$\int_0^T \left\| \frac{\partial[(-\triangle)^{-1}\dot{\phi}]}{\partial\nu} \right\|^2_{L^2(\Gamma_0)} dt \geq C_{T_0} \left(\|w^0\|^2_{L^2(\Omega)} + \|w^1\|^2_{H^{-1}(\Omega)} \right) \tag{15}$$

$$\text{for all } (w^0, w^1) \in H^1_0(\Omega) \times L^2(\Omega)$$

if and only if (T_0, Γ_0) satisfies the geometric control condition (BLR), i.e., every ray of geometric optics that propagates in Ω and is reflected on its boundary $\partial\Omega$ enters Γ_0 in time less than T_0.

Theorem

The energy satisfies the exponential estimate

$$E(t) \leq C\, e^{-\omega t}\, E(0),$$

for some positive constants ω *and* C *and all* $(w^0, w^1) \in \mathcal{H}$ *if and only if* (T_0, Γ_0) *satisfies the geometric control condition (BLR).*

Nonlinear stabilization

We consider the following second order differential equation

$$\begin{cases} \ddot{w}(t) + Aw(t) + a(.)\rho(., \dot{w}) = 0\,, & t \in (0,\infty)\,, x \in \Omega \\ w(0) = w^0\,, \dot{w}(0) = w^1\,. \end{cases} \tag{16}$$

where Ω is a bounded open set in $\mathbb{R}^N$, with a boundary Γ. We assume that Ω is either convex or of class $\mathcal{C}^{1,1}$.

We set $H = L^2(\Omega)$, with its usual scalar product denoted by $\langle \cdot, \cdot \rangle_H$ and the associated norm $\| \cdot \|_H$ and where $A : D(A) \subset H \to H$ is a densely defined self-adjoint linear operator satisfying

$$\langle Au, u \rangle_H \geq C \|u\|_H^2 \qquad \forall u \in D(A) \tag{17}$$

for some $C > 0$. We also introduce the scale of Hilbert spaces H_α, as follows: for every $\alpha \geq 0$, $H_\alpha = \mathcal{D}(A^\alpha)$, with the norm $\|z\|_\alpha = \|A^\alpha z\|_H$. The space $H_{-\alpha}$, is defined by duality with respect to the pivot space H as follows: $H_{-\alpha} = H_\alpha^*$, for $\alpha > 0$. The operator A can be extended (or restricted) to each H_α, such that it becomes a bounded operator

$$A : H_\alpha \to H_{\alpha-1} \quad \forall\, \alpha \in \mathbb{R}. \tag{18}$$

The equation (16) is understood as an equation in $H_{-1/2}$, i.e., all the terms are in $H_{-1/2}$. The energy of a solution is defined by

$$E_w(t) = \frac{1}{2}\Big(\|(w(t), \dot{w}(t))\|^2_{H_{1/2}\times H}\Big) \tag{19}$$

Suppose that $(w^0, w^1) \in H_{1/2} \times H$. Then the problem (16) admits a unique solution

$$w \in C([0, \infty); H_{1/2}) \cap C^1([0, \infty); H).$$

Moreover w satisfies, for all $t \geq 0$, the energy identity

$$\|(w^0, w^1)\|^2_{H_{1/2} \times H} - \|(w(t), \dot{w}(t))\|^2_{H_{1/2} \times H} =$$

$$2 \int_0^t \int_\Omega a(.)\rho(., \dot{w}(s))\dot{w}(s)\, dx\, ds. \tag{20}$$

The aim is to deduce sharp simple computable energy decay rates for the damped system (16) from observability estimates for the associated undamped system, that is

$$\begin{cases} \ddot{\phi}(t) + A\phi(t) = 0, \\ \phi(0) = \phi^0, \, \dot{\phi}(0) = \phi^1. \end{cases} \tag{21}$$

We make the following assumptions on the feedback ρ and on a:

Assumption (A1) : $\rho \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}; \mathbb{R})$ is a continuous monotone nondecreasing function with respect to the second variable on Ω such that $\rho(.,0) = 0$ on Ω and there exists a continuous strictly increasing odd function $g \in \mathcal{C}([-1,1];\mathbb{R})$, continuously differentiable in a neighbourhood of 0 and satisfying $g(0) = g'(0) = 0$, with

$$\begin{cases} c_1 g(|v|) \leq |\rho(.,v)| \leq c_2 g^{-1}(|v|)\,, |v| \leq 1\,, \text{ a.e. on} \Omega\,, \\ c_1 |v| \leq |\rho(.,v)| \leq c_2 |v|\,, |v| \geq 1\,, \text{ a.e. on} \Omega\,, \end{cases} \tag{22}$$

where $c_i > 0$ for $i = 1, 2$.

