Optimal design problems in a dynamical context : an overview

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joint works with F. Maestre (Sevilla), P. Pedregal (Ciudad Real) and F. Periago (Cartagena)

[AM, Pedegral, Periago, JDE 06]

Let $\Omega \subset \mathbb{R}^N$, $N = 1, 2$, $a \in L^\infty(\Omega, \mathbb{R}^+)$, $L \in (0,1)$, $T > 0$, $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$

$$(P_\omega^1): \quad \inf_{\mathcal{X}_\omega} I(\mathcal{X}_\omega) = \int_0^T \int_\Omega (|u_t|^2 + |\nabla u|^2) dxdt \tag{1}$$

subject to

$$\begin{cases} u_{tt} - \Delta u + a(\boldsymbol{x}) \mathcal{X}_\omega u_t = 0 & (0,T) \times \Omega, \\ u = 0 & (0,T) \times \partial\Omega, \\ u(0,\cdot) = u^0, \quad u_t(0,\cdot) = u^1 & \{0\} \times \Omega, \\ \mathcal{X}_\omega \in L^\infty(\Omega; \{0,1\}), & \\ \|\mathcal{X}_\omega\|_{L^1(\Omega)} = L \|\mathcal{X}_\Omega\|_{L^1(\Omega)} & \end{cases} \tag{2}$$

$\Longrightarrow$ [Fahroo-Ito, 97], [Freitas, 98], [Hebard-Henrot, 03, 05], [Henrot-Maillot, 05], [AM, AMCS 09]

(P^1_ω) IS A NONLINEAR PROBLEM.

(P^1_ω) IS A PROTOTYPE OF ILL-POSED PROBLEM : INFIMA ARE NOT REACHED IN THE CLASS OF CHARACTERISTIC FUNCTIONS.

MINIMIZING SEQUENCES $\{\mathcal{X}_{\omega_j}\}_{(j>0)}$ FOR I GENERATE FINER AND FINER MICRO-STRUCTURES.

FIND A RELAXATION, (RP^1_ω) OF (P^1_ω) SUCH THAT

$$(RP^1_\omega) \quad \text{is well-posed} \quad \text{and} \quad \min(RP^1_\omega) = \inf(P^1_\omega) \tag{3}$$

AND THEN EXTRACT FROM A MINIMIZER OF THE RELAXED PROBLEM (RP^1_ω) A MINIMIZING SEQUENCE FOR (P^1_ω) ?

$$(RP^1_\omega): \quad inf_{s\in L^\infty(\Omega)} \int_0^T \int_\Omega (u_t^2 + |\nabla u|^2)\, dx\, dt \tag{4}$$

subject to

$$\begin{cases} u_{tt} - \Delta u + a(x)s(x)u_t = 0 & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = u^0, \quad u_t(0,\cdot) = u^1 & \text{in } \Omega, \\ 0 \le s(x) \le 1, \quad \int_\Omega s(x)\,dx \le L\,|\Omega| & \text{in } \Omega. \end{cases} \tag{5}$$

The set of characteristic function $\{\mathcal{X} \in L^\infty(\Omega), \{0,1\}\}$ is simply replaced by its convex envelop for the L^∞ weak-$\star$ topology, i.e. $\{s \in L^\infty(\Omega), [0,1]\}$

Theorem (AM - Pedregal - Periago JDE 06)

Problem (RP^1_ω) is a full relaxation of (P^1_ω) in the sense that

1. *there are optimal solutions for (RP^1_ω);*
2. *the infimum of (P^1_ω) equals the minimum of (RP^1_ω);*
3. *if s is optimal for (RP^1_ω), then optimal sequences of damping subsets ω_j for (P^1_ω) are exactly those for which the Young measure associated with the sequence of their characteristic functions $\mathcal{X}_{\omega_j}$ is precisely*

$$s(x)\delta_1 + (1 - s(x))\delta_0 \tag{6}$$

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Step 1 of the proof (for $N = 1$) : Variational reformulation of (P_ω^1)

- Assuming ω time independent, we have (we note $Div = (\partial_t, \partial_x)$)

$$u_{tt} - \Delta u + a(x)\mathcal{X}_\omega u_t = 0 \Longleftrightarrow Div(u_t + a(x)\mathcal{X}_\omega u, -u_x) = 0 \tag{7}$$

$\Longrightarrow \exists v \in H^1((0,T)\times\Omega)$ such that $u_t + a(x)\mathcal{X}_\omega u = v_x$ and $-u_x = -v_t$

- $$A\nabla u + B\nabla v = -a\mathcal{X}_\omega \overline{u} \tag{8}$$

where $\nabla u = \begin{pmatrix} u_t \\ u_x \end{pmatrix}$, $\nabla v = \begin{pmatrix} v_t \\ v_x \end{pmatrix}$, $\overline{u} = \begin{pmatrix} u \\ 0 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

- $$\omega = \{x \in \Omega, A\nabla u + B\nabla v = -a(x)\overline{u}\} \quad \text{and} \quad \Omega\backslash\omega = \{x \in \Omega, A\nabla u + B\nabla v = 0\} \tag{9}$$

- Let the vector field $U(t,x) = (u(t,x), v(t,x)) \in (H^1((0,T)\times(0,1)))^2$ and the two sets of matrices

$$\begin{cases} \Lambda_0 = \left\{M \in \mathcal{M}^{2\times 2} : AM^{(1)} + BM^{(2)} = 0\right\} \\ \Lambda_{1,\lambda} = \left\{M \in \mathcal{M}^{2\times 2} : AM^{(1)} + BM^{(2)} = \lambda e_1\right\} \end{cases} \tag{10}$$

where $M^{(i)}$, $i = 1, 2$ stands for the i-th row of the matrix M, $\lambda \in \mathbb{R}$ and $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

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$$W(x,U,M) = \begin{cases} \left|M^{(1)}\right|^2, & M \in \Lambda_0 \cup \Lambda_{1,-a(x)U^{(1)}} \\ +\infty, & \text{else} \end{cases} \quad V(x,U,M) = \begin{cases} 1, & M \in \Lambda_{1,-a(x)U^{(1)}} \\ 0, & M \in \Lambda_0 \setminus \Lambda_{1,-a(x)U^{(1)}} \\ +\infty, & \text{else} \end{cases} \tag{12}$$

- the optimization problem (P^1_ω) is equivalent to the following vector variational problem

$$\left(VP^1_\omega\right) \quad m \equiv \inf_U \int_0^T \int_0^1 W(x, U(t,x), \nabla U(t,x))\, dx\, dt \tag{13}$$

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- This procedure transforms the scalar optimization problem $\left(P^1_\omega\right)$, with differentiable, integrable and pointwise constraints, into a non-convex, vector variational problem $\left(VP^1_\omega\right)$ with only pointwise and integral constraints.

