Constructive exact control of semilinear 1D wave equations by a least-squares approach

SORBONNE UNIVERSITÉ

Emmanuel Trélat

Work with Arnaud Münch

Journées Contrôle, Clermont-Ferrand, octobre 2021

Semilinear 1D wave equation

$\omega = (\ell_1, \ell_2)$ with $0 \leqslant \ell_1 < \ell_2 \leqslant 1, \qquad T > 0$

$$\begin{cases} \partial_{tt} y - \partial_{xx} y + g(y) = f 1_\omega & \text{in } Q_T = \Omega \times (0, T) \\ y = 0 & \text{on } \Sigma_T = \partial\Omega \times (0, T) \\ (y(\cdot, 0), \partial_t y(\cdot, 0)) = (u_0, u_1) & \text{in } \Omega = (0, 1) \end{cases} \tag{1}$$

$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega), \qquad f \in L^2(q_T)$ (control) $\quad$ where $q_T = \omega \times (0, T)$

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Theorem (Zuazua, AIHPC 1993)

Assume that $T > 2\max(\ell_1, 1 - \ell_2)$. There exists $\bar\beta = \bar\beta(\Omega, T) > 0$ such that, if

$$\limsup_{|x| \to +\infty} \frac{|g(x)|}{|x| \ln^2 |x|} < \bar\beta$$

then (1) is exactly controllable in time T, i.e., $\forall (u_0, u_1), (z_0, z_1) \in H_0^1(\Omega) \times L^2(\Omega)$ $\quad \exists f \in L^2(q_T)$ s.t. the solution of (1) satisfies $(y(\cdot, T), \partial_t y(\cdot, T)) = (z_0, z_1)$.

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- Nonconstructive proof by a Leray Schauder fixed point argument.
- Blow-up if $g(s) \simeq -s \ln^p(|s|)$ with $p > 2$ as $|s| \to +\infty$.
- Generalization to products of iterates of log by Cannarsa Komornik Loreti DCDS 2002.

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$(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$, $\qquad f \in L^2(q_T)$ (control) $\quad$ where $q_T = \omega \times (0, T)$

Leray Schauder fixed point argument (Zuazua, AIHPC 1993) :

Given $\xi \in L^\infty(Q_T)$, let f_ξ be the control of minimal norm such that

$$\partial_{tt} y_\xi - \partial_{xx} y_\xi + y_\xi \, \widehat{g}(\xi) = -g(0) + f_\xi 1_\omega \quad \text{in } Q_T, \qquad y_\xi = 0 \quad \text{on } \Sigma_T,$$

$$(y_\xi(\cdot, 0), \partial_t y_\xi(\cdot, 0)) = (u_0, u_1), \quad (y_\xi(\cdot, T), \partial_t y_\xi(\cdot, T)) = (z_0, z_1) \quad \text{in } \Omega,$$

where $\widehat{g}(x) = \frac{g(x) - g(0)}{x}$ if $x \neq 0$ and $\widehat{g}(0) = g'(0)$. Set $\boxed{K(\xi) = y_\xi \in L^\infty(Q_T)}$.

If β is small enough, then there exists $M = M(\|(u_0, u_1)\|, \|(z_0, z_1)\|) > 0$ such that

$$K\left(\bar{B}_{L^\infty(Q_T)}(0, M)\right) \subset \bar{B}_{L^\infty(Q_T)}(0, M).$$

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Objective: design an algorithm providing an explicit sequence $(f_k)_{k \in \mathbb{N}}$ that converges strongly to an exact control for (1).

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First idea: usual Picard fixed point iterations $y_{k+1} = K(y_k)$ with $f_{k+1} \in L^2(q_T)$ control of minimal $L^2(q_T)$ norm for y_{k+1} solution of

$$\partial_{tt} y_{k+1} - \partial_{xx} y_{k+1} + y_{k+1}\,\widehat{g}(y_k) = -g(0) + f_{k+1} 1_\omega \qquad \widehat{g}(x) = \begin{cases} \dfrac{g(x) - g(0)}{x} & \text{if } x \neq 0 \\ g'(0) & \text{if } x = 0 \end{cases}$$

$\rightarrow$ fails in general because K not contracting, even if g is globally Lipschitz (Fernández-Cara Münch MCRF 2012).

