Decay for the (nonlinear) KdV equations at critical lengths

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Objective and outline of the talk

- Objective : In this talk, we discuss the longtime behavior of the solution u of the system

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0 & (0,+\infty)\times(0,L), \\ u(t,0) = u(t,L) = u_x(t,L) = 0 & (0,+\infty), \\ u(0,\cdot) = u_0 & (0,L), \end{cases}$$

where u_0 is an initial data with $\|u_0\|_{L^2(0,L)}$ small.

- Outline of the talk :
  - The Korteweg-De Vries (KdV) equation.
  - Control of a KdV system.
  - Controllability vs the decay properties.
  - Decay properties of the KdV system for the critical lengths.
  - Conclusion and perspectives.

Part 1 : The Korteweg-De Vries (KdV) equation.

Korteweg-De Vries (KdV) equation

• The KdV equation

$$u_t(t,x) + u_x(t,x) + u_{xxx}(t,x) + u(t,x)u_x(t,x) = 0 \qquad (1)$$

is a mathematical model of waves on shallow water surfaces.

• The KdV equation also furnishes a very useful nonlinear approximation model including a balance between a weak nonlinearity and weak dispersive effects.

• The KdV equation was first introduced by Boussinesq in 1877 and rediscovered by Korteweg and de Vries in 1895. The history of the KdV equation started with experiments by Russell in 1834.

• The KdV equations has been investigated extensively : Miura, Gardner, Kruskal, Lax, Kato, Saut, Teman, Bourgain, Kenig, Tao

. . .

Part 2 : Control of a KdV system.

Control of the KdV equation

The control of the KdV equation has been attracted the control commnunity. Linearized approaches and non-linear methods have been proposed, see the survey of Rosier & Zhang 09, and Cerpa 14.

In this talk, we first discuss the following control problem

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0 & (0,T) \times (0,L), \\ u(\cdot, L) = u(\cdot, 0) = 0 & (0,T) \\ \partial_x u(\cdot, L) = U & (0,T) \\ u(0, \cdot) = u_0 & (0,L), \end{cases}$$

where u is the state and U is the control. This control problem was initially investigated by Rosier in 97.

To this end, Rosier introduced the following set of critical lengths

$$\mathcal{N} := \left\{ 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}};\ k, l \in \mathbb{N}_* \right\}.$$

and obtained the following remarkable results

- If $L \notin \mathcal{N}$, the linearized control KdV problem is controllable in small time.
- If $L \in \mathcal{N}$, there is a subspace $\mathcal{M}_L$ of $L^2(0, L)$ such that the linearized control problem is not null-controllable in finite time using controls in L^2 for initial data in $\mathcal{M}_L$.

As a consequence, he showed that the control KdV system is locally controllable in small time if $L \notin \mathcal{N}$.

On the linearized KdV system

Fix $T > 0$ and let u be a solution of the linearized KdV system

$$\begin{cases} u_t + u_x + u_{xxx} = 0 & (0,T) \times (0,L), \\ u(\cdot, L) = u(\cdot, 0) = \partial_x u(\cdot, L) = 0 & (0,T). \end{cases} \tag{2}$$

In this talk, we are mainly concentrated on solutions in

$$X_T := C([0,T]; L^2(0,L)) \cap L^2((0,T); H^1(0,L)).$$

Multiplying the equation of u with u and integrating by parts yield

$$\frac{d}{dt} \int_0^L |u(t,x)|^2 \, dx + |u_x(t,0)|^2 = 0.$$

Then

$$\int_0^L |u(t,x)|^2 \, dx + \int_0^t |u_x(s,0)|^2 \, ds = \int_0^L |u(0,x)|^2 \, dx,$$

which implies, in particular,

$$\int_0^T |u_x(t,0)|^2 \, dt \leqslant \int_0^L |u(0,x)|^2 \, dx.$$

On the control of the linearized KdV system

By the HUM, the linearized KdV system is null-controllable in time T with initial datum in $L^2(0, L)$ using the controls in $L^2(0, T)$ iff the following observability inequality