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$$J_\lambda(s) = J(s) + \lambda \,||\, s \,||_{L^1(\Omega)} \quad (\lambda \text{ Lagrange multiplier}) \tag{15}$$

Theorem

If $(u_0, u_1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$, then the derivative of J_λ with respect to s in any direction s_1 exists and takes the following expression

$$\frac{\partial J_\lambda(s)}{\partial s} \cdot s_1 = \int_\Omega s_1(x) \Big(\int_0^T a(x) u_t(t,x) p(t,x)\, dt + \lambda \Big) dx \tag{16}$$

where u is the solution of (45) and p is the solution in $C^1([0,T]; H^1_0(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ of the adjoint problem

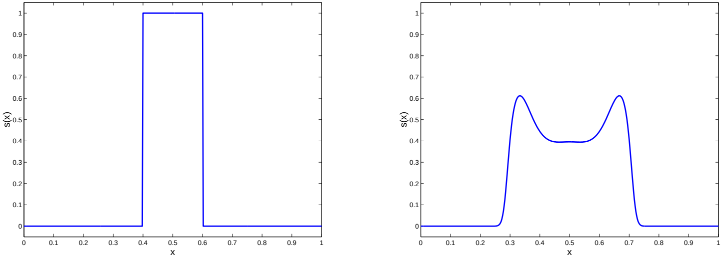
$$\begin{cases} p_{tt} - \Delta p - a(x)s(x)p_t = u_{tt} + \Delta u, & \text{in} \quad (0,T) \times \Omega, \\ p = 0, & \text{on} \quad (0,T) \times \partial\Omega, \\ p(T,\cdot) = 0, \quad p_t(T,\cdot) = u_t(T,\cdot) & \text{in} \quad \Omega. \end{cases} \tag{17}$$

■

$\Longrightarrow$ DO NOT USE HERE LEVEL SET OR TOPOLOGICAL ARGUMENT HERE

Some numerical results for (RP^1_ω)

$$\Omega = (0,1), \quad (u^0(x), u^1(x)) = (\sin(\pi x), 0), \quad L = 1/5, \quad T = 1 \tag{18}$$



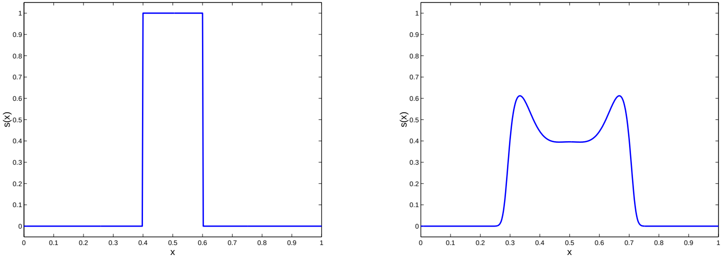
Optimal density for $a(x) = 1$ (Left) and $a(x) = 10$ (Right)

- If $a \leq a^\star(\Omega, L, u^0, u^1)$, $\{x \in \Omega, 0 < s(x) < 1\} = \emptyset$, $(P^1_\omega) = (RP^1_\omega)$ and is well-posed

  (Fourier Analysis and "topological" argument with respect to the amplitude of a)
- If $a > a^\star(\Omega, L, u^0, u^1)$, $\{x \in \Omega, 0 < s(x) < 1\} \neq \emptyset$, $(P^1_\omega) \neq (RP^1_\omega)$ and is NOT well-posed

$\Longrightarrow$ This property is very likely related to the over-damping phenomena)

$$\Omega = (0,1), \quad (u^0(x), u^1(x)) = (\sin(\pi x), 0), \quad L = 1/5, \quad T = 1 \tag{18}$$



Optimal density for $a(x) = 1$ (Left) and $a(x) = 10$ (Right)

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  (FOURIER ANALYSIS AND "TOPOLOGICAL" ARGUMENT WITH RESPECT TO THE AMPLITUDE OF a)

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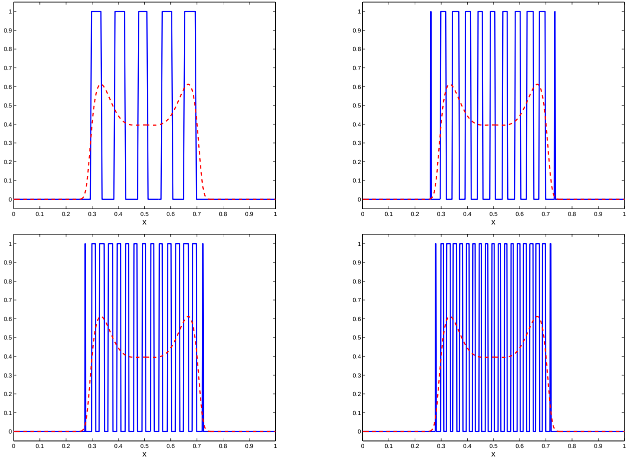
Some numerical results for (RP^1_ω)

$$\Omega = (0,1), \quad (u^0(x), u^1(x)) = (\sin(\pi x), 0), \quad L = 1/5, \quad T = 1 \tag{19}$$



$\sharp\omega_j$	10	20	30	40
$I(\mathcal{X}_{\omega_j})$	4.1331	3.7216	3.5413	3.4313

$$lim_{\sharp\omega_j \to \infty} I(\mathcal{X}_{\omega_j}) = I(s_{opt}) = 3.4212$$

A numerical illustration in 2-D: $\Omega = (0, 1)^2$

$$\Omega = (0,1)^2, \quad (y^0, y1) = (\sin(\pi x_1)\sin(\pi x_2), 0)$$

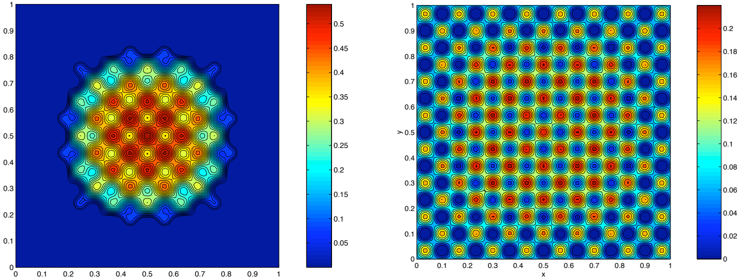


Figure: Iso-values of the optimal s in Ω for $a(\boldsymbol{x}) = 25\mathcal{X}_\Omega(\boldsymbol{x})$ (**Left**) and $a(\boldsymbol{x}) = 50\mathcal{X}_\Omega(\boldsymbol{x})$ (**Right**) - $T = 1$