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$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, $\qquad f \in L^2(q_T)$ (control) $\quad$ where $q_T = \omega \times (0,T)$

Objective: design an algorithm providing an explicit sequence $(f_k)_{k\in\mathbb{N}}$ that converges strongly to an exact control for (1).

Second idea: apply the Newton method to

$$\widetilde{F}(y,f) = \big(\partial_{tt} y - \partial_{xx} y + g(y) - f 1_\omega,\ y(\cdot\,,0) - u_0,\ \partial_t y(\cdot\,,0) - u_1, y(\cdot\,,T) - z_0,\ \partial_t y(\cdot\,,T) - z_1\big)$$

$\rightarrow$ fails to converge if the initial guess (y_0, f_0) is not close enough to a zero of $\tilde{F}$.

Semilinear 1D wave equation

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$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, $\quad f \in L^2(q_T)$ (control) $\quad$ where $q_T = \omega \times (0, T)$

Objective: design an algorithm providing an explicit sequence $(f_k)_{k\in\mathbb{N}}$ that converges strongly to an exact control for (1), for any initial guess.

Third idea (the good one): apply an optimal step descent method (with appropriate descent direction) to

$$\widetilde{E}(y, f) = \frac{1}{2}\|\widetilde{F}(y, f)\|^2$$

$\rightarrow$ This is a least-squares method (actually, equivalent to a damped Newton method).

(see related ideas in recent works by Lemoine Münch Pedregal 2021, for Navier-Stokes equations)

Least square algorithm

We consider the Hilbert space

$$\mathcal{H} = \Big\{(y,f) \in L^2(Q_T) \times L^2(q_T) \ \mid \ y \in \mathcal{C}([0,T]; H_0^1(0,1)) \cap \mathcal{C}^1([0,T]; L^2(0,1)), \\ \partial_{tt} y - \partial_{xx} y \in L^2(Q_T)\Big\}$$

endowed with the scalar product

$$((y_1,f_1),(y_2,f_2))_{\mathcal{H}} = (y_1,y_2)_{L^2(Q_T)} + ((y_1(\cdot,0),\partial_t y_1(\cdot,0)),(y_2(\cdot,0),\partial_t y_2(\cdot,0)))_{H_0^1(\Omega)\times L^2(\Omega)} \\ + \big(\partial_{tt} y_1 - \partial_{xx} y_1, \partial_{tt} y_2 - \partial_{xx} y_2\big)_{L^2(Q_T)} + (f_1,f_2)_{L^2(q_T)}$$

Let $\mathcal{A}$ and $\mathcal{A}_0$ be the subspaces of $\mathcal{H}$ defined by

$$\mathcal{A} = \Big\{(y,f) \in \mathcal{H} \ \mid \ (y(\cdot,0),\partial_t y(\cdot,0)) = (u_0,u_1), \ (y(\cdot,T),\partial_t y(\cdot,T)) = (z_0,z_1) \text{ in } \Omega\Big\}$$

$$\mathcal{A}_0 = \Big\{(y,f) \in \mathcal{H} \ \mid \ (y(\cdot,0),\partial_t y(\cdot,0)) = (0,0), \ (y(\cdot,T),\partial_t y(\cdot,T)) = (0,0) \text{ in } \Omega\Big\}.$$

We have $\quad \mathcal{A} = (\overline{y},\overline{f}) + \mathcal{A}_0 \quad \forall (\overline{y},\overline{f}) \in \mathcal{A}.$

Least square algorithm

We define the least-squares functional $E : \mathcal{A} \to \mathbb{R}$ ("error" functional) by

$$\boxed{E(y,f) = \frac{1}{2}\left\|\partial_{tt}y - \partial_{xx}y + g(y) - f1_\omega\right\|^2_{L^2(Q_T)}} \qquad \forall (y,f) \in \mathcal{A}$$

Least-squares minimization problem

Given $(\overline{y}, \overline{f}) \in \mathcal{A}$,

$$\boxed{\min_{(y,f)\in\mathcal{A}_0} E(\overline{y}+y, \overline{f}+f)}$$

In the framework of Zuazua's theorem, this minimum is zero and is reached.

