# *On the uniform controllability for a family of non-viscous and viscous convectively filtered Burgers equations*

Diego A. Souza

Joint work with R. K. C. Araújo (UFPE-Brazil) and E. Fernández-Cara (US-Spain)

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# *Outline*

1. Introduction

2. Main results
   - Global uniform controllability for nonviscous models
   - Global uniform controllability for viscous models

3. Additional results and comments

# Introduction: Fluid mechanics

# *Inhomogeneous Navier-Stokes equations*

Physical assumptions:

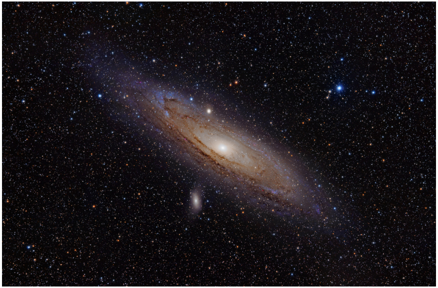
- Conservation of Mass;
- Newton's second law;
- Conservation of Volume;
- Newtonian Law;

$$\begin{cases} (\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) \ = \ \mu\Delta\mathbf{u} - \nabla p + \mathbf{f}, \\ \rho_t + \nabla(\rho\mathbf{u}) \ = 0, \\ \nabla\cdot\mathbf{u} = 0, \end{cases}$$

- $\rho$ – density of mass;
- $\mathbf{u}$ – velocity field;
- $p$ – pressure;
- $\mu$ – dynamic viscosity of the fluid;
- $\mathbf{f}$ – body force term;

# *Waves, tornados, motion of stars, smoke rings, etc*





# Turbulence

*Definition: turbulence or turbulent flow* is a flow regime characterized by chaotic property changes.

Main characteristics of turbulence:

- Fast variations in space and time of $p$ and $\mathbf{u}$ (wide range of length scales for eddies)
- Well behavior of (appropriately) averaged variables

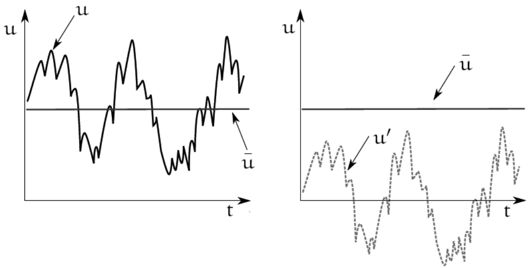
**Typically:** small (resp. large) Re $:= \frac{U_\infty L}{\nu} \Rightarrow$ laminar (resp. turbulent) flow.

# Large Eddy Simulation Models

1. Reynolds decomposition:
   $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$ and $p = \overline{p} + p'$, where $(\mathbf{u}, p)$ is a solution of

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0.$$

   Here: $\overline{\mathbf{u}}$ is the average velocity and $\mathbf{u}'$ is the fluctuation.




2. PDE's for $\overline{\mathbf{u}}$ and $\overline{p}$?

$$\overline{\mathbf{u}}_t + \nabla \cdot (\overline{\mathbf{u} \otimes \mathbf{u}}) + \nabla \overline{p} = \nu \Delta \overline{\mathbf{u}} + \overline{\mathbf{f}}, \quad \nabla \cdot \overline{\mathbf{u}} = 0.$$

3. Closure problem: assumptions relating $\overline{\mathbf{u} \otimes \mathbf{u}}$ and $\overline{\mathbf{u}}$.

# Reynolds hypothesis

**A particular closure hypothesis:**

$$\overline{\mathbf{u} \otimes \mathbf{u}} \approx \mathbf{z} \otimes \overline{\mathbf{u}}, \text{ with } \mathbf{z} = (\mathbf{Id} + \alpha^2 \mathbf{A})^{-1} \overline{\mathbf{u}}.$$

where $\alpha > 0$ is regularized parameter that introduces an energy "penalty" that inhibit the formation of eddies whose length-scale is smaller than $\alpha$.

**Leray-$\alpha$ model:**

$$\begin{cases} \overline{\mathbf{u}}_t + (\mathbf{z} \cdot \nabla)\overline{\mathbf{u}} + \nabla p = \nu \Delta \overline{\mathbf{u}} + \bar{\mathbf{f}}, & \nabla \cdot \overline{\mathbf{u}} = 0, \\ \mathbf{z} - \alpha^2 \Delta \mathbf{z} + \nabla \pi = \overline{\mathbf{u}}, & \nabla \cdot \mathbf{z} = 0. \end{cases}$$

*Remark:* **Leray-$\alpha$ solutions $\to$ NS solutions, as $\alpha \to 0^+$**

References:


- LERAY, J. Essai sur le mouvement d'un fluide visqueux emplissant l'espace. *Acta Math.* 63 (1934), 193-248.
- CHESKIDOV, A., HOLM, D. D., OLSON, E., AND TITI, E. S. On a Leray-$\alpha$ model of turbulence. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 461*, 2055 (2005), 629–649.