Some numerical results for (RP^1_ω) in 2D

$$\Omega = (0,1)^2, \quad (y^0, y1) = (\sin(\pi x_1)\sin(\pi x_2), 0)$$

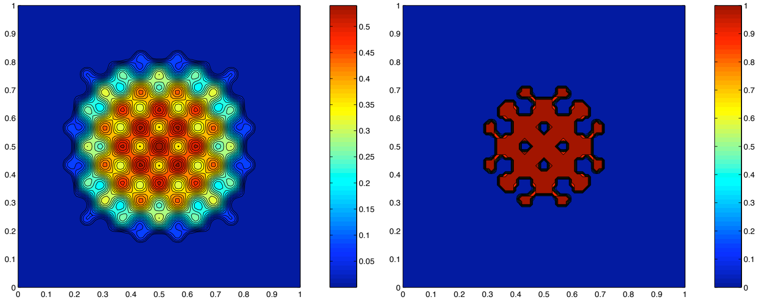


Figure: $T = 1$ - $a(x) = 25\mathcal{X}_\Omega(x)$ - density function $s_{lim} \in L^\infty(\Omega; [0,1])$ (left) and penalized density function $s_{pen} \in L^\infty(\Omega; \{0,1\})$ (right) - $J(s_{lim}) \approx 0.8881$ and $J(s_{pen}) \approx 0.9411$

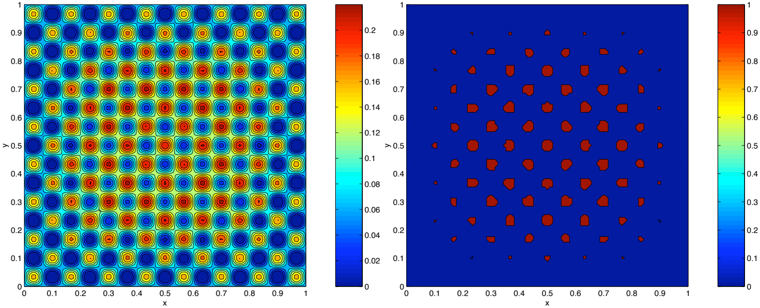


Figure: $T = 1$ - $a(x) = 50\mathcal{X}_\Omega(x)$ - density function $s_{lim} \in L^\infty(\Omega; [0, 1])$ (left) and penalized density function $s_{pen} \in L^\infty(\Omega; \{0, 1\})$ (right) - $J(s_{lim}) \approx 0.7839$ and $J(s_{pen}) \approx 0.8543$

[Maestre-AM-Pedegral, IFB 08][1]

- Let $\Omega \subset \mathbb{R},\ 0 < \alpha < \beta < \infty,\ L \in (0,1),\ T > 0,\ (u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

$$(P_\omega^2): \quad \inf_{\mathcal{X}_\omega} I(\mathcal{X}_\omega) = \int_0^T \int_\Omega (|u_t|^2 + a(t, \boldsymbol{x}, \mathcal{X}_\omega)|\nabla u|^2) dxdt \tag{20}$$

with for instance

$$a(t, \boldsymbol{x}, \mathcal{X}_\omega) = 1 \quad \text{(quadratic)} \quad \text{or} \quad a(t, \boldsymbol{x}, \mathcal{X}_\omega) = \alpha \mathcal{X}_\omega + \beta(1 - \mathcal{X}_\omega) \quad \text{(compliance)} \tag{21}$$

subject to

$$\begin{cases} u_{tt} - div\Big([\alpha \mathcal{X}_\omega + \beta(1 - \mathcal{X}_\omega)]\nabla u\Big) = 0 & (0,T) \times \Omega, \\ u = 0 & (0,T) \times \partial\Omega, \\ u(0,\cdot) = u^0, \quad u_t(0,\cdot) = u^1 & \Omega, \\ \mathcal{X}_\omega \in L^\infty((0,T) \times \Omega; \{0,1\}), & \\ \|\mathcal{X}_\omega\|_{L^1(\Omega)} = L \|\mathcal{X}_\Omega\|_{L^1(\Omega)} & (0,T) \end{cases} \tag{22}$$

- hyperbolic version of a well-known elliptic case considered by Kohn (1985), Tartar-Murat (1990), Pedregal (2003), Bellido (2004), ...
- ω depends on $\boldsymbol{x}$ AND on t: *Dynamical material* [K. Lurie 99, 00, 02].

[1] F. Maestre, AM, P. Pedregal, *Optimal design under the one-dimensional wave equation*, Interfaces and Free Boundaries (2008)

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- $h(t,x) = \beta a_\alpha(t,x) - \alpha a_\beta(t,x), \qquad a(t,x,\mathcal{X}) = \mathcal{X}(t,x)a_\alpha(t,x) + (1-\mathcal{X}(t,x))a_\beta(t,x)$
-

$$(RP^2_\omega): \qquad \min_{U,s} \int_0^T \int_\Omega CQW(t,\boldsymbol{x},\nabla U(t,\boldsymbol{x}), s(t,\boldsymbol{x}))dxdt$$

$$\begin{cases} U=(u,v)\in H^1([0,T]\times\Omega)^2, \ \ tr(\nabla U(t,\boldsymbol{x}))=0,\\ U^{(1)}(0,\boldsymbol{x})=u_0(\boldsymbol{x}), \ U^{(1)}_t(0,\boldsymbol{x})=u_1(\boldsymbol{x}) \quad \text{in} \quad \Omega,\\ U^{(1)}(t,1)=U^{(1)}(t,0)=0 \quad \text{in} \quad [0,T],\\ 0\le s(t,\boldsymbol{x})\le 1, \int_\Omega s(t,\boldsymbol{x})\,dx = L|\Omega| \ \ \forall t\in[0,T], \end{cases}$$

- $CQW(t,\boldsymbol{x},F,s)$ is defined by

$$\begin{cases} \frac{h}{s\beta(\beta-\alpha)^2}(\beta^2|F_{12}|^2+|F_{21}|^2+2\beta F_{12}F_{21})+|F_{11}|^2-\frac{a_\beta}{\beta}F_{12}F_{21} & \text{if } \boldsymbol{h(t,x)\ge 0, \psi(F,s)\le 0}\\ \frac{-h}{(1-s)\alpha(\beta-\alpha)^2}(\alpha^2|F_{12}|^2+|F_{21}|^2+2\alpha F_{12}F_{21})+|F_{11}|^2-\frac{a_\alpha}{\alpha}F_{12}F_{21}, & \text{if } \boldsymbol{h(t,x)\le 0, \psi(F,s)\le 0}\\ -detF+\frac{1}{s(1-s)(\beta-\alpha)^2}\Big(((1-s)\beta^2(\alpha+a_\alpha)+s\alpha^2(\beta+a_\beta))|F_{12}|^2 & \\ \qquad +((1-s)(\alpha+a_\alpha)+s(\beta+a_\beta))|F_{21}|^2+2((\alpha+a_\alpha)\beta-sh)F_{12}F_{21}\Big) & \text{if } \boldsymbol{\psi(F,s)\ge 0}.\\ +\infty & \text{if } \boldsymbol{Tr(F)\neq 0} \end{cases}$$

$$\psi(F,s)=\frac{(\alpha(1-s)+\beta s)}{(\beta-\alpha)}\Big(F_{21}+\lambda^-_{\alpha,\beta}(s)F_{12}\Big)\Big(F_{21}+\lambda^+_{\alpha,\beta}(s)F_{12}\Big)$$