Least square algorithm

Assume that $T > 2\max(\ell_1, 1-\ell_2)$. Given any $(y,f) \in \mathcal{A}$, $\exists!(Y^1, F^1) \in \mathcal{A}_0$ with F^1 of minimal $L^2(q_T)$ norm, solving

$$\begin{cases} \partial_{tt} Y^1 - \partial_{xx} Y^1 + g'(y)\, Y^1 = F^1 1_\omega + \Big(\partial_{tt} y - \partial_{xx} y + g(y) - f 1_\omega\Big) & \text{in } Q_T, \\ Y^1 = 0 & \text{on } \Sigma_T, \\ (Y^1(\cdot,0), \partial_t Y^1(\cdot,0)) = (0,0), \quad (Y^1(\cdot,T), \partial_t Y^1(\cdot,T)) = (0,0) & \text{in } \Omega, \end{cases}$$

called *the solution of minimal control norm*. It satisfies

$$\|Y^1\|_{L^\infty(Q_T)} \leqslant C e^{C\sqrt{\|g'(y)\|_{L^\infty}}} \sqrt{E(y,f)}.$$

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$$\|Y^1\|_{L^\infty(Q_T)} \leqslant C e^{C\sqrt{\|g'(y)\|_{L^\infty}}} \sqrt{E(y,f)}.$$

This follows from the general estimate (variant of Zuazua 1993):

Potential $A \in L^\infty(Q_T)$, source $B \in L^2(Q_T)$, $T > 2\max(\ell_1, 1-\ell_2)$.
The (unique) L^2-minimal norm control $u \in L^2(q_T)$ such that

$$\partial_{tt} z - \partial_{xx} z + Az = u 1_\omega + B \quad \text{in } Q_T, \qquad z = 0 \quad \text{on } \Sigma_T,$$

$$(z(\cdot,0), \partial_t z(\cdot,0)) = (z_0, z_1), \quad (z(\cdot,T), \partial_t z(\cdot,T)) = (0,0) \quad \text{in } \Omega,$$

satisfies

$$\|u\|_{L^2(q_T)} + \|(z, \partial_t z)\|_{L^\infty(0,T;H_0^1 \times L^2)} \leqslant C\Big(\|B\|_{L^2}\, e^{(1+C)\sqrt{\|A\|_{L^\infty}}} + \|z_0, z_1\|_{H_0^1 \times L^2}\Big) e^{C\sqrt{\|A\|_{L^\infty}}}$$

for some constant $C > 0$ only depending on Ω and T.

Least square algorithm

Assume that $T > 2\max(\ell_1, 1-\ell_2)$. Given any $(y,f) \in \mathcal{A}$, $\exists!(Y^1, F^1) \in \mathcal{A}_0$ with F^1 of minimal $L^2(q_T)$ norm, solving

$$\begin{cases} \partial_{tt} Y^1 - \partial_{xx} Y^1 + g'(y)\, Y^1 = F^1 1_\omega + \Big(\partial_{tt} y - \partial_{xx} y + g(y) - f 1_\omega\Big) & \text{in } Q_T, \\ Y^1 = 0 & \text{on } \Sigma_T, \\ (Y^1(\cdot,0), \partial_t Y^1(\cdot,0)) = (0,0), \quad (Y^1(\cdot,T), \partial_t Y^1(\cdot,T)) = (0,0) & \text{in } \Omega, \end{cases}$$

called *the solution of minimal control norm*. It satisfies

$$\|Y^1\|_{L^\infty(Q_T)} \leqslant C e^{C\sqrt{\|g'(y)\|_{L^\infty}}} \sqrt{E(y,f)}.$$

Lemma

The (Gateaux) derivative of E at $(y,f) \in \mathcal{A}$ along the direction (Y^1, F^1) of minimal control norm is

$$E'(y,f) \cdot (Y^1, F^1) = \lim_{\substack{\lambda \to 0 \\ \lambda \neq 0}} \frac{E((y,f) + \lambda(Y^1, F^1)) - E(y,f)}{\lambda} = 2E(y,f)$$

Consequence: $-(Y^1, F^1)$ is a descent direction for E.