$$\int_0^T |u_x(t, 0)|^2 \, dx \geqslant C_{T,L} \int_0^L |u(0, x)|^2 \, dx \tag{3}$$

holds for all solutions u of the linearized system (2).
One can show that for $L \in \mathcal{N}$, the observability inequality does not hold for any $T > 0$. For example, in the case $L = k2\pi$ $(k = l)$, $u(t, x) = 1 - \cos x$ is a solution of the linearized KdV equation satisfying

$$u(t, 0) = u(t, L) = u_x(t, 0) = u_x(t, L) = 0.$$

On the critical lengths

One can show that if the observability inequality does not hold, then $\exists$ a sequence of solutions (u_n) of the linearized KdV system s.t.

$$n \int_0^T |u_{n,x}(t,0)|^2 \, dx \leqslant \int_0^L |u_n(0,x)|^2 \, dx = 1.$$

One can then prove that $\exists$ a non-zero solution of

$$\begin{cases} u_t + u_x + u_{xxx} = 0 \quad (0,T) \times (0,L), \\ u(\cdot,0) = u(\cdot,L) = \partial_x u(\cdot,0) = \partial_x u(\cdot,L) = 0 \ \ (0,T). \end{cases} \tag{4}$$

via a compactness argument. One can also show that this set of solutions is of finite dimension and derive the existence of a non-zero solution of the system

$$\begin{cases} -ip v + v_x + v_{xxx} = 0 \qquad (0,L), \\ v(0) = v(L) = v_x(0) = v_x(L) = 0. \end{cases} \tag{5}$$

This characterizes all critical lengths (Rosier 97). Set

$$\mathcal{M}_L = \text{span}\Big\{u(0,\cdot);\ u \text{ is a solution of (4)}\Big\}.$$

Control properties for the critical lengths

• $\dim \mathcal{M}_L = 1$: Coron & Crépeau 04, the control KdV system is locally controllable in small time.
• $\dim \mathcal{M}_L \geqslant 2$, Cerpa 07, and Cerpa & Crépeau 09, the control KdV system is locally controllable in finite time.
• $\dim \mathcal{M}_L \geqslant 2$, Jean-Michel Coron & Armand Koenig & H.M. Ng. 20, for a class of critical lengths, the KdV system is NOT locally controllable in small time ($\exists k, l : k - l \notin 3\mathbb{N}$). This is surprising when compared with internal controls : for any $L > 0$, the KdV system is locally controllable in small time when the control is located in any open subset of $(0, L)$; this result is due to Menzala & Vasconcellos & Zuazua 02, Pazoto 05.

∗ The starting point of the analysis is the power expansion method introduced by Jean-Michel Coron and Emmanuelle Crépeau :

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \dots .$$

The idea is to obtain further information using the nonlinear term.

Part 3 : Controllability vs the decay properties.

Controllability vs the decay property

Recall that the linearized control KdV system is controllable iff

$$\int_0^T |u_x(t,0)|^2\,dx \geqslant C_{T,L}\int_0^L |u(0,x)|^2\,dx$$

holds for all solutions u of the linearized KdV system

$$\begin{cases} u_t + u_x + u_{xxx} = 0 & (0,T)\times(0,L), \\ u(\cdot,0) = u(\cdot,L) = \partial_x u(\cdot,L) = 0 & (0,T). \end{cases}$$

Concerning u, one also has

$$\int_0^L |u(T,x)|^2\,dx + \int_0^T |u_x(s,0)|^2\,ds = \int_0^L |u(0,x)|^2\,dx.$$

Thus if the linearized control KdV system is controllable then u decays exponentially, and hence a similar fact holds for the (nonlinear) KdV system locally around 0. This interesting connection was observed by Menzala & Vasconcellos & Zuazua 02.

Part 4 : Decay properties of the KdV system for the critical lengths.