$\mathbf{A}$ is the Stokes operator, i.e. $\mathbf{A} = -\mathbb{P}\Delta$, where $\mathbb{P}$ is the Leray's projector

# *One dimensional fluid models*

Nonviscous Burgers equation:

$$u_t + uu_x = f.$$

- Viscous Burgers equation:

$$u_t + uu_x = \nu u_{xx} + f.$$

- Benjamin-Bona-Mahony equation:

$$u_t - u_{xxt} + u_x + uu_x = f$$

- Korteweg-de Vries equation:

$$u_t - u_{xxx} + 6uu_x = f$$

- Degasperis-Procesi equation:

$$u_t + 2\kappa u_x - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = f$$

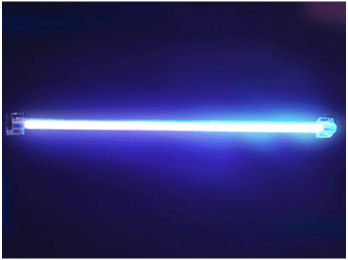
- Camassa-Holm equation:

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = f$$

- $b$-family:

$$u_t + 2\kappa u_x - u_{xxt} + (b+1)uu_x - bu_x u_{xx} - uu_{xxx} = f$$

*Motion in a neon tube, traffic motion, surface waves of long wavelength, Shallow water, etc*




# *Convectively filtered Burgers equation*

Nonviscous Burgers-$\alpha$:

$$\begin{cases} u_t + zu_x = f, \\ z - \alpha^2 z_{xx} = u. \end{cases}$$

Viscous Burgers-$\alpha$:

$$\begin{cases} u_t + zu_x = \nu u_{xx} + f, \\ z - \alpha^2 z_{xx} = u. \end{cases}$$

Motivations:

- A "toy model" for Leray-$\alpha$
- Applications: models that capture shock formation

References:


- Bhat and Fetecau, A Hamiltonian Regularization of the Burgers Equation, JNS ('06).
- —, The Riemann problem for the Leray-Burgers equation, JDE, (2009).
- G. Norgard and K. Mohseni, A regularization of the Burgers equation using a filtered convective velocity, J. Phys. A ('08).
- —, On the convergence of the convectively filtered Burgers equation to the entropy solution of the inviscid Burgers equation. MMS ('09).

# Introduction: Control problems

# *Controllability problem*

**Control system** is a **dynamical system** involving two variables, the **state** and the **control**, i.e.

$$\begin{cases} u_t = f(t,u,v), \\ u(0) = u_0, \end{cases}$$

where $u \in C^0([0,+\infty);\mathcal{S})$ is the **state** and $v \in \mathcal{C}$ is the **control**.

**Goal:** to find a **control** such that the **associated state** behaves in an appropriate manner in a given final time.

*Exact controllability at time $T$:*

For any $u_0, u_T \in \mathcal{S}$, find $v \in \mathcal{C}$ such that $u(T) = u_T$;

*Particular cases:*

- Exact controllability to the trajectories: $u_T \equiv \widehat{u}(T)$, where $(\widehat{u},\widehat{v})$ is a trajectory;
- Null controllability: $u_T \equiv 0$;

*Approximate controllability at time $T$:*

For any $u_0, u_T \in \mathcal{S}$ and $\varepsilon > 0$, find $v \in \mathcal{C}$ such that $\|u(T) - u_T\|_{\mathcal{S}} \leq \varepsilon$.

# *Heat equation*

We assume: $\kappa > 0$, $\mathcal{O} \subset \Omega$, $\gamma \subset \partial\Omega$ and $T > 0$.
The controlled linear heat equation:

$$\left\{\begin{array}{lll} u_t - \kappa\Delta u = v1_{\mathcal{O}} & in & \Omega \times (0,T), \\ u = 0 & on & \partial\Omega \times (0,T), \\ u(0) = u_0 & in & \Omega. \end{array}\right. \qquad \left\{\begin{array}{lll} u_t - \kappa\Delta u = 0 & in & \Omega \times (0,T), \\ u = h1_{\gamma} & on & \partial\Omega \times (0,T), \\ u(0) = u_0 & in & \Omega. \end{array}\right.$$

*Remark:* **Regularizing effect $\Longrightarrow$ EC does not hold**.

*Distributed (boundary) null controllability:*

$\forall\, u_0 \in L^2(\Omega)\ \exists\, v \in L^2(\mathcal{O} \times (0,T))$ or $h \in L^2(\gamma \times (0,T))$ s. t. $u(\cdot\,,T) \equiv 0$.

*Distributed (boundary) observability inequality:*

$\exists C_w > 0$ s. t. $\|\varphi(0)\|_{L^2(\Omega)} \leq C_w \|\varphi\|_{L^2(\mathcal{O}\times(0,T))}, \quad \forall \varphi_T \in L^2(\Omega)$.

$$\left( \|\varphi(0)\|_{L^2(\Omega)} \leq C_w \left\| \frac{\partial\varphi}{\partial\nu} \right\|_{L^2(\gamma\times(0,T))}, \quad \forall \varphi_T \in L^2(\Omega). \right)$$

References:

- Russell ('78) : method of moments;
- Lebeau & Robbiano ('95) : spectral inequalities for the low frequencies;
- Fursikov & Imanuvilov ('96) : global Carleman inequalities.

# *Known results for the Navier-Stokes equations*

Let $\Omega$ be a smooth bounded domain and $T > 0$.