- $h(t,x) = \beta a_\alpha(t,x) - \alpha a_\beta(t,x), \qquad a(t,x,\mathcal{X}) = \mathcal{X}(t,x)a_\alpha(t,x) + (1-\mathcal{X}(t,x))a_\beta(t,x)$
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The relaxed formulation of

$$(P^2_\omega): \quad \inf_{\mathcal{X}_\omega} I(\mathcal{X}_\omega) = \int_0^T \int_\Omega (|u_t|^2 + [\alpha \mathcal{X}_\omega + \beta(1 - \mathcal{X}_\omega)] |\nabla u|^2) dxdt \tag{23}$$

subject to

$$\begin{cases} u_{tt} - div\Big([\alpha \mathcal{X}_\omega + \beta(1 - \mathcal{X}_\omega)] \nabla u \Big) = 0 & (0, T) \times \Omega, \\ u = 0 & (0, T) \times \partial\Omega, \\ u(0, \cdot) = u^0, \quad u_t(0, \cdot) = u^1 & \Omega, \\ \mathcal{X}_\omega \in L^\infty((0, T) \times \Omega; \{0, 1\}), & \\ \|\mathcal{X}_\omega\|_{L^1(\Omega)} = L \|\mathcal{X}_\Omega\|_{L^1(\Omega)} & (0, T) \end{cases} \tag{24}$$

(Maestre-AM-Pedregal 08)

$$m = \min_{u,s} \int_0^T \int_\Omega \left(u_t(t, x)^2 + \frac{1}{(\alpha^{-1} s + \beta^{-1}(1 - s))} u_x(t, x)^2 \right) dxdt \tag{25}$$

subject to

$$\begin{cases} u_{tt} - div\left(\frac{1}{\alpha^{-1} s(t,x) + \beta^{-1}(1 - s(t,x))} \nabla u\right) = 0 & in \quad (0, T) \times \Omega, \\ u = 0 & on \quad (0, T) \times \partial\Omega, \\ u(0, x) = u^0(x), \ u_t(0, x) = u^1(x) & in \quad \Omega, \\ 0 \le s(t, x) \le 1, \int_\Omega s(t, x)\, dx \le L|\Omega| & in \quad [0, T] \end{cases} \tag{26}$$

and the optimal measure is recovered with first order laminates with normal $(0, 1)$.

Problem (P^2_ω): Compliance case : $(a_\alpha, a_\beta) = (\alpha, \beta)$

The relaxed formulation of

$$(P^2_\omega): \quad \inf_{\mathcal{X}_\omega} I(\mathcal{X}_\omega) = \int_0^T \int_\Omega (|u_t|^2 + [\alpha \mathcal{X}_\omega + \beta(1 - \mathcal{X}_\omega)]|\nabla u|^2) dxdt \tag{23}$$

subject to

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Theorem (Maestre-AM-Pedregal 08)

$$m = \min_{u,s} \int_0^T \int_\Omega \left(u_t(t,x)^2 + \frac{1}{(\alpha^{-1}s + \beta^{-1}(1-s))} u_x(t,x)^2 \right) dxdt \tag{25}$$

subject to

$$\begin{cases} u_{tt} - div\big(\frac{1}{\alpha^{-1}s(t,x) + \beta^{-1}(1-s(t,x))} \nabla u\big) = 0 & in \quad (0,T) \times \Omega, \\ u = 0 & on \quad (0,T) \times \partial\Omega, \\ u(0,x) = u^0(x), \ u_t(0,x) = u^1(x) & in \quad \Omega, \\ 0 \le s(t,x) \le 1, \ \int_\Omega s(t,x)\, dx \le L|\Omega| & in \quad [0,T] \end{cases} \tag{26}$$

and the optimal measure is recovered with first order laminates with normal $(0,1)$.

Problem (P_ω^2): Quadratic case : $(a_\alpha, a_\beta) = (1,1)$

The relaxed formulation of

$$(P_\omega^2): \quad \inf_{\mathcal{X}_\omega} I(\mathcal{X}_\omega) = \int_0^T \int_\Omega (|u_t|^2 + |\nabla u|^2) dxdt \tag{27}$$

subject to

$$\begin{cases} u_{tt} - div\Big([\alpha \mathcal{X}_\omega + \beta(1 - \mathcal{X}_\omega)] \nabla u \Big) = 0 & (0,T) \times \Omega, \\ u = 0 & (0,T) \times \partial\Omega, \\ u(0,\cdot) = u^0, \quad u_t(0,\cdot) = u^1 & \Omega, \\ \mathcal{X}_\omega \in L^\infty((0,T) \times \Omega; \{0,1\}), & \\ \|\mathcal{X}_\omega\|_{L^1(\Omega)} \le L \|\mathcal{X}_\Omega\|_{L^1(\Omega)} & (0,T) \end{cases} \tag{28}$$

(Maestre-AM-Pedregal 08)

$$m = \min_{u,s} \int_0^T \int_\Omega \Big(u_t(t,x)^2 + \big[\alpha s(t,x) + \beta(1 - s(t,x))\big] u_x(t,x)^2 \Big) dxdt \tag{29}$$

subject to

$$\begin{cases} u_{tt} - div([\alpha s(t,x) + \beta(1 - s(t,x))]\nabla u) = 0 & in \quad (0,T) \times \Omega, \\ u = 0 & on \quad (0,T) \times \partial\Omega, \\ u(0,x) = u^0(x), \; u_t(0,x) = u^1(x) & in \quad \Omega, \\ 0 \le s(t,x) \le 1, \; \int_\Omega s(t,x)\, dx \le L|\Omega| & in \quad [0,T] \end{cases} \tag{30}$$

and the optimal measure is recovered with first order laminates with normal $(1,0)$.

Problem (P^2_ω): Quadratic case : $(a_\alpha, a_\beta) = (1, 1)$

The relaxed formulation of

$$(P^2_\omega): \quad \inf_{\mathcal{X}_\omega} I(\mathcal{X}_\omega) = \int_0^T \int_\Omega (|u_t|^2 + |\nabla u|^2) dxdt \tag{27}$$

subject to

$$\begin{cases} u_{tt} - div\Big([\alpha \mathcal{X}_\omega + \beta(1 - \mathcal{X}_\omega)] \nabla u \Big) = 0 & (0, T) \times \Omega, \\ u = 0 & (0, T) \times \partial\Omega, \\ u(0, \cdot) = u^0, \quad u_t(0, \cdot) = u^1 & \Omega, \\ \mathcal{X}_\omega \in L^\infty((0, T) \times \Omega; \{0, 1\}), & \\ \|\mathcal{X}_\omega\|_{L^1(\Omega)} \leq L \|\mathcal{X}_\Omega\|_{L^1(\Omega)} & (0, T) \end{cases} \tag{28}$$

Theorem (Maestre-AM-Pedregal 08)

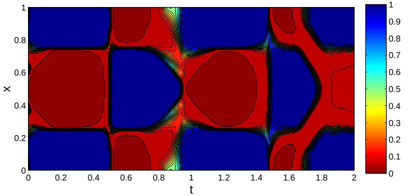
$$m = \min_{u,s} \int_0^T \int_\Omega \left(u_t(t, x)^2 + \Big[\alpha s(t, x) + \beta(1 - s(t, x)) \Big] u_x(t, x)^2 \right) dxdt \tag{29}$$