Moreover $a \in \mathcal{C}(\overline{\Omega})$, with $a \geq 0$ on Ω and

$$\exists\ a_- > 0 \text{ such that } a \geq a_- \text{ on } \omega\,. \tag{23}$$

Here ω stands for the subregion of Ω on which the feedback ρ is active and $U = L^2(\omega)$.
We define a function H by

$$H(x) = \sqrt{x}g(\sqrt{x})\,,\ x \in [0, r_0^2]. \tag{24}$$

Thanks to assumption $(A1)$, H is of class $\mathcal{C}^1$ and is strictly convex on $[0, r_0^2]$, where $r_0 > 0$ is a sufficiently small number. We denote by $\widehat{H}$ the extension of H to $\mathbb{R}$ where $\widehat{H}(x) = +\infty$ for $x \in \mathbb{R}\backslash[0, r_0^2]$.

We also define a function L by

$$L(y) = \begin{cases} \dfrac{\widehat{H}^\star(y)}{y} & , \text{ if } y \in (0, +\infty), \\ 0 & , \text{ if } y = 0, \end{cases} \tag{25}$$

where $\widehat{H}^\star$ stands for the convex conjugate function of $\widehat{H}$, i.e.: $\widehat{H}^\star(y) = \sup_{x\in\mathbb{R}}\{xy - \widehat{H}(x)\}$.

L is strictly increasing continuous and onto from $[0,+\infty)$ on $[0,r_0^2)$.
We define a function Λ_H on $(0,r_0^2]$ by

$$\Lambda_H(x)=\frac{H(x)}{xH'(x)}\,. \tag{26}$$

We also define

$$\psi_r(x)=\frac{1}{H'(r_0^2)}+\int_{\frac{1}{x}}^{H'(r_0^2)}\frac{1}{v^2\Big(1-\Lambda_H\big((H')^{-1}(v)\big)\Big)}\,dv\,,\quad x\geq\frac{1}{H'(r_0^2)}. \tag{27}$$

- Assume that ρ and a satisfy the assumption $(A1)$ and that there exists $r_0 > 0$ sufficiently small so that the function H defined by (24) is strictly convex on $[0, r_0^2]$.
- Assume that

$$\lim_{x \to 0^+} \frac{H'(x)}{\Lambda_H(x)} = 0 \tag{28}$$

where Λ_H is defined by (26).

- Moreover assume that there exists $T > 0$ such that the following observability inequality is satisfied for the linear conservative system (21).

$$c_T E_\phi(0) \leq \int_0^T |\sqrt{a}\dot{\phi}|_H^2 \, dt \,, \quad \forall (\phi_0, \phi_1) \in H_{1/2} \times H \,. \tag{29}$$

with a certain $c_T > 0$.

Main Theorem (A - Alabau Boussouira)

The energy of the solution of (16) satisfies

$$E_w(t) \leq \beta T L\Big(\frac{1}{\psi_r^{-1}(\frac{t-T}{T_0})}\Big) \,, \qquad \text{for } t \text{ sufficiently large} . \tag{30}$$

If further $\limsup_{x\to 0^+} \Lambda_H(x) < 1$, then we have the simplified decay rate

$$E_w(t) \leq \beta T (H')^{-1}\Big(\frac{D T_0}{t-T}\Big) \,, \tag{31}$$

for t sufficiently large. Here D is a positive constant which is independent of $E_w(0)$ and T, whereas T_0 depends on T and β is a positive constant chosen such as $\beta > \max\left(\frac{2\alpha T}{C_T}, \frac{E_w(0)}{L(H'(r_0^2))}, \frac{E_w(0)}{\delta}\right)$, where $C_T > 0$, α and $\delta > 0$ are constants.