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p + \mathbf{v}1_{\mathcal{O}}, \quad \nabla\cdot\mathbf{u} = 0 & \textit{in} \quad \Omega\times(0,T), \\ \mathbf{u} = \mathbf{0} & \textit{on} \quad \partial\Omega\times(0,T), \\ \mathbf{u}(0) = \mathbf{u}_0 & \textit{in} \quad \Omega. \end{cases}$$

**Theorem [LECT: Fernández–Cara, Guerrero, Imanuvilov, Puel (2004)]**

$\forall(\widehat{\mathbf{u}},\widehat{p})$ with $\widehat{\mathbf{u}} \in \mathbf{L}^\infty$: $\exists\,\delta > 0$ s.t. $\|\mathbf{u}_0 - \widehat{\mathbf{u}}_0\| < \delta \Rightarrow \exists\,\mathbf{v}\in\mathbf{L}^2$ s.t. $\mathbf{u}(T) = \widehat{\mathbf{u}}(T)$.

Controllability with a reduced number of controls:

- $N-1$ controls - $\mathcal{O}$ touches the boundary [F–C, G, I, P (2006)];
- $N-1$ **null** controls - with no assumption on $\mathcal{O}$ [Carreño, G (2013)];
- $N=3$, only one **null** control - with no assumption on $\mathcal{O}$ [Coron, Lissy (2014)].

*Remark: Global: ECT? NC? AC?*

- **Open problem** for **Dirichlet** BC;
- **Solved** for **Navier-slip with friction** BC [ Coron, Marbach, Sueur (2020)]

# *Known results for the Burgers equations*

Let: $L, T > 0$, $Q = (0,T) \times (0,L)$ and $\mathcal{O} \subset (0,L)$.

$$(B_\nu) \begin{cases} u_t + uu_x = \nu u_{xx} + p, & \text{in} \quad Q, \\ u(0,\cdot) = v_l, \ \ u(L,\cdot) = v_r, & \text{on} \quad (0,T), \\ u(\cdot,0) = y_0, & \text{in} \quad (0,L), \end{cases} \qquad (B_0) \begin{cases} u_t + uu_x = p, & \text{in} \quad Q, \\ u(0,\cdot) = v_l, \ \ u(L,\cdot) = v_r, & \text{on} \quad (0,T), \\ u(\cdot,0) = u_0, & \text{in} \quad (0,L). \end{cases}$$

**Theorem [LECT-$B_\nu$: Fursikov-Imanuvilov (1996)]**
For any $\widehat{u}$, with $\widehat{u} \in L^\infty$: $\exists\, \delta > 0$ such that $\|u_0 - \widehat{u}_0\| < \delta \Rightarrow \exists p \in L^2$, with $\operatorname{supp} p(\cdot,t) \subset \mathcal{O}$ and $v_l \equiv v_r \equiv 0$, such that $u(T) = \widehat{u}(T)$.

*Remark: Global ($B_\nu$ and $B_0$) : ECT? NC? AC?*

- **Lack of NC for $B_\nu$:** using only $v_l$ [ Fernández-Cara, Guerrero (2007)]
- **Lack of NC for $B_\nu$:** using only $v_l$ and $v_r$ [ Guerrero-Imanuvilov (2007)]
- **GEC for $B_0$ and GECT for $B_\nu$:** using $p = p(t)$, $v_l$ and $v_r$ [ Chapouly (2009)]
- **GNC for $B_0$ and GNC for $B_\nu$:** using only $p = p(t)$ and $v_r$ [ Marbach (2014)]
- **Lack of NC for $B_\nu$:** using only $p = p(t)$ [ Marbach (2018)]

# Main results: Global uniform controllability for nonviscous models

M. CHAPOULY, Global controllability of nonviscous and viscous Burgers-type equations, SIAM J. Control Optim., 48 (3), 1567-1599, (2009).

# *Inviscid Burgers-$\alpha$ system*

Given a time $T > 0$ and a length $L > 0$ the *inviscid Burgers-$\alpha$ system* is given by:

$$
(B_0^\alpha)\begin{cases}
u_t + zu_x = p(t) & \text{in} \quad (0,T)\times(0,L),\\
z - \alpha^2 z_{xx} = u & \text{in} \quad (0,T)\times(0,L),\\
z(\cdot,0) = v_l, \quad z(\cdot,L) = v_r & \text{on} \quad (0,T),\\
u(\cdot,0) = v_l & \text{on} \quad I_l,\\
u(\cdot,L) = v_r & \text{on} \quad I_r,\\
u(0,\cdot) = u_0 & \text{in} \quad (0,L),
\end{cases}
$$

where $I_l = \{t \in [0,T] : \; v_l(t) > 0\}$ and $I_r = \{t \in [0,T] : \; v_r(t) < 0\}$;

- The triplet $(p, v_l, v_r)$ are the *controls* and the couple $(u,z)$ is the *associated state*;

- Motivation: Regularization of the *inviscid Burgers equation*, member $b = 0$ of the so-called $b$-family...