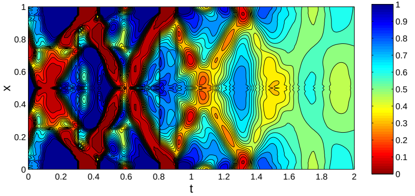
subject to

$$\begin{cases} u_{tt} - div([\alpha s(t, x) + \beta(1 - s(t, x))] \nabla u) = 0 & \textit{in} \quad (0, T) \times \Omega, \\ u = 0 & \textit{on} \quad (0, T) \times \partial\Omega, \\ u(0, x) = u^0(x), \;\; u_t(0, x) = u^1(x) & \textit{in} \quad \Omega, \\ 0 \leq s(t, x) \leq 1, \; \int_\Omega s(t, x)\, dx \leq L|\Omega| & \textit{in} \quad [0, T] \end{cases} \tag{30}$$

and the optimal measure is recovered with first order laminates with normal $(1, 0)$.

Let $\Omega = (0,1)$, $T = 2$ and $(u^0, u^1) = (\sin(\pi x), 0)$ and $L = 0.5$

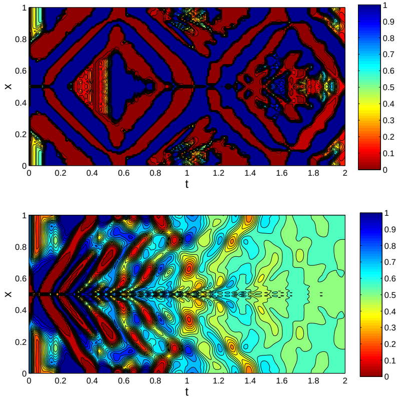




Iso-value of the optimal density s on $(0,T) \times \Omega$ Top: $(\alpha, \beta) = (1, 1.1)$ -Bottom:$(\alpha, \beta) = (1, 10)$

Some numerical results for (RP^2_ω)

Let $\Omega = (0,1)$, $T = 2$ and $(u^0, u^1) = (e^{-0.5(x-0.5)^2}, 0)$ and $L = 0.5$



Iso-value of the optimal density s on $(0,T) \times \Omega$ Top: $(\alpha, \beta) = (1, 1.1)$ -Bottom:$(\alpha, \beta) = (1, 10)$

[Maestre, AM, Pedregal, SIAM Appl. Math. 07][2]

- Simultaneous optimization w.r.t. to $\omega_1 \subset (0,T) \times \Omega$ et $\omega_2 \subset \Omega$

$$(P^3_\omega): \quad \inf_{\mathcal{X}_{\omega_1}, \mathcal{X}_{\omega_2}} I(\mathcal{X}_{\omega_1}, \mathcal{X}_{\omega_2}) = \int_0^T \int_\Omega (|u_t|^2 + a(t, \boldsymbol{x}, \mathcal{X}_{\omega_1})|\nabla u|^2) dxdt \tag{31}$$

subject to

$$\begin{cases} u_{tt} - div\Big([\alpha \mathcal{X}_{\omega_1} + \beta(1 - \mathcal{X}_{\omega_1})]\nabla u\Big) + a(\boldsymbol{x})\mathcal{X}_{\omega_2} u_t = 0 & (0,T) \times \Omega, \\ u = 0 & (0,T) \times \partial\Omega, \\ u(0,\cdot) = u^0, \quad u_t(0,\cdot) = u^1 & \{0\} \times \Omega, \\ \mathcal{X}_{\omega_1} \in L^\infty(\Omega \times (0,T); \{0,1\}), & \\ \mathcal{X}_{\omega_2} \in L^\infty(\Omega; \{0,1\}), & \\ \|\mathcal{X}_{\omega_1}(t,\cdot)\|_{L^1(\Omega)} \le L_{des} \|\mathcal{X}_\Omega\|_{L^1(\Omega)}, & (0,T) \\ \|\mathcal{X}_{\omega_2}\|_{L^1(\Omega)} = L_{dam} \|\mathcal{X}_\Omega\|_{L^1(\Omega)}, & \end{cases} \tag{32}$$

$L_{dam}, L_{des} \in (0,1)$.

$\Longrightarrow$ SMOOTHING EFFECT ON THE (α, β) DISTRIBUTION DUE TO THE DISSIPATION

[2] F. Maestre, AM, P. Pedregal *A spatio-temporal design problem for a damped wave equation*, SIAM Appl. Math (2007)

$$(\mathrm{P}_t) \quad \text{Minimize over } \mathcal{X} : \quad J_t(\mathcal{X}) = \frac{1}{2}\int_0^T\int_\Omega K(t,x)\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt$$

$$\begin{cases} (\beta(t,x)u(t,x))' - \operatorname{div}\,(K(t,x)\nabla u(t,x)) = f(t,x) & \text{in} \quad (0,T)\times\Omega, \\ u = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{33}$$

with

$$\beta(t,x) = \mathcal{X}(t,x)\beta_1 + (1-\mathcal{X}(t,x))\beta_2, \; K(t,x) = \mathcal{X}(t,x)k_1 I_N + (1-\mathcal{X}(t,x))k_2 I_N,$$

(AM, Pedregal, Periago, JMPA 2008)

$$(RP_t) \quad \text{Minimize over } \left(\theta, \overline{G}, u\right) : \quad \overline{J}_t(\theta,\overline{G},u) = \frac{1}{2}\int_0^T\int_\Omega \left[k_1\frac{\left|\overline{G}-k_2\nabla u\right|^2}{\theta(k_1-k_2)^2} + k_2\frac{\left|\overline{G}-k_1\nabla u\right|^2}{(1-\theta)(k_2-k_1)^2}\right]dxdt$$

$$\begin{cases} G\in L^2\left((0,T)\times\Omega;\mathbb{R}^{N+1}\right), & u\in H^1((0,T)\times\Omega;\mathbb{R}), \\ ((\theta\beta_1+(1-\theta)\beta_2)u)' - div\,\overline{G} = 0 & dans\ H^{-1}((0,T)\times\Omega), \\ u|_{\partial\Omega} = 0 \quad p.p.\ t\in[0,T], & u(0) = u_0 \quad dans\ \Omega, \\ \theta\in L^\infty((0,T)\times\Omega;[0,1]), \quad \int_\Omega\theta(t,x)\,dx = L|\Omega| \quad p.p.\ t\in(0,T). \end{cases}$$

is a relaxation of (P_t) in the following sense :

Optimization of the heat flux: Div-Rot Young Measure

$$(\mathrm{P}_t) \quad \text{Minimize over } \mathcal{X}: \quad J_t(\mathcal{X}) = \frac{1}{2}\int_0^T\int_\Omega K(t,x)\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt$$

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$$\Omega = (0,1)^2 \quad u^0(\boldsymbol{x}) = (\sin(\pi x_1)\sin(\pi x_2), 0) \quad T = 0.5 \quad L = 1/2. \tag{34}$$

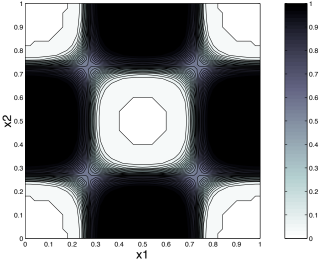

Figure: Resolution of (RP_t) - $(\beta_1, \beta_2) = (10, 10.2)$, $(k_1, k_2) = (0.1, 0.102)$ - Iso-values of θ - $\overline{J}(\theta, \overline{G}, u) \approx \mathbf{0.1126}$.