Decay property of the KdV systems for the critical lengths

Motivated by this connection, we next deal with the long time behavior of the solutions of the KdV system, for critical lengths,

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0 & (0, +\infty) \times (0, L), \\ u(t, 0) = u(t, L) = u_x(t, L) = 0 & (0, +\infty), \\ u(0, \cdot) = u_0 & (0, L), \end{cases}$$

where $u_0 \in L^2(0, L)$ is the initial data assumed to be small.

This is the main topic of the talk.

This problem is open for a while. Previous works :

- Chu & Coron & Shang 15 : $\dim \mathcal{M}_L = 1$.
- Tang & Chu & Shang & Coron 18 : $k = l = 2$.

The analysis is based on establishing the existence and smoothness of a center manifold associated with the (non-linear) KdV system.

- The smallness of initial datum is necessary, Doronin & Natali 14.

New results and approach

We first introduce some notations/conventions. Let $L \in \mathcal{N}$. There exists exactly $n_L \in \mathbb{N}_*$ pairs $(k_m, l_m) \in \mathbb{N}_* \times \mathbb{N}_*$ $(1 \leqslant m \leqslant n_L)$ such that $k_m \geqslant l_m$, and

$$L = 2\pi\sqrt{\frac{k_m^2 + k_m l_m + l_m^2}{3}}. \tag{6}$$

For $1 \leqslant m \leqslant n_L$, set

$$p_m = p(k_m, l_m) = \frac{(2k_m + l_m)(k_m - l_m)(2l_m + k_m)}{3\sqrt{3}(k_m^2 + k_m l_m + l_m^2)^{3/2}}. \tag{7}$$

For $1 \leqslant m \leqslant n_L$, set

$$\eta_{1,m} = -\frac{2\pi i(2k_m + l_m)}{3L}, \ \eta_{2,m} = \eta_{1,m} + \frac{2\pi i}{L}k_m, \ \eta_{3,m} = \eta_{2,m} + \frac{2\pi i}{L}l_m.$$

Then, the following useful properties hold :

$$e^{\eta_{1,m}L} = e^{\eta_{2,m}L} = e^{\eta_{3,m}L} \tag{8}$$

and $\eta_{1,m}, \eta_{2,m}, \eta_{3,m}$ are the three solutions of

$$z^3 + z - ip_m = 0. \tag{9}$$

Define

$$\begin{cases} \psi_m(x) = \sum_{j=1}^{3}(\eta_{j+1,m} - \eta_{j,m})e^{\eta_{j+2,m}x} & [0, L], \\ \Psi_m(t,x) = e^{-itp_m}\psi_m(x) & \mathbb{R} \times [0, L], \end{cases}$$

One can check that Ψ_m is a solution of the linearized KdV equation satisfying

$$\Psi_m(t,0) = \Psi_m(t,L) = \Psi_{m,x}(t,L) = 0 = \Psi_{m,x}(t,0).$$

Then, the unreachable space for the linearized KdV system,

$$\mathcal{M}_L = \text{span}\Big\{\{\mathfrak{R}(\psi_m(x)); 1 \leqslant m \leqslant n_L\} \cup \{\mathfrak{I}(\psi_m(x)); 1 \leqslant m \leqslant n_L\}\Big\}.$$

For $1 \leqslant m \leqslant n_L$ with $k_m \neq l_m$, let $\sigma_{j,m}$ $(1 \leqslant j \leqslant 3)$ be the solutions of

$$\sigma^3 - 3(k_m^2 + k_m l_m + l_m^2)\sigma + 2(2k_m + l_m)(2l_m + k_m)(k_m - l_m) = 0,$$

and set, with the convention $\sigma_{j+3,m} = \sigma_{j,m}$ for $j \geqslant 1$,

$$s_m = s(k_m, l_m) := \sum_{j=1}^{3} \sigma_{j,m}(\sigma_{j+2,m} - \sigma_{j+1,m}) \times \left(e^{\frac{4\pi i(k_m - l_m)}{3}} e^{2\pi i \sigma_{j,m}} + e^{-2\pi i \sigma_{j,m}} \right). \quad (10)$$