# *Global uniform exact controllability result*

*Theorem [Global uniform exact controllability result in $C^1$]*

Let $\alpha, T, L > 0$ be given. The *inviscid Burgers-$\alpha$ system* is *globally exactly controllable* in $C^1$. That is, for any given $u_0, u_T \in C^1([0,L])$, there exist a time-dependent control $p^\alpha \in C^0([0,T])$, a couple of boundary controls $(v_l^\alpha, v_r^\alpha) \in C^1([0,T];\mathbb{R}^2)$ and an associated state $(u^\alpha, z^\alpha) \in C^1([0,T] \times [0,L];\mathbb{R}^2)$ satisfying $(B_0^\alpha)$ in the classical sense and

$$u^\alpha(T,\cdot) = u_T \quad \text{in} \quad (0,L).$$

Moreover, there exists a positive constant $C > 0$ (depending on $u_0$ and $u_T$ but **independent of** $\alpha$) such that

$$\|(z^\alpha, u^\alpha)\|_{C^1([0,T]\times[0,L];\mathbb{R}^2)} + \|p^\alpha\|_{C^0([0,T])} + \|(v_l^\alpha, v_r^\alpha)\|_{C^1([0,T];\mathbb{R}^2)} \le C.$$

# *Strategy of the proof. Return method*

The **return method** has been introduced by J.-M. **Coron**.

**The principle of the method is the following:**

- find a trajectory of the nonlinear system such that the **linearized system** around it is **controllable**.
- Then, one hope to construct a solution of the nonlinear controllability problem close to such trajectory.

# *Strategy of the proof. Return method*

The proof is splitted in 4 steps:

- **Step 1**: we linearize the system around a suitable trajectory;
- **Step 2**: we prove the global controllability for the linearized system;
- **Step 3**: we deduce the local controllability for the nonlinear system;
- **Step 4**: we use a scaling argument to deduce the desired global result.

# *Local null-controllability of the "perturbed" system*

For $k \geq 1$, let us introduce the set:

$$\Lambda_{L,T,k} := \left\{ \lambda \in C_0^k([0,T];[0,\infty)) \ : \ \|\lambda\|_{L^1(0,T)} > L \right\}.$$

Then, $(\widehat{u}, \widehat{z}) = (\lambda, \lambda)$ is a trajectory for $(B_0^\alpha)$ with $(\widehat{p}, \widehat{v}_l, \widehat{v}_r) = (\lambda', \lambda, \lambda)$. The linearization around $(\widehat{u}, \widehat{z}) = (\lambda, \lambda)$ is:

$$u_t + \lambda(t) u_x = 0, \quad z - \alpha^2 z_{xx} = u \text{ in } (0,T) \times \mathbb{R},$$

which is globally controllable to zero.

**One can see that the velocity $\lambda(t)$ is "fast enough" such that $\Phi_\lambda(T;0,x) \in \mathbb{R} \backslash \mathrm{supp}\, u(\cdot,0)$, for all $x \in [0,L]$.**

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Consider the the flux $\Phi_\lambda \in C^1([0,T] \times [0,T] \times \mathbb{R})$, solution to:

$$\begin{cases} \phi'_\lambda(s;t,x) = \lambda(t), \\ \phi_\lambda(s;s,x) = x. \end{cases}$$

# *Local null-controllability of the "perturbed" system*

Then, one may expect the null-controllability for the "perturbed" system:

$$
(PB_0^\alpha)\begin{cases}
y_t + (\lambda(t) + w)y_x = 0, \quad w - \alpha^2 w_{xx} = y & \text{in} \quad (0,T)\times(0,L),\\
w(\cdot,0) = q_l, \quad w(\cdot,L) = q_r & \text{on} \quad (0,T),\\
y(\cdot,0) = q_l & \text{on} \quad J_l,\\
y(\cdot,L) = q_r & \text{on} \quad J_r,\\
y(0,\cdot) = y_0 & \text{in} \quad (0,L).
\end{cases}
$$

$J_l = \{t \in [0,T] :\ q_l(t) > 0\}$ and $J_r = \{t \in [0,T] :\ q_r(t) < 0\}$

# *Local null-controllability of the “perturbed” system*

*Theorem [Local null control for the “perturbed” inviscid Burgers-$\alpha$ system]*

Let $T$, $L > 0$ be given and assume that $\lambda \in \Lambda_{L,T,0}$. Then, there exist $\delta > 0$ and $C > 0$ (both independent of $\alpha$) such that, for any $y_0 \in C^1([0,L])$ with $\|y_0\|_{C^1([0,L])} \leq \delta$, there exists $(q_l, q_r) \in C^1([0,T];\mathbb{R}^2)$ and an associated state $(y,w) \in C^1([0,T] \times [0,L];\mathbb{R}^2)$ solution to $(PB_0^\alpha)$ and

$$y(T,\cdot) = 0 \ \text{ in } \ (0,L).$$

Moreover, there exists $C > 0$ (independent of $\alpha$) such that

$$\|y\|_{C^1([0,T]\times[0,L])} \leq C\|y_0\|_{C^1([0,L])} \quad \forall \alpha > 0.$$

*Remark:*

- $(u,z) := (y,w) + (\lambda,\lambda)$, $(v_l, v_r) := (q_l, q_r) + (\lambda,\lambda)$ and $p := \lambda'$. Then, a **local uniform null controllability** result holds for $(u,z)$.
- **Scaling arguments** $+$ **time-reversibility** $\Longrightarrow$ **Global EC in** $C^1$.