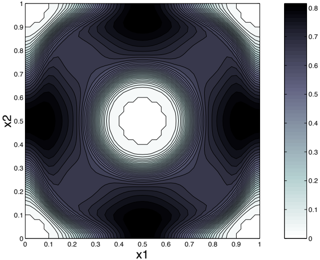

Figure: Resolution of (RP_t) - $(\beta_1, \beta_2) = (10, 20)$, $(k_1, k_2) = (0.1, 1)$ - Iso-values of θ - $\overline{J}(\theta, \overline{G}, u) \approx \mathbf{0.1806}$.

Asymptotic in time ($T \to \infty$) of the optimal density for the heat

[Allaire, AM, Periago 09], [Trelat Zuazua 14], [Porreta Zuazua 14]

$$\text{Minimize over } \mathcal{X}: \quad J_t(\mathcal{X}) = \frac{1}{T}\int_0^T\int_\Omega K(t,x)\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt$$

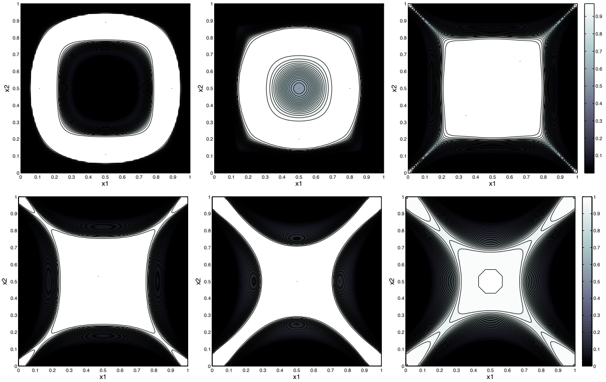

Figure: $f = 1$ - Isovalues of θ for $T = 0.5$, $T = 1.5$ $T = 1$, $T = 2$, $T = 4$ and the limit case $T = \infty$.

Relaxation commutes with limit $T \to \infty$

[AM 06,07,08] [Asch-Lebeau 99], [Chambolle-Santosa 03], [Periago 09], [Privat-Trelat-Zuazua 13, 14]

- Let $\Omega \subset \mathbb{R}^N$, $N = 1, 2$, $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, $L \in (0,1)$, $T > 0$ [3]

$$(P_\omega^4): \quad \inf_{\mathcal{X}_\omega} \|v_\omega\|^2_{L^2(\omega\times(0,T))} \tag{35}$$

where v_ω is an exact control, supported on $\omega \times (0,T)$ for

$$\begin{cases} u_{tt} - \Delta u = v_\omega \mathcal{X}_\omega & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = u^0, \quad u_t(0,\cdot) = u^1 & \text{in } \Omega \end{cases} \tag{36}$$

and subject to

$$\begin{cases} \text{The system (45) may be observed from} \quad \omega \times (0,T), \\ \|\mathcal{X}_\omega\|_{L^1(\Omega)} \le L \|\mathcal{X}_\Omega\|_{L^1(\Omega)} \end{cases} \tag{37}$$

[3] AM, *Optimal design of the support of the control for the 2-D wave equation*, C.R.Acad Sci., Paris Serie I (2006)

$$(RP^4_\omega): \quad \inf_{s\in L^\infty(\Omega)} \frac{1}{2}\int_\Omega s(\boldsymbol{x}) \int_0^T v_s^2(\boldsymbol{x},t)dtdx \tag{38}$$

where v_s (function of the density s) is such that sv_s if the HUM control associated to the unique solution of

$$\begin{cases} y_{tt} - \Delta y = s(x)v_s & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = y^0, \quad y_t(0,\cdot) = y^1 & \text{in } \Omega, \\ 0 \le s(x) \le 1, \quad \int_\Omega s(x)\,dx = L\,|\Omega| & \text{in } \Omega. \end{cases} \tag{39}$$

$\Longrightarrow$ The set $\{\mathcal{X}_\omega \in L^\infty(\Omega,\{0,1\})\}$ is replaced by it convex envelop $\{s \in L^\infty(\Omega,[0,1])\}$ for the weak-$\star$ topology.

Theorem (Periago 09)

Problem (RP^4_ω) is a full relaxation of (P^4_ω) in the sense that

- *there are optimal solutions for (RP^4_ω);*
- *the infimum of (P^4_ω) equals the minimum of (RP^4_ω);*

$\Longrightarrow$ THE PROOF REQUIRES A UNIFORM OBSERVABILITY CONSTANT WITH RESPECT TO ω.

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Well-Posed Relaxation of (P^4_ω)

$$(RP^4_\omega): \quad \inf_{s \in L^\infty(\Omega)} \frac{1}{2}\int_\Omega s(\boldsymbol{x}) \int_0^T v_s^2(\boldsymbol{x}, t)\,dtdx \tag{38}$$

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$$\begin{cases} y_{tt} - \Delta y = s(x)v_s & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = y^0, \quad y_t(0,\cdot) = y^1 & \text{in } \Omega, \\ 0 \le s(x) \le 1, \quad \int_\Omega s(x)\,dx = L\,|\Omega| & \text{in } \Omega. \end{cases} \tag{39}$$

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$\Longrightarrow$ THE PROOF REQUIRES A UNIFORM OBSERVABILITY CONSTANT WITH RESPECT TO ω.

$$(\mathcal{CP}_\omega): \min_{s\in L^\infty(\Omega;[0,1])} \overline{J}_\lambda(s) \quad \text{avec} \quad \overline{J}_\lambda(s) = \frac{1}{2}\int_\Omega s(\boldsymbol{x})\int_0^T v_s^2(t,\boldsymbol{x})dtdx + \lambda\int_\Omega s(\boldsymbol{x})dx \tag{40}$$

où v_s (fonction de la densité s) est telle que sv_s est le contrôle de norme L^2-minimale associée à

$$\begin{cases} y_{tt} - \Delta y = s(\boldsymbol{x})v_s, & (0,T)\times\Omega, \\ y = 0, & (0,T)\times\partial\Omega, \\ (y(0,\cdot), y_t(0,\cdot)) = (y^0,y^1), & \Omega. \end{cases} \tag{41}$$