Here are the new results

Theorem (H.M.Ng. 20)

Let $L \in \mathcal{N}$. *Assume that either* $\dim \mathcal{M}_L = 1$ *or*

$$(k_m \neq l_m \quad \textit{and} \quad s_m \neq 0) \quad \forall 1 \leqslant m \leqslant n_L. \tag{11}$$

Then $\exists \varepsilon_0 = \varepsilon_0(L) > 0$ *s.t.* $\forall u_0$ *(real) with* $\|u_0\|_{L^2(0,L)} \leqslant \varepsilon_0$, *the unique solution* u *of the KdV system satisfies*

$$\lim_{t \to \infty} \|u(t, \cdot)\|_{L^2(0,L)} = 0.$$

More precisely, $\exists C = C(L) > 0$ *s.t., for* $t \geqslant C/\|u_0\|^2_{L^2(0,L)}$ *and* $\|u_0\|_{L^2(0,L)} \leqslant \varepsilon_0$, *it holds*

$$\|u(t, \cdot)\|_{L^2(0,L)} \leqslant \frac{1}{2}\|u(0, \cdot)\|_{L^2(0,L)}.$$

Consequently, for some $c = c(L) > 0$,

$$\|u(t, \cdot)\|_{L^2(0,L)} \leqslant c/t^{1/2} \textit{ for } t \geqslant 0. \tag{12}$$

Comments

• We conclude that $\|u(t,\cdot)\|_{L^2(0,L)} \leqslant c/t^{1/2}$ for $t \geqslant 0$ not the fact that

$$\|u(t,\cdot)\|_{L^2(0,L)} \leqslant c\|u(0,\cdot)\|_{L^2(0,L)}/t^{1/2} \text{ for } t \geqslant 0, \tag{13}$$

which is not true.

• When $k_m \neq l_m$ for all $1 \leqslant m \leqslant n_L$, the condition $s_m \neq 0$ for all $1 \leqslant m \leqslant n_L$ is almost equivalent to the fact that the second order approximation of solutions with initial conditions in $\mathcal{M}_L$ decays. (the first oder approximation conserves the L^2-norm).

• The condition $k_m \neq l_m$ for all $1 \leqslant m \leqslant n_L$ means that $\dim \mathcal{M}_L$ is even.

Condition (11) can be checked numerically easily. We have

Corollary (H.M.Ng 20)

Let $L \in \mathcal{N}$. *Assume that either* $\dim \mathcal{M}_L = 1$ *or* $1 \leqslant k_m, l_m \leqslant 1000$ *for some* $1 \leqslant m \leqslant n_L$. *Then the decay properties hold if* $k_m \neq l_m$ *for all* $1 \leqslant m \leqslant n_L$.

One can show that $s_m \neq 0$ for a class of infinite elements of (k_m, l_m).

The optimality of the decay rate $1/t^{1/2}$ is open ; however, we can prove that

Proposition (H.M.Ng 20)

Let $L \in \mathcal{N}$. *Then* $\exists c > 0$ *s.t.* $\forall \varepsilon > 0$, $\exists u_0 \in L^2(0, L)$ *s.t.*

$$\|u_0\|_{L^2(0,L)} \leqslant \varepsilon \text{ and } \|u(t, \cdot)\|_{L^2(0,L)} \geqslant c \ln(t+2)/t \text{ for some } t > 0,$$

where u *is the solution of the KdV system corresponding to* u_0.

Ideas of the proof - Slide 1

The analysis is inspired from the power expansion method.