# *Sketch of the proof. I*

- Since $\lambda \in \Lambda_{L,T,0}$ then there exists $\eta \in (0, L/2)$ such that

$$\int_0^T \lambda(t)\, dt > L + 2\eta.$$

- Let the **extension operators:**
  - $\pi_1 : C^1([0,L]) \mapsto C^1(\mathbb{R})$ and $\pi_2 : C^2([0,L]) \mapsto C^2(\mathbb{R})$;
  - $\pi_i(f) = f$ in $[0,L]$, for $i = 1, 2$;
  - $\operatorname{supp} \pi_i(f) \subset (-\eta, L+\eta)$, for $i = 1, 2$.
- We will use a **fixed-point argument**. For such, let us introduce the Banach space $F := C^0([0,T]; C^0([0,L])) \cap L^\infty(0,T; C^{0,1}([0,L]))$ and the set

$$B_R := \{h \in F \text{ such that } \|h\|_F \leq R\}$$

where $R > 0$ will be chosen appropriately.

# *Sketch of the proof. II*

- Given $h \in B_R$ there exists a unique $w \in C^0([0,T];C^2([0,L]))$ solution to the **time-dependent elliptic problem:**

$$\begin{cases} w - \alpha^2 w_{xx} = h & \text{in} \quad (0,T)\times(0,L), \\ w(\cdot,0) = h(\cdot,0), \;\; w(\cdot,L) = h(\cdot,L) & \text{on} \quad (0,T). \end{cases}$$

Using the **maximum principle**: $\|w\|_{C^0([0,T];C^1([0,L]))} \leq \|h\|_F \leq R.$

Let $w^* \in C^0([0,T];C^2(\mathbb{R}))$ given by $w^*(t,\cdot) := \pi_2(w)(t,\cdot)$. Then, there exists $C_1 > 0$ **(independent of** $\alpha$**)** such that

$$(\star) \qquad \|w^*\|_{C^0([0,T];C^1(\mathbb{R}))} \leq C_1 \|w\|_{C^0([0,T];C^1([0,L]))}.$$

- Consider the the flux $\Phi^* \in C^1([0,T]\times[0,T]\times\mathbb{R})$, solution to:

$$\begin{cases} \phi_t^*(s;t,x) = \lambda(t) + w^*(t,\phi^*(s;t,x)), \\ \phi^*(s;s,x) = x. \end{cases}$$

Then, if we consider the flux $\Phi_\lambda \in C^1([0,T]\times[0,T]\times\mathbb{R})$ associated with the ODE $\xi'(t) = \lambda(t)$, using $(\star)$ and choosing $R := \dfrac{\eta}{C_1 T}$, we get:

$$(\star\star) \qquad \|\Phi^* - \Phi_\lambda\|_{C^0([0,T]\times[0,T]\times\mathbb{R})} \leq T\|w^*\|_{C^0([0,T];C^0(\mathbb{R}))} \leq \eta.$$

# *Sketch of the proof. III*

- We define $y_0^* := \pi_1(y_0) \in C^1(\mathbb{R})$ and search for $y \in C^1([0,T] \times \mathbb{R})$ solution to the transport equation:

$$\begin{cases} y_t + (\lambda(t) + w^*(t,x))y_x = 0 & \text{in} \quad (0,T) \times \mathbb{R}, \\ y(0,\cdot) = y_0^* & \text{in} \quad \mathbb{R}, \\ y(T,\cdot) = 0 & \text{in} \quad [0,L]. \end{cases}$$

Using the **method of characteristic** we deduce that

$$(\star\star\star) \qquad y(t,x) := y_0^*(\Phi^*(t;0,x)), \;\; \forall (t,x) \in [0,T] \times \mathbb{R},$$

is a **classical solution** to the transport equation above.

- Using $(\star\star)$, we get $\Phi^*(T;0,\cdot) < -\eta$ in $(-\infty, L]$.
Since $\operatorname{supp} y_0^* \subset (-\eta, L+\eta)$, we have $y(T,\cdot) \equiv 0$ in $[0,L]$.

- Moreover,

$$\|y\|_{C^1([0,T]\times[0,L])} \leq C_2 \|y_0\|_{C^1([0,L])}$$

for some $C_2 > 0$, **independent of** $\alpha$**.**

# *Sketch of the proof. IV*

- For $y_0 \in C^1([0,L])$ small enough we can define the **fixed-point mapping** $\mathcal{F} : B_R \mapsto B_R$ by:

$$\mathcal{F}(h)(t,x) := y(t,x), \ \ \forall\, (t,x) \in [0,T] \times [0,L],$$

  where $y$ is the function defined in $(\star\star\star)$.

- By an **induction argument**, there exists $C > 0$

$$\|\mathcal{F}^m(h_1) - \mathcal{F}^m(h_2)\|_E \leq \frac{C^m T^m}{m!} \|h_1 - h_2\|_E \quad \forall m \geq 1.$$

- Finally:
  1. $\mathcal{F}^m$ is a **contraction** in $E$ for $m$ large enough;
  2. Consider $\widetilde{B}_R := \overline{B}_R^{\|\cdot\|_E}$ and $\widetilde{\mathcal{F}}^m$ being the extension of $\mathcal{F}^m$ to $\widetilde{B}_R$;
  3. $\widetilde{\mathcal{F}}^m(\widetilde{B}_R) \subset B_R \cap C^1([0,T] \times [0,L])$;
  4. There exists a unique $y \in \widetilde{B}_R$ such that $\widetilde{\mathcal{F}}^m(y) = y$.