Remark

If

$$0 < \liminf_{x\to 0^+} \Lambda_H(x) \tag{32}$$

holds, then since $\lim_{x\to 0^+} H'(x) = 0$, (28) *holds.*
Moreover, under the above hypotheses, we have

$$L\left(\frac{1}{\psi_r^{-1}\left(\frac{t-T}{T_0}\right)}\right) \to 0 \text{ as } t \to \infty\,.$$

We now give an important result showing that sharp energy decay rates for the case of arbitrary nonlinear damping is a consequence of exponential stabilization for the case of linear damping.

Linear damped system

$$\begin{cases} \ddot{z}(t) + Az(t) + a(.)\dot{z} = 0\,, & t \in (0,\infty)\,, x \in \Omega \\ z(0) = z^0\,, \dot{z}(0) = z^1\,. \end{cases} \tag{33}$$

We define the energy of a solution z of (33) by E_z as in (19) replacing w by z and for initial date $(z_0, z_1) \in H_{1/2} \times H$.

Main Corollary

Assume that ρ and a satisfy the assumption $(A1)$ and that there exists $r_0 > 0$ sufficiently small so that the function H defined by (24) is strictly convex on $[0, r_0^2]$. Assume also that (28) holds. We moreover assume that the system (33) is exponentially stable, that is there exist $\mu > 0$ and $C > 0$ such that

$$E_z(t) \leq C E_z(0) e^{-\mu t}, \quad \forall \, (z_0, z_1) \in H_{1/2} \times H \,. \tag{34}$$

Then there exists $T > 0$ such that the energy of the solution of (16) satisfies (30). If further $\limsup_{x \to 0^+} \Lambda_H(x) < 1$, then E_w satisfies the simplified decay rate (31)

The proof of the main Theorem relies on the next theorem. For this, we consider the following assumption.

Assumption (A2) :

H is a continuously differentiable strictly convex function on $[0, r_0^2]$ with $H(0) = H'(0) = 0$.

The function M defined by

$$M(x) = xL^{-1}(x)\,, \quad x \in [0, r_0^2)\,. \tag{35}$$

is such that $\lim_{x \to 0^+} M'(x) = 0$, where L is defined by (25).

Remark

Thanks to assumption (A2), for all positive constant κ, there exists $\delta \in (0, r_0^2]$ such that the function $x \mapsto x - \kappa M(x)$ is strictly increasing on $[0, \delta]$.

- Assume that assumption $(A2)$ holds and let $T > 0$ and $\rho_T > 0$ be given. Let $\delta > 0$ be such that the function defined by $x \mapsto x - \rho_T M(x)$ is strictly increasing on $[0, \delta]$.
- Assume that $\widehat{E}$ is a nonnegative, nonincreasing function defined on $[0, \infty)$ with $\widehat{E}(0) < \delta$ and satisfying

$$\widehat{E}((k+1)T) \leq \widehat{E}(kT)\Big(1-\rho_T L^{-1}(\widehat{E}(kT))\Big), \quad \forall\, k \in \mathbb{N}. \quad (36)$$

Discret - Main Theorem

$\widehat{E}$ satisfies the upper estimate

$$\widehat{E}(t) \leq TL\Big(\frac{1}{\psi_r^{-1}\big(\frac{(t-T)\rho_T}{T}\big)}\Big)\,, \quad \text{for } t \text{ sufficiently large,} \tag{37}$$

If moreover $\limsup_{x\to 0^+} \Lambda_H(x) < 1$, then we have the simplified decay rate

$$\widehat{E}(t) \leq T(H')^{-1}\Big(\frac{D\,T}{\rho_T(t-T)}\Big)\,, \tag{38}$$

for t sufficiently large and where D is a positive constant independent of $\widehat{E}(0)$ and of T.

Preliminary intermediate results

The initial data $(w(0), \dot{w}(0))$ will be kept fixed. We extend H by $+\infty$ on $\mathbb{R}\backslash[0, r_0^2]$ and still denote this extension by H. We define the convex conjugate of H and denote it by $H^\star$. Moreover we define a weight function f such that

$$H^\star(f(s)) = \frac{sf(s)}{\beta}, \qquad s \in [0, \beta r_0^2), \tag{39}$$

where $\beta > \max(\frac{\alpha T}{C_T}, \frac{E_w(0)}{L(H'(r_0^2))}, \frac{E_w(0)}{\delta})$ where $C_T > 0$, α and $\delta > 0$ are the constants.