Least square algorithm

Lemma (followed)

- $$\frac{1}{\sqrt{2}\max\left(1, \|g'(y)\|_{L^\infty}\right)}\|E'(y,f)\|_{\mathcal{A}_0'} \leqslant \sqrt{E(y,f)} \leqslant \frac{1}{\sqrt{2}}Ce^{C\sqrt{\|g'(y)\|_{L^\infty}}}\|E'(y,f)\|_{\mathcal{A}_0'}$$
- Assume that $g \in \mathcal{C}^{1,s}(\mathbb{R})$ for some $s \in [0,1]$. Then
$$E((y,f)-\lambda(Y^1,F^1)) \leqslant \left(|1-\lambda| + \lambda^{1+s}K(y)E(y,f)^{\frac{s}{2}}\right)^2 E(y,f) \qquad \forall \lambda \in \mathbb{R}$$
where $K(y) = C\,[g']_s\left(Ce^{C\sqrt{\|g'(y)\|_{L^\infty}}}\right)^{1+s}$ with $[g']_s = \sup\limits_{\substack{a,b\in\mathbb{R}\\ a\neq b}} \dfrac{|g'(a)-g'(b)|}{|a-b|^s} < +\infty.$

Consequence: any *critical* point $(y,f) \in \mathcal{A}$ of E (i.e., $E'(y,f) = 0$) is a zero of E, and thus is a pair solution of the controllability problem. Moreover:

> *given any sequence $(y_k, f_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ such that $\|E'(y_k,f_k)\|_{\mathcal{A}_0'} \to 0$ and such that $\|g'(y_k)\|_{L^\infty}$ is uniformly bounded, we have $E(y_k,f_k) \to 0$.*

Thanks to this instrumental property, a minimizing sequence for E cannot be stuck in a local minimum, and this, even though E fails to be convex (it has multiple zeros).

Least square algorithm

This leads to define, for any (arbitrarily) fixed $m \geqslant 1$, the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ by

$$\boxed{\begin{cases} (y_0, f_0) \in \mathcal{A} \\ (y_{k+1}, f_{k+1}) = (y_k, f_k) - \lambda_k (Y_k^1, F_k^1) \\ \lambda_k = \underset{\lambda\in[0,m]}{\operatorname{argmin}}\ E\big((y_k, f_k) - \lambda(Y_k^1, F_k^1)\big) \end{cases}}$$

(optimal step along the descent direction $-(Y_k^1, F_k^1)$)

where $(Y_k^1, F_k^1) \in \mathcal{A}_0$ is the solution of minimal control norm of

$$\boxed{\begin{cases} \partial_{tt} Y_k^1 - \partial_{xx} Y_k^1 + g'(y_k)\cdot Y_k^1 = F_k^1 1_\omega + (\partial_{tt} y_k - \partial_{xx} y_k + g(y_k) - f_k 1_\omega) & \text{in } Q_T \\ Y_k^1 = 0 & \text{on } \Sigma_T \\ (Y_k^1(\cdot,0), \partial_t Y_k^1(\cdot,0)) = (0,0) & \text{in } \Omega \end{cases}}$$

Least square algorithm: unconditional convergence result

Theorem (Münch Trélat, 2021)

Assume that $T > 2\max(\ell_1, 1-\ell_2)$, that $g \in \mathcal{C}^{1,s}(\mathbb{R})$ for some $s \in [0,1]$, and that

$$\exists \alpha \geqslant 0 \quad \exists \beta \in [0, \tfrac{s^2}{C^2(2s+1)^2}) \quad | \quad |g'(x)| \leqslant \alpha + \beta \ln^2(1+|x|) \qquad \forall x \in \mathbb{R}. \tag{2}$$