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$F := C^0([0,T]; C^0([0,L])) \cap L^\infty(0,T; C^{0,1}([0,L]))$ and $E := C^0([0,T]; C^0([0,L]))$.

# Main results: Global uniform controllability for viscous models

# *Viscous Burgers-$\alpha$ system*

- Given a time $T > 0$ and a length $L > 0$ the *Viscous Burgers-$\alpha$ system* is given by:

$$
(B_1^\alpha)\begin{cases}
u_t + z u_x = u_{xx} + p(t) & \text{in} \quad (0,T)\times(0,L),\\
z - \alpha^2 z_{xx} = u & \text{in} \quad (0,T)\times(0,L),\\
z(\cdot,0) = u(\cdot,0) = v_l, \quad z(\cdot,L) = u(\cdot,L) = v_r & \text{on} \quad (0,T),\\
u(0,\cdot) = u_0 & \text{in} \quad (0,L).
\end{cases}
$$

- The triplet $(p, v_l, v_r)$ are the *controls* and the couple $(u, z)$ is the *associated state*;

- Motivation: Regularization of the *viscous Burgers equation*, 1$D$-version of the *Leray-$\alpha$ model*, member $b = 0$ of the so-called viscous $b$-family...

# *Global exact controllability to constant trajectories*

*Theorem [Global exact controllability to constant trajectories]*

Let $\alpha, T, L > 0$ be given. The viscous *Burgers-$\alpha$ system* is **globally exactly controllable** in $L^\infty$ to constant trajectories. That is, for any $u_0 \in L^\infty(0,L)$ and $N \in \mathbb{R}$, there exist controls $p^\alpha \in C^0([0,T])$ and $(v_l^\alpha, v_r^\alpha) \in H^{3/4}(0,T;\mathbb{R}^2)$ and states $(u^\alpha, z^\alpha) \in L^2(0,T;H^1(0,L;\mathbb{R}^2)) \cap L^\infty(0,T;L^\infty(0,L;\mathbb{R}^2))$ satisfying $(B_1^\alpha)$,

$$u^\alpha(T,\cdot) = N \quad \text{in} \quad (0,L)$$

and the estimates

$$\|p^\alpha\|_{C^0([0,T])} + \|(v_l^\alpha, v_r^\alpha)\|_{H^{3/4}([0,T];\mathbb{R}^2)} \leq C,$$

where $C$ is a positive constant (depending on $u_0$ and $N$ but independent of $\alpha$). Moreover, if $u_0 \in H_0^1(0,L)$ then the same conclusion holds with $(u^\alpha, z^\alpha) \in L^2(0,T;H^2(0,L;\mathbb{R}^2)) \cap H^1(0,T;L^2(0,L;\mathbb{R}^2))$.

The proof is splitted in three steps:

- Step 1: **Smoothing effect**;
- Step 2: **Uniform approximate controllability**;
- Step 3: **Uniform local exact controllability to the trajectories**.

# *Smoothing effect*

*Proposition [Smoothing effect]*

Let $u_0 \in L^\infty(0, L)$ be given and let $(u^\alpha, z^\alpha)$ be the solution to the **uncontrolled** *viscous Burgers-$\alpha$ system* (that is, $p^\alpha = v_l^\alpha = v_r^\alpha = 0$). Then, there exist $T^* \in (0, T/2)$ and $C > 0$ (independent of $\alpha$) such that the solution $u^\alpha$ belongs to $C^0([T^*, T]; C^2([0, L]))$ and satisfies

$$\|u^\alpha\|_{C^0([T^*,T];C^2([0,L]))} \leq C\Lambda(\|u_0\|_{L^\infty}),$$

where $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and satisfies $\Lambda(s) \to 0$ as $s \to 0^+$.

**Sketch of the proof:**

- **Smoothing effect for parabolic operators**
- **Energy estimates**.

# Uniform approximate controllability

*Proposition [Uniform approximate controllability]*

Let $u_0, u_f \in C^2([0,L])$ be given. There exist positive constants $\tau_*$ and $K > 0$, independent of $\alpha$, such that, for any $\tau \in (0, \tau_*]$, there exist $p^\alpha \in C^0([0,\tau])$, $(v_l^\alpha, v_r^\alpha) \in H^{3/4}(0,\tau;\mathbb{R}^2)$ and associated states $(u^\alpha, z^\alpha)$, in appropriated functional spaces, solution to $(B_1^\alpha)$, with $T$ replaced by $\tau$, and satisfying

$$\|u^\alpha(\tau) - u_f\|_{H^1} \le K\sqrt{\tau} \text{ and } \|p^\alpha\|_{C^0} + \|(v_l^\alpha, v_r^\alpha)\|_{H^{3/4}} \le C \quad \forall \alpha > 0.$$

**Strategy of the proof**:

- **Heuristic procedure:** let us consider the decomposition

$$\begin{cases} u^\alpha := y^{\alpha,\tau} + \lambda^\tau + r^{\alpha,\tau}, \\ z^\alpha := w^{\alpha,\tau} + \lambda^\tau + q^{\alpha,\tau}. \end{cases}$$