We recall that f is defined by

$$f(s) = L^{-1}\left(\frac{s}{\beta}\right), \quad \forall\ s \in [0, \beta r_0^2),$$

where L is the continuous strictly increasing function defined from $[0, +\infty)$ onto $[0, r_0^2)$ by (25).
One can show that f is a strictly increasing function from $[0, \beta r_0^2)$ onto $[0, \infty)$.

- We start by a key Lemma which relies on the optimal-weight convexity method.
- Assume that ρ and a satisfy the assumption $(A1)$ and that there exists $r_0 > 0$ sufficiently small so that the function H defined by (24) is strictly convex on $[0, r_0^2]$.
- Let $(w^0, w^1) \in H_{1/2} \times H$ be given and $(\phi^0, \phi^1) = (w^0, w^1)$ and w and ϕ be the respective solutions of (16) and of (21).

Key Lemma

The following inequality holds

$$\int_0^T f(E_\phi(0)) \int_\Omega \Big(a(x)|\dot{w}|^2 + a(x)|\rho(x,\dot{w})|^2 \Big)\, dx\, dt \leq$$

$$c_5\, TH^\star(f(E_\phi(0))) + c_6 \Big(f(E_\phi(0)) + 1 \Big) \int_0^T \int_\Omega a(x)\rho(x,\dot{w})\dot{w}\, dx\, dt\,, \tag{40}$$

where

$$c_5 = |\Omega|(1 + c_2^2)\,,\; c_6 = \left(\frac{1}{c_1} + c_2 \right),$$

and $|\Omega| = \int_\Omega d\sigma$, with $d\sigma = a(.)dx$.

- The next Lemma compares the localized kinetic damping of the linearly damped equation with the localized linear and nonlinear kinetic energies of the nonlinearly damped equation.
- Assume that $\rho \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}; \mathbb{R})$ is a continuous monotone nondecreasing function with respect to the second variable on Ω such that $\rho(.,0) = 0$ on Ω. Let w be the solution of (16) with initial data $(w^0, w^1) \in H_{1/2} \times H$.
- Let us introduce z solution of the linear locally damped problem

$$\begin{cases} \ddot{z} + Az + a(x)\dot{z} = 0\,, \\ z(0) = w^0, \dot{z}(0) = w^1\,. \end{cases}$$

Comparison Lemma

The following inequality holds

$$\int_0^T \int_\Omega a(x)|\dot{z}|^2 \, dx \, dt \leq 2 \int_0^T \int_\Omega \left(a(x)|\dot{w}|^2 + a(x)|\rho(x, \dot{w}|^2 \right) dx \, dt. \tag{41}$$

The next Lemma compares the localized observation for the conservative undamped equation with the localized damping of the linearly damped equation.

Lemma

Assume that $a \in \mathcal{C}(\overline{\Omega})$*, with* $a \geq 0$ *on* Ω*. Let* $T > 0$ *be given, then there exists* $k_T > 0$ *such that for all* $(w^0, w^1) \in H_{1/2} \times H$

$$\int_0^T \int_\Omega a|\dot{\phi}|^2 \, dx \, dt \leq k_T \int_0^T \int_\Omega a|\dot{z}|^2 \, dx \, dt \qquad (42)$$

where ϕ *is the solution of the conservative equation* (21) *with* $(\phi^0, \phi^1) = (w^0, w^1)$*.*

Theorem

We assume the hypotheses of Key Lemma and denote by w *and* ϕ *the respective solutions of* (16) *and* (21) *where* $(w^0, w^1) = (\phi^0, \phi^1) \in H_{1/2} \times H$*. We set* $\widehat{E}_w = \frac{E_w}{\beta}$*. Then, the following inequality holds*

$$\widehat{E}_w(T) \leq \widehat{E}_w(0)\Big(1 - \rho_T L^{-1}(\widehat{E}_w(0))\Big)\,. \tag{43}$$

where

$$\rho_T = \frac{c_T}{4k_T(c_6 H'(r_0^2) + 1)}\,. \tag{44}$$

Key Corollary

Assume the hypotheses of Key Lemma. We set

$$E_k = \widehat{E}_w(kT)\,, \quad \forall\ k \in \mathbb{N}\,. \tag{45}$$

We define M as in (35). Then the following inequalities hold

$$E_{k+1} - E_k + \rho_T M(E_k) \leq 0\,, \quad \forall\ \ k \in \mathbb{N}\,, \tag{46}$$

with

$$E_0 = \widehat{E}_w(0)\,. \tag{47}$$

Proposition

Assume the hypotheses of Main Theorem and define ψ by

$$\psi(x) = x - \rho_T M(x)\,, \qquad x \in [0, r_0^2) \tag{48}$$

where ρ_T is defined by (44). Then, there exists $\delta > 0$ such that ψ is strictly increasing on $[0, \delta]$.