In the case where $s = 0$ (i.e., $g' \in L^\infty(\mathbb{R})$) but $g' \notin \mathcal{C}^{1,s}(\mathbb{R})$ for any $s \in (0,1]$, we assume moreover that $2\|g'\|_{L^\infty} C^2 e^{C\sqrt{\|g'\|_{L^\infty}}} < 1$. Then:

- The sequence $(y_k, f_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$, initialized at any $(y_0, f_0) \in \mathcal{A}$, converges to $(\overline{y}, \overline{f}) \in \mathcal{A}$, solution of (1) such that $(y(\cdot, T), \partial_t y(\cdot, T)) = (z_0, z_1)$.
- $\lambda_k > 0 \quad \forall k \in \mathbb{N} \quad$ and $\quad \lambda_k \to 1$.
- $E(y_k, f_k) \to 0 \quad$ (decreasing).

Moreover, the convergence of all these sequences is at least linear, and is of order $\geqslant 1 + s$ after a finite number of iterations.

$g \in \mathcal{C}^{1,s}(\mathbb{R})$ means that $g \in \mathcal{C}^1(\mathbb{R})$ and g' is uniformly Hölder continuous with exponent s, i.e.,

$$[g']_s = \sup_{\substack{a,b\in\mathbb{R}\\ a\neq b}} \frac{|g'(a) - g'(b)|}{|a-b|^s} < +\infty.$$

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- $\lambda_k > 0 \quad \forall k \in \mathbb{N} \quad$ and $\quad \lambda_k \to 1$.
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Moreover, the convergence of all these sequences is at least linear, and is of order $\geqslant 1+s$ after a finite number of iterations.

- The convergence is unconditional.
- The limit $(\overline{y}, \overline{f}) = (y_0, f_0) - \sum_{k=0}^{+\infty} \lambda_k (Y_k^1, F_k^1)$ depends on the initialization.

It also depends on the selection criterion: F_k^1 is the control of minimal norm.

Least square algorithm: unconditional convergence result

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Assume that $T > 2\max(\ell_1, 1-\ell_2)$, that $g \in \mathcal{C}^{1,s}(\mathbb{R})$ for some $s \in [0,1]$, and that

$$\exists \alpha \geqslant 0 \quad \exists \beta \in [0, \tfrac{s^2}{C^2(2s+1)^2}) \quad | \quad |g'(x)| \leqslant \alpha + \beta \ln^2(1+|x|) \qquad \forall x \in \mathbb{R}. \qquad (2)$$

In the case where $s = 0$ (i.e., $g' \in L^\infty(\mathbb{R})$) but $g' \notin \mathcal{C}^{1,s}(\mathbb{R})$ for any $s \in (0,1]$, we assume moreover that $2\|g'\|_{L^\infty} C^2 e^{C\sqrt{\|g'\|_{L^\infty}}} < 1$. Then:

- The sequence $(y_k, f_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$, initialized at any $(y_0, f_0) \in \mathcal{A}$, converges to $(\overline{y}, \overline{f}) \in \mathcal{A}$, solution of (1) such that $(y(\cdot, T), \partial_t y(\cdot, T)) = (z_0, z_1)$.
- $\lambda_k > 0 \quad \forall k \in \mathbb{N} \quad$ and $\quad \lambda_k \to 1$.
- $E(y_k, f_k) \to 0 \quad$ (decreasing).

Moreover, the convergence of all these sequences is at least linear, and is of order $\geqslant 1 + s$ after a finite number of iterations.

- (2) implies that $\displaystyle \limsup_{|x|\to+\infty} \frac{|g(x)|}{|x| \ln^2 |x|} < \frac{s^2}{(2s+1)^2 C^2}$

"Limit case": $g(x) = a + bx + \frac{1}{9C^2+\varepsilon} x \ln^2(1+|x|)$ for any $\varepsilon > 0$ and $a, b \in \mathbb{R}$.

There exist cases covered by Zuazua's theorem but not by the above.