- Replacing them in $(B_1^\alpha)$:

$$\begin{cases} u_t^\alpha - u_{xx}^\alpha + z^\alpha u_x^\alpha = \lambda_t^\tau(t) + \left[y_t^{\alpha,\tau} + (\lambda^\tau(t) + w^{\alpha,\tau}) y_x^{\alpha,\tau}\right] \\ \qquad + \left[r_t^{\alpha,\tau} - r_{xx}^{\alpha,\tau} + (q^{\alpha,\tau} + w^{\alpha,\tau} + \lambda^\tau) r_x^{\alpha,\tau} \right. \\ \qquad \left. + q^{\alpha,\tau} y_x^{\alpha,\tau} - y_{xx}^{\alpha,\tau}\right] \\ z^\alpha - \alpha^2 z_{xx}^\alpha = \left[(w^{\alpha,\tau} + \lambda^\tau) - \alpha^2 (w^{\alpha,\tau} + \lambda^\tau)_{xx}\right] + \left[q^{\alpha,\tau} - \alpha^2 q_{xx}^{\alpha,\tau}\right]. \end{cases}$$

- Then, we have: **Inviscid part + viscous remainder**.

# *Sketch of the proof. I*

For $\tau > 0$ small enough and $\lambda^\tau \in C_0^1(0,\tau)$ taken appropriately, we get controls $(v_l^{\alpha,\tau}, v_r^{\alpha,\tau}) \in C^2([0,\tau])$ such that

$$\begin{cases} y_t^{\alpha,\tau} + (\lambda^\tau(t) + w^{\alpha,\tau}) y_x^{\alpha,\tau} = 0 & \text{in} \quad (0,\tau)\times(0,L), \\ w^{\alpha,\tau} - \alpha^2 w_{xx}^{\alpha,\tau} = y^{\alpha,\tau} & \text{in} \quad (0,\tau)\times(0,L), \\ y^{\alpha,\tau}(\cdot,0) = w^{\alpha,\tau}(\cdot,0) = v_l^{\alpha,\tau} & \text{on} \quad (0,\tau), \\ y^{\alpha,\tau}(\cdot,L) = w^{\alpha,\tau}(\cdot,L) = v_r^{\alpha,\tau} & \text{on} \quad (0,\tau), \\ y^{\alpha,\tau}(0,\cdot) = u_0 & \text{in} \quad (0,L), \\ y^{\alpha,\tau}(\tau,\cdot) = u_f & \text{in} \quad (0,L). \end{cases}$$

*Remark:*
We note that $\|u^\alpha(\tau,\cdot) - u_f\|_{H^1(0,L)} = \|r^{\alpha,\tau}(\tau,\cdot)\|_{H^1(0,L)}$.

Then, the remaining question is:
*Is it possible to find $K > 0$ (independent of $\alpha$ and $\tau$) such that*

$$\|r^{\alpha,\tau}(\tau,\cdot)\|_{H^1(0,L)} \le K\sqrt{\tau}\ ?$$

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$\lambda \in C_0^1(0,1;\mathbb{R}_+)$ with $\|\lambda\|_{L^1(0,1/2)} > L$ and $\lambda(t) = \lambda(1-t)$. We set $\lambda^\tau(t) = \frac{1}{\tau}\lambda(\frac{t}{\tau})$.

# *Sketch of the proof. II*

Indeed, it is possible to prove that (using Faedo-Galerkin method and energy estimates) that there exists a unique couple $(r^{\alpha,\tau}, q^{\alpha,\tau})$, in appropriated functional spaces, solution to the **viscous remainder system:**

$$\begin{cases} r_t^{\alpha,\tau} + (q^{\alpha,\tau} + w^{\alpha,\tau} + \lambda^\tau) r_x^{\alpha,\tau} - r_{xx}^{\alpha,\tau} + q^{\alpha,\tau} y_x^{\alpha,\tau} - y_{xx}^{\alpha,\tau} = 0 & \text{in} \quad (0,\tau) \times (0,L), \\ q^{\alpha,\tau} - \alpha^2 q_{xx}^{\alpha,\tau} = r^{\alpha,\tau} & \text{in} \quad (0,\tau) \times (0,L), \\ r^{\alpha,\tau}(\cdot,0) = 0, \quad r_x^{\alpha,\tau}(\cdot,L) = 0, & \text{on} \quad (0,\tau), \\ q^{\alpha,\tau}(\cdot,0) = 0, \quad q^{\alpha,\tau}(\cdot,L) = r^{\alpha,\tau}(\cdot,L), & \text{on} \quad (0,\tau), \\ r^{\alpha,\tau}(0,\cdot) = 0 & \text{in} \quad (0,L). \end{cases}$$

Moreover, there exists $K > 0$ (independent of $\alpha$ and $\tau$) such that

$$\|r^{\alpha,\tau}\|_{C^0([0,\tau];H^1(0,L))} \leq K\sqrt{\tau}.$$

The controls:
$p^\alpha := \lambda^\tau$, $v_l := y^{\alpha,\tau}(\cdot,0) + \lambda^\tau$ and $v_r := y^{\alpha,\tau}(\cdot,L) + \lambda^\tau + r^{\alpha,\tau}(\cdot,L)$.