Proposition

We assume that (A1) holds. Then

$$\liminf_{x\to 0^+} \frac{H'(x)}{\Lambda_H(x)} = 0 \tag{49}$$

where Λ_H is defined by (26).

Remark

Hence the only situation where (28) can be violated occurs if $\liminf_{x\to 0^+} \Lambda_H(x) = 0$ and $\lim_{x\to 0^+} \frac{H'(x)}{\Lambda_H(x)}$ does not exist.

Comparison with an Euler scheme

We start by a first comparison result between the energy evaluated at time kT and a sequence $\widetilde{y_k}$ which is a numerical approximation obtained by a standard Euler scheme applied to an appropriate ordinary differential equation.

Lemma

Assume the hypotheses of the above theorem. We set

$$E_k = \widehat{E}(kT), \quad \forall\ k \in \mathbb{N}. \tag{50}$$

We consider the sequence $(\widetilde{y_k})_k$ *defined by induction as follows*

$$\begin{cases} \widetilde{y_{k+1}} - \widetilde{y_k} + \rho_T M(\widetilde{y_k}) = 0\,, k \in \mathbb{N}\,, \\ \widetilde{y_0} = E_0. \end{cases} \tag{51}$$

Then the following inequality holds

$$E_k \leq \widetilde{y_k}\,, \tag{52}$$

for all $k \in \mathbb{N}$.

We now compare the sequence $(\widetilde{y_k})$ obtained using an Euler scheme to the solution of the associated ordinary differential equation at time kT.

Lemma

Assume the hypotheses of Discret - Main Theorem. We define E_k as in (50). We consider the ordinary differential equation

$$\begin{cases} y'(s) + \frac{\rho_T}{T} M(y(s)) = 0\,, & s \geq 0\,, \\ y(0) = E_0 \end{cases} \tag{53}$$

and set

$$s_k = kT\,, y_k = y(s_k)\,, \quad \forall\ k \in \mathbb{N}. \tag{54}$$

Then we have for all k in $\mathbb{N}$, $\widetilde{y_k} \leq y_k$, where $(\widetilde{y_k})_k$ is defined by (51).

Remark

As mentioned before, the sequence $(\widetilde{y}_k)_k$ is a numerical approximation of the sequence $(y(s_k))_k$ thanks to the Euler scheme applied to (53).

We deduce from the above lemmas the following result

Corollary

Assume the hypotheses of Discret - Main Theorem. Then we have

$$E_k \leq y(s_k) \,, \quad \forall\, k \in \mathbb{N} \,. \tag{55}$$

➥ We can now give the proof of our two main results. We start by

Sketch of proof of Main Theorem

Since (28) holds, we have that $\lim\limits_{x \to 0^+} M'(x) = 0$. This, together with the assumptions of Main Theorem imply that assumption $(A2)$ holds. We set $\widehat{E} = E_w/\beta$. Then thanks to our choice of β we have $\widehat{E}_0 = E_w(0)/\beta < \delta$. Thus, thanks to our assumptions we can apply Key Corollary, so that the sequence $(E_k)_k$ defined by (45) satisfies (46). This implies that $\widehat{E}$ satisfies (36). We can therefore apply the Discret main theorem to $\widehat{E}$, so that $\widehat{E}$ satisfies (37). If additionnally $\limsup_{x \to 0^+} \Lambda_H(x) < 1$ we obtain (38). Going back to the definition of E we conclude.

Proof of Main Corollary

The exponential stabilization for system (33) implies that there exists $T > 0$ and $c_T > 0$ such that (29) holds for (21). We can thus apply Main Theorem to conclude.

Remark

The fact that exponential stabilization implies controllability is the generalization of Russell's principle.

Applications to examples of PDE's and dampings

- Now, we give applications of Main Theorem and Main Corollary. In the next result, we denote by $C_T(E(0))$ a positive (explicit) constant depending on $E(0)$ and T whereas K_T is a positive constant depending on T.
- We also only give the expression of g in a right neighbourhood of 0, since as long as g has a linear growth at infinity, the asymptotic behavior of the energy depends only on the behavior of g close to 0.