Theorem

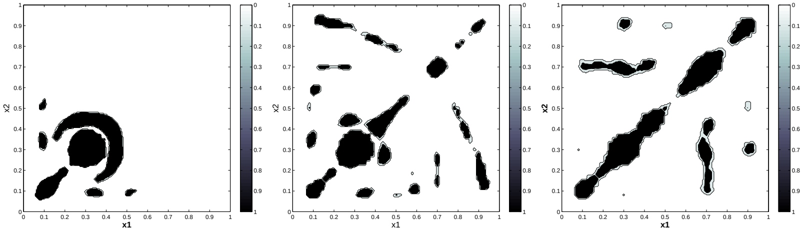
The derivative of J_λ with respect to s is given by the following expression :

$$\frac{\partial J_\lambda(s)}{\partial s}\cdot\overline{s} = \int_\Omega \left(-\frac{1}{2}\int_0^T v_s^2(x,t)dt + \lambda\right)\overline{s}\,dx \tag{42}$$

where v_s is the HUM control (of minimal L^2-norm) with support on s which drives to the rest at time $t = T$ the solution u of the wave eq. ■

$\Longrightarrow$ THE DERIVATIVE IS INDEPENDENT OF ANY ADJOINT PROBLEM !

Let $\Omega = (0,1)^2$, and $(u^0, u^1) = (e^{-80(x_1-0.3)^2-80(x_2-0.3)^2}, 0)$ and $L = 1/10$



Iso-value of the optimal density s on Ω for $T = 0.5$, $T = 1$, $T = 3$

$\Longrightarrow \{x \in \Omega, 0 < s(x) < 1\} = \emptyset$, $(P^4_\omega) = (RP^4_\omega)$ AND IS WELL-POSED.

Let $\Omega = (0,1)^2$, and $(y^0, y^1) = (e^{-80(x-0.3)^2}, 0)$ and $L = 1/10$

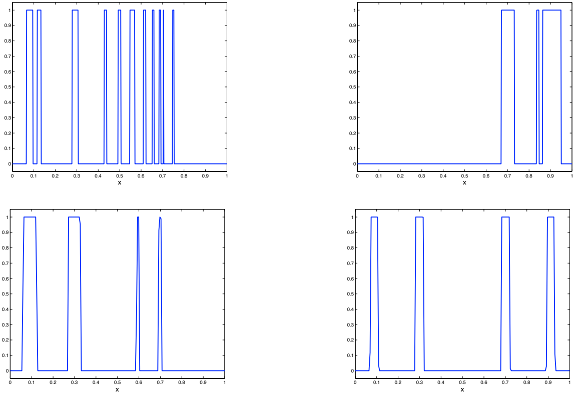


Figure: Limit density function s^{lim} for $T = 0.5$ (top left), $T = 1.5$ (top right), $T = 2.5$ (bottom left) and $T = 3$ (bottom right) initialized with $s^0 = L = 0.15$ on $\Omega = (0,1)$

Remark

T AND $|\omega|$ MAY BE ARBITRARILY SMALL !!!

$(u^0, u^1) = (e^{-80(x_1-0.3)^2-80(x_2-0.3)^2}, 0)$

$$\int_\Omega s(x,t)\,dx = L|\Omega|, \forall t \in (0,T) \quad (43)$$

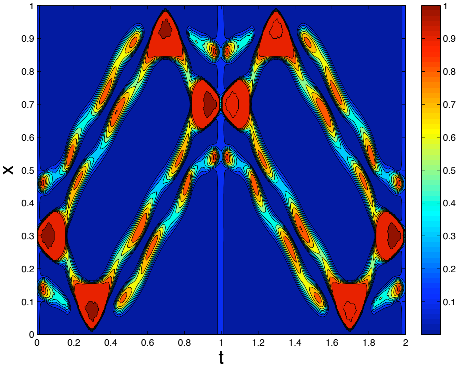
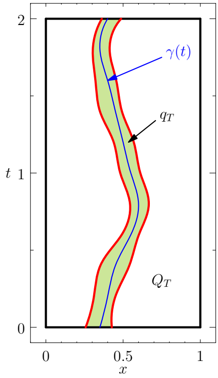


Figure: Optimal time dependent density along $(0,1) \times (0,T)$

$\Longrightarrow$ CONJECTURE : THE OPTIMAL DISTRIBUTION FOLLOWS THE CHARACTERISTICS GENERATED BY THE INITIAL DATA.

$Q_T = (0,1) \times (0,T)$, $q_T \subset Q_T$, $\boldsymbol{V} := H_0^1(0,1) \times L^2(0,1)$

$$\begin{cases} y_{tt} - y_{xx} = v 1_{q_T}, & (x,t) \in Q_T \\ y = 0, & (x,t) \in \partial\Omega \times (0,T) \\ (y(\cdot,0), y_t(\cdot,0)) = (y_0, y_1) \in \boldsymbol{V}, & x \in (0,1). \end{cases}$$



Dependent domains q_T included in Q_T.

Theorem (Cindea, Castro, Munch 14)

Let $T > 0$. Assume that $q_T \subset (0,1) \times (0,T)$ is a finite union of connected open sets and satisfies the geometric optic condition . Then, the null controllability holds.

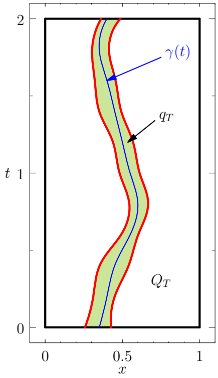


Figure: Dependent domains q_T included in Q_T.

Let $\Omega \subset \mathbb{R}$, $u^0 \in L^2(\Omega)$, $L \in (0,1)$, $T > 0$

$$(P^4_\omega): \quad \inf_{\mathcal{X}_\omega} \|v_\omega\|^2_{L^2(\omega\times(0,T))} \tag{44}$$

where v_ω is an exact control, supported on $\omega \times (0,T)$ for

$$\begin{cases} u_t - div(a(x)u_x) = v_\omega \mathcal{X}_\omega & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = u^0 & \text{in } \Omega \end{cases} \tag{45}$$

and subject to $\|\mathcal{X}_\omega\|_{L^1(\Omega)} \leq L\|\mathcal{X}_\Omega\|_{L^1(\Omega)}$

$\Longrightarrow$ THE RELAXATION IS PERFORMED IN [AM-PERIAGO JDE 11] [4] BASED ON UNIFORM OBSERVABILITY INEQUALITY W.R.T. ω;

$\Longrightarrow$ ONCE AGAIN, $\mathcal{X}_\omega$ IS SIMPLY REPLACED BY A DENSITY $s \in S_L = \{s \in L^\infty(\Omega,[0,1])\}$

[4] Optimal distribution of the internal null control for the one-dimensional heat equation, J. Diff. Equations

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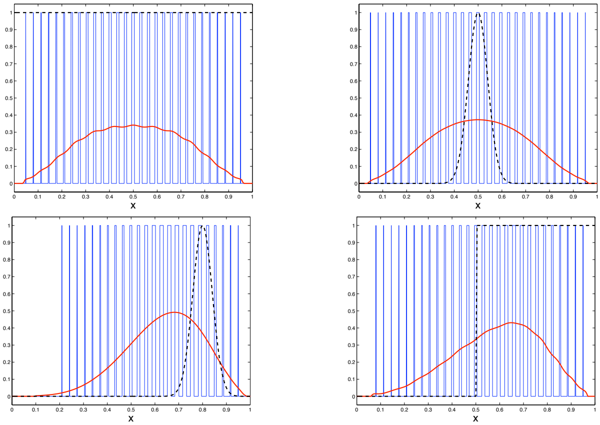

Figure: $a(x) := 1/10$ - $T = 1/2$ - Optimal density θ_{lim} and associated characteristic function $\mathcal{X}_{\omega^{30}}$ for $u^0(x) = 1$ (Top Left), $u^0(x) = e^{-300(x-0.5)^2}$, $u^0(x) = e^{-300(x-0.8)^2}$ and $u^0(x) = \mathcal{X}_{[1/2,1[}(x)$.