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Theorem (Münch Trélat, 2021)

Assume that $T > 2\max(\ell_1, 1-\ell_2)$, that $g \in \mathcal{C}^{1,s}(\mathbb{R})$ for some $s \in [0,1]$, and that

$$\exists \alpha \geqslant 0 \quad \exists \beta \in [0, \frac{s^2}{C^2(2s+1)^2}) \quad | \quad |g'(x)| \leqslant \alpha + \beta \ln^2(1+|x|) \qquad \forall x \in \mathbb{R}. \tag{2}$$

In the case where $s=0$ (i.e., $g' \in L^\infty(\mathbb{R})$) but $g' \notin \mathcal{C}^{1,s}(\mathbb{R})$ for any $s \in (0,1]$, we assume moreover that $2\|g'\|_{L^\infty} C^2 e^{C\sqrt{\|g'\|_{L^\infty}}} < 1$. Then:

- The sequence $(y_k, f_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$, initialized at any $(y_0, f_0) \in \mathcal{A}$, converges to $(\overline{y}, \overline{f}) \in \mathcal{A}$, solution of (1) such that $(y(\cdot, T), \partial_t y(\cdot, T)) = (z_0, z_1)$.
- $\lambda_k > 0 \quad \forall k \in \mathbb{N} \quad$ and $\quad \lambda_k \to 1$.
- $E(y_k, f_k) \to 0 \quad$ (decreasing).

Moreover, the convergence of all these sequences is at least linear, and is of order $\geqslant 1+s$ after a finite number of iterations.

- The finite number of iterations is $k_0 = 0$ if $(1+s)\, c\, E(y_0, f_0)^{\frac{s}{2}} < 1$ and otherwise

$$k_0 = \left\lfloor \frac{(1+s)^{1+\frac{1}{s}}}{s} \left(c^{\frac{1}{s}} \sqrt{E(y_0, f_0)} - 1 \right) \right\rfloor + 1 \text{ with } c = [g']_s\, C^{2+s} e^{(1+s)C\sqrt{\alpha}} (1+M)^{(1+s)C\sqrt{\beta}}.$$

Least square algorithm: unconditional convergence result

Theorem (Münch Trélat, 2021)

Assume that $T > 2\max(\ell_1, 1-\ell_2)$, that $g \in \mathcal{C}^{1,s}(\mathbb{R})$ for some $s \in [0,1]$, and that

$$\exists \alpha \geqslant 0 \quad \exists \beta \in [0, \frac{s^2}{C^2(2s+1)^2}) \quad | \quad |g'(x)| \leqslant \alpha + \beta \ln^2(1+|x|) \qquad \forall x \in \mathbb{R}. \tag{2}$$

In the case where $s = 0$ (i.e., $g' \in L^\infty(\mathbb{R})$) but $g' \notin \mathcal{C}^{1,s}(\mathbb{R})$ for any $s \in (0,1]$, we assume moreover that $2\|g'\|_{L^\infty} C^2 e^{C\sqrt{\|g'\|_{L^\infty}}} < 1$. Then:

- The sequence $(y_k, f_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$, initialized at any $(y_0, f_0) \in \mathcal{A}$, converges to $(\overline{y}, \overline{f}) \in \mathcal{A}$, solution of (1) such that $(y(\cdot, T), \partial_t y(\cdot, T)) = (z_0, z_1)$.
- $\lambda_k > 0 \quad \forall k \in \mathbb{N} \quad$ and $\quad \lambda_k \to 1$.
- $E(y_k, f_k) \to 0 \quad$ (decreasing).

Moreover, the convergence of all these sequences is at least linear, and is of order $\geqslant 1 + s$ after a finite number of iterations.

- If $s = 0$, we add a smallness condition on $\|g'\|_{L^\infty}$.

Steps of proof

Denote $E_k = E(y_k, f_k)$. By the lemma we have (with $K(y_k) = [g']_s\, C^{2+s}\, e^{(1+s)C\sqrt{\|g'(y_k)\|_{L^\infty}}}$)

$$E_{k+1} \leqslant \min_{\lambda \in [0,m]} \left(|1-\lambda| + \lambda^{1+s} K(y_k) E_k^{\frac{s}{2}} \right)^2 E_k \qquad (3)$$

Main difficulty

Prove that the sequence $(\|y_k\|_{L^\infty})_{k \in \mathbb{N}}$ remains uniformly bounded.

