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Mini-Workshop: Adaptive Methods for Control Problems Constrained by Time-Dependent PDEs

Organised by
Max Gunzburger, Tallahassee
Karl Kunisch, Graz
Angela Kunoth, Köln

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Abstract. Optimization problems constrained by time-dependent PDEs (Partial Differential Equations) are challenging from a computational point of view: even in the simplest case, one needs to solve a system of PDEs coupled *globally* in time and space for the unknown solutions (the *state*, the *costate* and the *control* of the system). Typical and practically relevant examples are the control of nonlinear heat equations as they appear in laser hardening or the thermic control of flow problems (Boussinesq equations). Specifically for PDEs with a *long time horizon*, conventional time-stepping methods require an enormous storage of the respective other variables. In contrast, *adaptive methods* aim at distributing the available degrees of freedom in an a-posteriori-fashion to capture singularities and are, therefore, most promising.

Mathematics Subject Classification (2010): 65xx, 49xx.

Introduction by the Organisers

In January 2004, two of us organized an Oberwolfach–Miniworkshop "Numerical Methods for Instationary Control Problems", see Report 03/2004. Citing from that report, we wrote "The topic for the current Miniworkshop organized by Karl Kunisch (Graz), Angela Kunoth (Bonn) and Rolf Rannacher (Heidelberg) emerged from the Oberwolfach-Workshop "Numerical Techniques for Optimization Problems with PDE Constraints" which was held February 16-22, 2003. It was realized that numerically solving control problems which are constrained by time-dependent nonlinear PDEs are particularly challenging with respect to the complexity of the problem. Mathematically one has to minimize a functional under

PDE constraints and possibly additional constraints on the state and the control. Standard discretizations on uniform grids in space and time will only yield solutions where the inherent structures of the problem (nonlinearity, constraints) are not sufficiently captured. Certain optimization problems for large coupled systems of partial differential equations are currently not yet numerically treatable or do not satisfy the time constraints required in practice. Overcoming this barrier can only be achieved by designing new mathematically founded algorithmic approaches. The road towards this goal leads to many interesting problems in optimization, linear algebra, numerics, analysis, and approximation theory." In that report, we also wrote "Different modern approaches to overcome the complexity issues in numerical simulations for PDE-constrained optimization have been presented and discussed. One of the approaches is to employ fast iterative solvers like multigrid on uniform grids. The methodology which conceptually provides the largest potential is to introduce adaptivity. This drastically reduces complexity but depending on the context may require solving an additional problem. Wavelet approaches particularly allow to resolve each of the variables separately and in addition provide a built-in preconditioning."

By now, robust preconditioners for the fast solution of the resulting linear systems of stationary elliptic PDEs for the (at least) three variables state, costate and control exist in a variety of forms for discretizations on uniform grids. Twelve years later, however, we find that several issues have still not yet been systematically investigated and that some promising approaches have not been exploited to its full potential. Specifically, the issue to reduce the complexity of the problem by introducing adaptivity in space and time is not well understood for finite element or other standard discretizations. Most importantly, the issues of discretizations for systems of PDEs with

1. hierarchical spaces and adaptivity in space and time;
2. error estimation, convergence and complexity estimates on different grids for the different variables state, costate and control;
3. exchange of information from different grids while maintaining accuracy;
4. stability issues for problems with long time horizons;

have not been understood. Although by-now full weak space-time formulations of the constraining PDEs have become increasing popular, they still lack a mathematical foundation and rigorous proofs except in the case when wavelet schemes are employed. Even in the case of a single parabolic PDE, there does not yet exist an adaptive finite-element based scheme for which convergence and optimal complexity (when compared to a reference best N-term approximation) has been proved. We think that some aspects from the wavelet methodology may be helpful, like when proving optimal complexity of adaptive finite element methods for elliptic PDEs ten years ago which introduced new theoretical paradigms. While one can fully exploit with wavelets the functional analytic framework, they are still difficult to construct and to implement which, therefore, leads us to search for concepts which are easier to realize.

Before addressing the issue of adaptivity in time and space, one needs to be aware in which situations this concept applies. This raises the question in which function spaces (particular, Besov spaces) one can expect the different solution variables to be. *Regularity* of solutions of PDEs in Besov or other function spaces which allow for isolated singularities in the data or the domain have become a major topic for the solution of a single elliptic or parabolic PDE in recent years. Simulations for elliptic control problems based on wavelets reveiled that the state may be in a smooth (Sobolev) space while the adjoint state inherits non-regularity from the way the optimization problem is posed. In addition, in the case of PDE-constrained control problems with additional state inquality constraints, the adjoint variable is no longer a function but a Borel measure. However, recent results for establishing L_∞ error estimates for finite element schemes lead to insights that a norm from a space like BMO might be more appropriate.

We have seen in the past years an abundance of manuscripts on "a priori-" and "a posteriori error estimates" for PDE-constrained control problems including varieties of additional control and state constraints which mainly follow the principle of employing one error estimator and one grid for all variables. In addition, another large amount of publications stems from introducing uncertainty quantification into the area and therefore, another level of complexity, i.e., PDE-constrained control problems with stochastic coefficients, see, e.g., Report 04/2013 for the Oberwolfach Workshop "Numerical Methods for PDE Constrained Optimization with Uncertain Data" organized by Matthias Heinkenschloss (Houston) and Volker Schulz (Trier). We explicitly did not want to thematize uncertain data this time.

The time was perfect for organizing a Miniworkshop which focussed on the issue of *adaptive methods* for PDE-constrained control problems with a possibly long time horizon with some leading mathematicians in an atmosphere of a small workshop with not too many participants, to attack in a systematical way the underlying theoretical and resulting practical issues sketched above. The workshop was well attended with experts from various backgrounds in PDE-constrained control problems, regularity of solutions of PDEs, finite elements and wavelet methods.

We invited Martin Gander who spent this week at the institute within a "Research in Pairs" program to also give a talk during our workshop and include his extended abstract as well.

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