Examples of dampings

Theorem

We assume that $\rho \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}; \mathbb{R})$ *is a continuous monotone nondecreasing function with respect to the second variable on* Ω *such that* $\rho(.,0) = 0$ *on* Ω *and satisfying* (22). *We assume that* $a \in \mathcal{C}(\overline{\Omega})$ *satisfies* (23) *with* $a \geq 0$ *on* Ω. *We assume that there exists* $T > 0$ *such that the solution of* (21) *satisfies the observability inequality* (29).

Example 1

let g be given by $g(x) = x^p$ where $p > 1$ on $(0, r_0]$.

Then the energy of solution of (16) satisfies the estimate

$$E(t) \leq C_T(E(0))t^{\frac{-2}{p-1}}, \quad (56)$$

for t sufficiently large and for all $(u_0, u_1) \in H_{1/2} \times L^2(\Omega)$.

Example 2

let g be given by $g(x) = x^p(ln(\frac{1}{x}))^q$ where $p > 2$ and $q > 1$ on $(0, r_0]$.

Then the energy of solution of (16) satisfies the estimate

$$E(t) \leq C_T(E(0))t^{-2/(p-1)}(\ln(t))^{-2q/(p-1)},$$

for t sufficiently large and for all $(u_0, u_1) \in H_{1/2} \times L^2(\Omega)$.

Example 3

let g be given by $g(x) = e^{-\frac{1}{x^2}}$ on $(0, r_0]$.

Then the energy of solution of (16) satisfies the estimate

$$E(t) \leq C_T(E(0))(\ln(t))^{-1}, \qquad (57)$$

for t sufficiently large and for all $(u_0, u_1) \in H_{1/2} \times L^2(\Omega)$.

Example 4

let g be given by $g(x) = e^{-(ln(\frac{1}{x}))^p}$ where $1 < p < 2$ on $(0, r_0]$.

Then the energy of solution of (16) satisfies the estimate

$$E(t) \leq C_T(E(0))e^{-2(\ln(K_T\, t))^{1/p}},$$

for t sufficiently large and for all $(u_0, u_1) \in H_{1/2} \times L^2(\Omega)$.

Example 5

let g be given by $g(x) = x(ln(\frac{1}{x}))^{-p}$ where $p > 0$.

Then the energy of solution of (16) satisfies the estimate

$$E(t) \leq C_T(E(0))e^{-K_T \, t^{1/(p+1)}}(\frac{1}{t})^{1/(p+1)} \qquad (58)$$

for t sufficiently large and for all $(u_0, u_1) \in H_{1/2} \times L^2(\Omega)$.

Stabilization of the nonlinear damped wave equation

We consider the following initial and boundary problem:

$$\begin{cases} u_{tt} - \Delta u + a(x)\rho(x, u_t) = 0, \, , \quad (x,t) \in \Omega \times (0, +\infty) \\ u = 0 \, , \quad \text{on } \partial\Omega \times (0, +\infty), \\ u(x,0) = u^0(x), \; u_t(x,0) = u^1(x) \, , \quad \text{on } \Omega, \end{cases} \tag{59}$$

where ρ and a satisfy $(A1)$.

Hence u satisfies an equation of the form (16) with:

$$A = -\Delta : D(A) \subset H = L^2(\Omega) \to L^2(\Omega),$$
$$D(A) = \left\{ u \in L^2(\Omega),\, \Delta u \in L^2(\Omega),\, u_{|\partial\Omega} = 0 \right\},$$

$H_{1/2} = H_0^1(\Omega)$ It is well-known that A is a self-adjoint operator satisfying (17). The conservative equation (21) becomes in this case:

$$\begin{cases} \phi_{tt} - \Delta\phi = 0\,, \quad \Omega \times (0, +\infty)\,, \\ \phi = 0, \quad \partial\Omega \times (0, +\infty), \\ \phi(x,0) = u^0(x), \phi_t(x,0) = u^1(x), \quad \Omega. \end{cases} \tag{60}$$

We consider the control geometric condition, also called the condition of geometric optics of Bardos Lebeau Rauch :

(G.C.C)

the generalized ray of $\overline{\Omega}$ has a finite order contact with the boundary $\partial\Omega$ and there exist $T_0 > 0$ such that every generalized ray of Ω with length greater than T_0 hits the open set ω.