This is done by *a priori / a posteriori* arguments:

A priori assumption: $\|y_k\|_{L^\infty} \leqslant M \qquad \forall k \in \mathbb{N}$

Keeping track of all estimates in the chain of arguments $\Rightarrow$ a posteriori satisfied (for M large enough).

Steps of proof

Denote $E_k = E(y_k, f_k)$. By the lemma we have (with $K(y_k) = [g']_s\, C^{2+s}\, e^{(1+s)C\sqrt{\|g'(y_k)\|_{L^\infty}}}$)

$$E_{k+1} \leqslant \min_{\lambda \in [0,m]} \left(|1-\lambda| + \lambda^{1+s} K(y_k) E_k^{\frac{s}{2}} \right)^2 E_k \tag{3}$$

Step 1

There exists $k_0 \in \mathbb{N}$ such that $(E_k)_{k \geqslant k_0}$ decays to 0 with order $\geqslant 1+s$.

Idea: use the a priori estimate $\|g'(y_k)\|_{L^\infty} \leqslant \alpha + \beta \ln^2(1+M)$ and infer from (3)

$$c^{\frac{1}{s}} \sqrt{E_{k+1}} \leqslant \begin{cases} \left(c^{\frac{1}{s}} \sqrt{E_k}\right)^{1+s} & \text{if} \quad (1+s)^{\frac{1}{s}} c^{\frac{1}{s}} \sqrt{E_k} < 1 \\ c^{\frac{1}{s}} \sqrt{E_k} - \frac{s}{(1+s)^{1+\frac{1}{s}}} & \text{if} \quad (1+s)^{\frac{1}{s}} c^{\frac{1}{s}} \sqrt{E_k} \geqslant 1 \end{cases}$$

with $c = [g']_s\, C^{2+s} e^{(1+s)C\sqrt{\alpha}} (1+M)^{(1+s)C\sqrt{\beta}}$

Steps of proof

Denote $E_k = E(y_k, f_k)$. By the lemma we have (with $K(y_k) = [g']_s\, C^{2+s}\, e^{(1+s)C\sqrt{\|g'(y_k)\|_{L^\infty}}}$)

$$E_{k+1} \leqslant \min_{\lambda\in[0,m]} \left(|1-\lambda| + \lambda^{1+s} K(y_k) E_k^{\frac{s}{2}}\right)^2 E_k \tag{3}$$

Step 1

There exists $k_0 \in \mathbb{N}$ such that $(E_k)_{k\geqslant k_0}$ decays to 0 with order $\geqslant 1+s$.

Step 2

$\lambda_k \to 1$ as $k \to +\infty$ at least with order $1+s$.

Step 3

$\sum_{k\geqslant 0} \sqrt{E_k} < +\infty$ and $\displaystyle\sum_{k=p}^{+\infty} \sqrt{E_k} \leqslant \mathrm{Cst}\sqrt{E_p}$ for every $p \in \mathbb{N}$.

Step 4

$(\overline{y}, \overline{f}) = (y_0, f_0) - \displaystyle\sum_{k=0}^{+\infty} \lambda_k (Y_k^1, F_k^1)$ with convergence of order $\geqslant 1+s$ after k_0 iterations.

Further comments

Minimization functional

The descent direction $-(Y^1, F^1)$ is designed by minimizing $J(v) = \|v\|^2_{2,q_T}$.
The analysis remains true for $J(y, v) = \|w_1 v\|^2_{2,q_T} + \|w_2 y\|^2_2$ for some weights.

Local controllability when removing the growth condition on g

If $g \in \mathcal{C}^{1,s}(\mathbb{R})$ for some $s \in (0, 1]$ and if $E(y_0, f_0)$ is small snough, then the convergence result remains true.

Multi-dimensional case

Generalization to 2D and 3D under the stricter growth condition $|g(x)| \leqslant \beta|x| \ln^{1/2} |x|$ at infinity, for $\beta > 0$ small enough (Bottois Lemoine Münch 2021).

Boundary control

Open issue.

Semilinear heat equations

Treated in Lemoine Marín-Gayte Münch COCV 2021.