# *Uniform local exact controllability to the trajectories*

*Theorem[Uniform local exact controllability to the $C^1$ trajectories]*

Let $T, L, \alpha > 0$ and $\widehat{m} \in C^1([0,T])$ be given. There exists $\delta > 0$ (independent of $\alpha$) such that, for any initial data $u_0 \in H^1(0,L)$ satisfying $\|u_0 - \widehat{m}(0)\|_{H^1} \leq \delta$ there exist $p^\alpha \in C^0([0,T])$ and $(v_l^\alpha, v_r^\alpha) \in H^{3/4}(0,T;\mathbb{R}^2)$ and associated states $(u^\alpha, z^\alpha) \in L^2(0,T;H^2(0,L;\mathbb{R}^2)) \cap H^1(0,T;L^2(0,L;\mathbb{R}^2))$ satisfying $(B_1^\alpha)$ and

$$u^\alpha(T,\cdot) \equiv \widehat{m}(T).$$

Moreover, $p^\alpha = \widehat{m}'$ and the following estimates hold:

$$\|p^\alpha\|_{C^0([0,T])} + \|(v_l^\alpha, v_r^\alpha)\|_{H^{3/4}([0,T];\mathbb{R}^2)} \leq C \quad \forall \alpha > 0,$$

where $C > 0$ is a positive constant independent of $\alpha$.

# *Uniform local exact controllability to the trajectories*

We write $(u^\alpha, z^\alpha) = (y^\alpha + \widehat{m}, w^\alpha + \widehat{m})$ and $p^\alpha = \widehat{m}'$. Then, $(y^\alpha, w^\alpha)$ must satisfy

$$\begin{cases} y_t^\alpha - y_{xx}^\alpha + (w^\alpha + \widehat{m})y_x^\alpha = 0 & \text{in} \quad (0,T)\times(0,L), \\ w^\alpha - \alpha^2 w_{xx}^\alpha = y^\alpha & \text{in} \quad (0,T)\times(0,L), \\ y^\alpha(\cdot,0) = w^\alpha(\cdot,0) = h_l^\alpha & \text{in} \quad (0,T), \\ y^\alpha(\cdot,0) = w^\alpha(\cdot,L) = h_r^\alpha & \text{in} \quad (0,T), \\ y^\alpha(0,\cdot) = y_0 & \text{in} \quad (0,L), \end{cases}$$

where $y_0 := u_0 - \widehat{m}(0)$ and $(h_l^\alpha, h_r^\alpha) := (v_l^\alpha - \widehat{m}, v_r^\alpha - \widehat{m})$. Therefore, the result in the previous slide is equivalent to the **local null-controllability to the above system.**

References:


- F.D. Araruna, E. Fernández-Cara and DS, On the control of the Burgers-$\alpha$ model, *Advances in Differential Equations* 18(2013), 935-954.
- E. Fernández-Cara and DS, Remarks on the control of a family of b-equations, *Trends in Control Theory and Partial Differential Equations* 32 (2019), 123-138.

# *Proof (Global exact controllability to constant trajectories)*

- **First step:** Let $u_0 \in L^\infty(0,L)$ and $(u_1^\alpha, z_1^\alpha)$ the solution to the uncontrolled *viscous Burgers-$\alpha$ system*. Then, for $\tau > 0$ small enough, $u_1^\alpha(T/2-\tau,\cdot) \in C^2([0,L])$.

- **Second step:** Let us define $u_{2,0}^\alpha := u_1^\alpha(T/2-\tau,\cdot)$. Using the **approximate controllability result** for $u_{2,0}^\alpha$ and $N$ we get controls $(p^\alpha, v_{l,2}^\alpha, v_{r,2}^\alpha)$ and states $(u_2^\alpha, z_2^\alpha)$ such that, $u_2^\alpha(0,\cdot) = u_{2,0}^\alpha$ and
$$\|u_2^\alpha(\tau,\cdot) - N\|_{H^1(0,L)} \leq K\sqrt{\tau}.$$

- **Third step:** Let us define $u_{3,0}^\alpha := u_2^\alpha(\tau,\cdot)$. For $\tau > 0$ small enough, $\|u_{3,0}^\alpha - N\|_{H^1(0,L)} \leq K\sqrt{\tau} \leq \delta$. Using the **local controllability result to trajectories** for $u_{3,0}^\alpha$ and $N$ we get controls $(v_{l,3}^\alpha, v_{r,3}^\alpha)$ and states $(u_3^\alpha, z_3^\alpha)$ such that, $u_3^\alpha(0,\cdot) = u_{3,0}^\alpha$ and
$$u_3^\alpha(T/2,\cdot) = N.$$

# *Final comments*

- Other similar $\alpha$-models: (viscous Camassa-Holm, Degasperis-Procesi, etc.). Local NC? YES!
- Global control results? open question.
- Reducing the number of controls?
- Extension to high dimensions? Local null control for Leray-$\alpha$ by Araruna, Fernández-Cara and S (2014).
- Global control results? Open question. first step: Euler-$\alpha$ models...

**Thank you very much for your attention!!!**