- We assume that Ω is a $\mathcal{C}^\infty$ bounded open set with a boundary of class $\mathcal{C}^\infty$.
- We assume that ρ and a satisfy the assumption $(A1)$ with $a \in \mathcal{C}^\infty(\overline{\Omega}; [0, \infty))$.
- Assume that there exists $r_0 > 0$ sufficiently small so that the function H defined by (24) is strictly convex on $[0, r_0^2]$ and that (28) is satisfied.
- Moreover assume that the geometric condition **(G.C.C)** is valid.

➥ The stability result can now be stated as follows.

Theorem

$\exists T > 0$ such that the energy of the solution of (59) satisfies

$$E_u(t) \leq \beta T L\Big(\frac{1}{\psi_r^{-1}(\frac{t-T}{T_0})}\Big)\,, \qquad \text{for } t \text{ sufficiently large}\,. \tag{61}$$

If further $\limsup_{x\to 0^+} \Lambda_H(x) < 1$, then we have the simplified decay rate

$$E_u(t) \leq \beta T (H')^{-1}\Big(\frac{DT_0}{t-T}\Big)\,, \text{for sufficiently large } t. \tag{62}$$

Here D is a positive constant which is independent of $E_u(0)$ and T, whereas T_0 depends on T and β is a positive constant chosen such as $\beta > \max\left(\frac{2\alpha T}{C_T}, \frac{E_u(0)}{L(H'(r_0^2))}, \frac{E_u(0)}{\delta}\right)$, where $C_T > 0$, α and $\delta > 0$ are constants.

Second example: stabilization of a nonlinear Bernoulli-Euler plate equation

We consider the following initial and boundary value problem :

$$\begin{cases} u_{tt} + \Delta^2 u + a(x)\rho(x, u_t) = 0\,, \quad \Omega \times (0, +\infty)\,, \\ u = 0,\ \Delta u = 0, \quad \partial\Omega \times (0, +\infty), \\ u(x, 0) = u^0(x), u_t(x, 0) = u^1(x), \quad \Omega, \end{cases} \tag{63}$$

where ρ and a satisfy $(A1)$ and Ω is a bounded smooth domain of $\mathbb{R}^n$, $n \geq 2$.

In this case :

$$A = \Delta^2,\ \mathcal{D}(A) = \Big\{ u \in L^2(\Omega),\ \Delta^2 u \in L^2(\Omega),\ u_{|\partial\Omega} = 0,\ \Delta u_{|\partial\Omega} = 0 \Big\}. \tag{64}$$

Moreover the conservative equation (21) becomes in this case

$$\begin{cases} \phi_{tt} + \Delta^2 \phi = 0\,, & \Omega \times (0, +\infty)\,, \\ \phi = 0,\ \Delta\phi = 0, & \partial\Omega \times (0, +\infty), \\ \phi(x, 0) = u^0(x), \phi_t(x, 0) = u^1(x), & \Omega. \end{cases} \tag{65}$$

Prior to give the geometrical assumptions on ω and Ω based on the piecewise multiplier method, we need some notation.

- If $\Omega_j \subset \Omega$ is a Lipschitz domain, we denote by Γ_j its boundary and by ν_j the outward unit normal to Γ_j.
- Moreover, if U is a subset of $\mathbb{R}^N$ and $x \in \mathbb{R}^N$, we set $d(x, U) = \inf_{y \in U} |x - y|$, and $\mathcal{N}_\varepsilon(U) = \{x \in \mathbb{R}^N, d(x, U) \leq \varepsilon\}$.

We make the following geometric assumptions on Ω and ω:

$$
\textbf{(HG)} \begin{cases} \exists\ \varepsilon > 0, \text{ domains } \Omega_j \subset \Omega \text{ with Lipschitz boundary } \Gamma_j \\ \text{ for } 1 \leq j \leq J \text{ and points } x_j \text{ in } \mathbb{R}^N \\ \text{ such that } \Omega_i \cap \Omega_j = \emptyset \quad \text{ if } i \neq j\,, \\ \Omega \cap \mathcal{N}_\varepsilon\Big[\cup_j \gamma_j(x_j) \cup (\Omega \backslash \cup_j \Omega_j) \Big] \subset \omega\,, \end{cases}
$$

where $\gamma_j(x_j) = \{x \in \Gamma_j\,, (x - x_j) \cdot \nu_j(x) > 0\}$.