$\Longrightarrow$ RELAXATION PHENOMENON FOR THE HEAT CASE !

Minimization of the observ ability constant w.r.t. the control support ω

$$\inf_{\omega\subset\Omega, |\omega|=L|\Omega|} \underbrace{\sup_{\phi_T\in L^2(\omega)} \frac{\|\phi(0,\cdot)\|^2_{L^2(\Omega)}}{\int_\omega\int_0^T \phi^2(x,t)dxdt}}_{C(\omega)}, \tag{46}$$

where ϕ solves the homogeneous backward heat equation with final data ϕ_T.

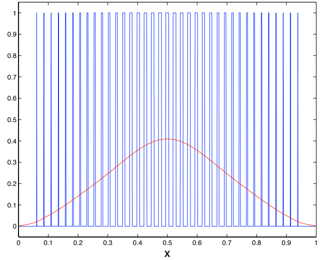

Figure: $|\omega| = 1/5$ - $T = 1$ - $a(x) := 0.1$ - Optimal density and a penallized characteristic function.

We get $C(\omega_{opt}) \approx 1.179 < C(]1/2 - L/2, 1/2 + L/2[) \approx 2.301$

$\Longrightarrow$ UP TO THE BOUNDARY, THE OPTIMAL CONTROL IS UNIFORMLY DISTRIBUTED OVER THE SPATIAL DOMAIN !

[Munch-Periago, System and Control letters 2013]

$$\begin{cases} y_t - div(a(x)\nabla y) + A\,y = v\,1_\omega, & (x,t) \in Q_T \\ y(\sigma,t) = 0, \quad (\sigma,t) \in \Sigma, & y(x,0) = y_0(x), \quad x \in \Omega. \end{cases} \tag{47}$$

$$(P_\alpha) \begin{cases} \text{Minimize } J_\alpha(v) = \|v\|_{L^\infty(q_T)} \\ \text{subject to } \; v \in \mathcal{C}_\alpha(y_0, T) \end{cases}$$

where $\mathcal{C}_\alpha(y_0,T) = \{v \in L^\infty(q_T) : y$ solves (47) and satisfies $\|y(T)\|_{L^2(\Omega)} \leq \alpha\}$.

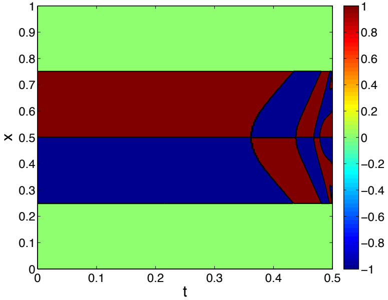

Figure: $y_0(x) = \sin(2\pi x)$ - $A(x) := 1/10$ - $\omega = (0.2, 0.8)$ - Iso-values of the control function $v \in Q_T$.

$\Longrightarrow$ Set $v = [\lambda\mathcal{X}_{\mathcal{O}} + (-\lambda)(1-\mathcal{X}_{\mathcal{O}})]1_{\omega}$

$\Longrightarrow$ Reformulate (P_α) as follows :

$$(T_\alpha)\begin{cases}\text{Minimize } \lambda^2\\ \text{Subject to } \ (\lambda,\mathcal{X}_{\mathcal{O}})\in\mathcal{D}(y_0,T)\end{cases}$$

$\mathcal{D}(y_0,T) = \{(\lambda,\mathcal{X}_{\mathcal{O}})\in\mathbb{R}^+\times L^\infty(Q_T,\{0,1\}) \ y = y(\lambda,\mathcal{X}_{\mathcal{O}})$ solves (48) and $\|y(T)\|_{L^2(\Omega)}\leq\alpha\}$

with

$$\begin{cases} y_t-(a(x)y_x)_x = [\lambda\mathcal{X}_{\mathcal{O}}+(-\lambda)(1-\mathcal{X}_{\mathcal{O}})]1_\omega, & (x,t)\in Q_T\\ y(x,t)=0, & (x,t)\in\{0,1\}\times(0,T)\\ y(x,0)=y_0(x), & x\in(0,1).\end{cases}\tag{48}$$

$\Longrightarrow$ Relaxation of the (time dependent) optimal design problem (T_α) and "capture" of the oscillation near T via time-dependent density [5].

[5] F. Periago, AM, *Numerical approximation of bang-bang controls for the heat equation: an optimal design approach* SCL, 2013

$$
(RNB_\alpha)\begin{cases}
\text{Minimize in } (\lambda, s): & \overline{J}_\alpha(\lambda, 1_{\mathcal{O}}) = \frac{1}{2}\left(\lambda^2 + \frac{1}{\alpha}\|y(T)\|^2_{L^2(\Omega)}\right) \\
\text{subject to} & \\
& y_t - \Delta y + ay = 0 \quad \text{in } Q_T \\
& \partial_\nu y(\sigma, t) = \lambda\left[2s(\sigma, t) - 1\right] 1_{\Sigma_0} \quad \text{on } \Sigma_T \\
& y(0) = y_0 \quad \text{in } \Omega \\
& (\lambda, s) \in \mathbb{R}^+ \times L^\infty(\Sigma_T; [0, 1]).
\end{cases}
$$

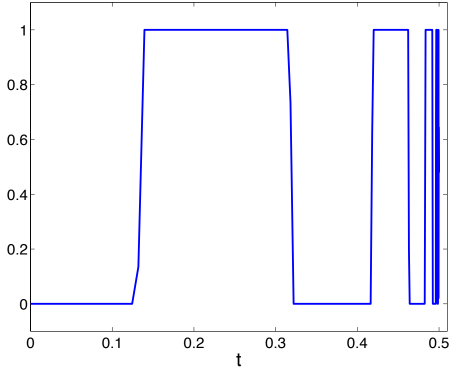

Figure: Neumann case - The optimal density s for $t \in [0, T]$ - $\alpha = 10^{-6}$.

THE END

... AND MANY OTHER PROBLEMS ..... !

THANK YOU FOR YOUR ATTENTION