Theorem

Assume that there exists $r_0 > 0$ sufficiently small so that the function H defined by (24) is strictly convex on $[0, r_0^2]$ and that (28) is satisfied. Moreover assume that the geometric condition **(HG)** *is valid. Then, the energy of the solution of (63) satisfies*

$$E_u(t) \leq \beta T L\Big(\frac{1}{\psi_r^{-1}(\frac{t-T}{T_0})}\Big) , \quad \text{for } t \text{ sufficiently large} . \tag{66}$$

If further $\limsup_{x \to 0^+} \Lambda_H(x) < 1$, then we have the simplified decay rate for t sufficiently large

$$E_u(t) \leq \beta T (H')^{-1}\big(\frac{D T_0}{t - T}\big). \tag{67}$$

Here D is a positive constant which is independent of $E_u(0)$ and T, whereas $T_0 = \frac{T}{\rho_T}$ depends on T, β is a positive constant chosen such as $\beta > \max\Big(\frac{2\alpha T}{C_T}, \frac{E_u(0)}{L(H'(r_0^2))}, \frac{E_u(0)}{\delta}\Big)$, where $C_T > 0$, α and $\delta > 0$ are the constants.

Remark

① In the case where $a \in C^\infty(\overline{\Omega})$ and under a geometric condition like **(G.C.C)** we obtain the same stability result (by decomposed the plate-like operator in two Schrödinger-like operators $\partial_t^2 + \Delta^2 = (i\partial + \Delta)(-i\partial_t + \Delta)$).

② By using the equivalence between exact internal controllability of the Kirchhoff plate-like equation (68) and the wave equation, we obtain a stability result as the above Theorem for the following system, under the same geometric condition **(G.C.C)** in the case where $a \in \mathcal{C}(\overline{\Omega})$ and under condition (**HG**) in the case where $a \in \mathcal{C}(\overline{\Omega})$.

$$\begin{cases} u_{tt} - \gamma\, \Delta u_{tt} + \Delta^2 u + a(x)\rho(x, u_t) = 0, \,, \quad (x,t) \in \Omega \times (0, +\infty) \\ u = 0, \, \Delta u = 0, \quad \text{on } \partial\Omega \times (0, +\infty), \\ u(x,0) = u^0(x), \, u_t(x,0) = u^1(x)\,, \quad \text{on } \Omega, \end{cases} \tag{68}$$

where ρ and a satisfy $(A1)$, $\gamma > 0$ is a constant and Ω is a bounded smooth domain of $\mathbb{R}^n$, $n \geq 2$.

Some references

- F. Alabau-Boussouira, Convexity and weighted integral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems, *Appl. Math. and Optimization*, **51** (2005), 61–105.
- F. Alabau-Boussouira and K. Ammari, Sharp energy estimates for nonlinearly locally damped PDE's via observability for the associated undamped system, *Journal of Functional Analysis*, **260** (2011), 2424-2450.
- K. Ammari and S. Nicaise, Stabilization of elastic systems by collocated feedback, Lectures Notes in Mathematics, Vol 2124, Springer-Verlag, Berlin, 2015.
- K. Ammari and M. Tucsnak, Stabilization of Bernoulli-Euler beams by means of a pointwise feedback force, *SIAM. J. Control. Optim.*, **39** (2000), 1160-1181.

- K. Ammari and M. Tucsnak, Stabilization of second order evolution equations by a class of unbounded feedbacks, *ESAIM. COCV.*, 6 (2001), 361-386.
- C. Bardos, G. Lebeau and J. Rauch, Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, *SIAM J. Control Optim.*, **30** (1992), 1024-1065.
- M. Daoulatli, I. Lasiecka and D. Toundykov, Uniform energy decay for a wave equation with partially supported nonlinear boundary dissipation without growth conditions, *Discrete Contin. Dyn. Syst. Ser. S*, **2** (2009), 67–94.
- I. Lasiecka and D. Tataru, Uniform boundary stabilization of semilinear wave equation with nonlinear boundary damping, *Differential and Integral Equations*, **8** (1993), 507–533.

G. Lebeau, *Equation des ondes amorties,* Algebraic and geometric methods in mathematical physics (Kaciveli, 1993), 73-109, Math. Phys. Stud., 19, Kluwer Acad. Publ., Dordrecht, 1996.