The key of the analysis is to (observe and) establish the following fact : Let $L \in \mathcal{N}$. Under the stated conditions (either $\dim \mathcal{M}_L = 1$ or $\Big((k_m \neq l_m \text{ and } s_m \neq 0) \forall 1 \leqslant m \leqslant n_L\Big)$), there exist two constants $T_0 > 0$ and $C > 0$ depending only on L such that for $T \geqslant T_0$, one has, for all u_0 with $\|u_0\|_{L^2(0,L)}$ sufficiently small, the corresponding solution u of the (nonlinear) KdV system satisfies

$$\|u(T,\cdot)\|_{L^2(0,L)} \leqslant \|u_0\|_{L^2(0,L)}\Big(1 - C\|u_0\|^2_{L^2(0,L)}\Big) \text{ for } T \geqslant T_0. \quad (14)$$

Note that, for non-critical lengths, one can prove that, for any $T > 0$,

$$\|u(T,\cdot)\|_{L^2(0,L)} \leqslant \|u_0\|_{L^2(0,L)}\Big(1 - C_{T,L}\|u_0\|^1_{L^2(0,L)}\Big). \quad (15)$$

Ideas of the proof - Slide 2

To get an idea of how to prove (14), let consider the case $u_0 \in \mathcal{M}_L \setminus \{0\}$ with $\|u_0\|_{L^2(0,L)}$ small, which is somehow the worst case. Let $\widetilde{u}_1$ be the unique solution of

$$\begin{cases} \widetilde{u}_{1,t} + \widetilde{u}_{1,x} + \widetilde{u}_{1,xxx} = 0 & (0,+\infty) \times (0,L), \\ \widetilde{u}_1(\cdot,0) = \widetilde{u}_1(\cdot,L) = \widetilde{u}_{1,x}(\cdot,L) = 0 & (0,+\infty), \\ \widetilde{u}_1(0,\cdot) = u_0/\varepsilon & (0,L), \end{cases} \tag{16}$$

with $\varepsilon = \|u_0\|_{L^2(0,L)} > 0$, and let $\widetilde{u}_2$ be the unique solution of

$$\begin{cases} \widetilde{u}_{2,t} + \widetilde{u}_{2,x} + \widetilde{u}_{2,xxx} + \widetilde{u}_{1,x}\widetilde{u}_1 = 0 & (0,+\infty) \times (0,L), \\ \widetilde{u}_2(\cdot,0) = \widetilde{u}_2(\cdot,L) = \widetilde{u}_{2,x}(\cdot,L) = 0 & (0,+\infty), \\ \widetilde{u}_2(0,\cdot) = 0 & (0,L). \end{cases} \tag{17}$$

Then, with $u_\varepsilon = \varepsilon\widetilde{u}_1 + \varepsilon^2\widetilde{u}_2$,

$$u_{\varepsilon,t} + u_{\varepsilon,x} + u_{\varepsilon,xxx} + u_\varepsilon u_{\varepsilon,x} = \varepsilon^3(\widetilde{u}_1\widetilde{u}_2)_x + \varepsilon^4\widetilde{u}_2\widetilde{u}_{2,x}. \tag{18}$$

Ideas of the proof - Slide 3

By considering the system of $\varepsilon\widetilde{u}_1 + \varepsilon^2\widetilde{u}_2 - u$, we can prove that, for $\tau > 0$ (arbitrary),

$$\|(\varepsilon\widetilde{u}_1 + \varepsilon^2\widetilde{u}_2 - u)_x(\cdot, 0)\|_{L^2(0,\tau)} \leqslant c_\tau \varepsilon^3, \qquad (19)$$

for some $c_\tau > 0$ depending only on τ and L, provided that ε is sufficiently small. Since $\widetilde{u}_1(t, \cdot) \in \mathcal{M}_L$ for all $t > 0$, one has

$$\widetilde{u}_{1,x}(t, 0) = 0 \text{ for } t \geqslant 0.$$

Thus, if one can show that, for some $\tau_0 > 0$ and for some $c_0 > 0$

$$\|\widetilde{u}_{2,x}(\cdot, 0)\|_{L^2(0,\tau_0)} \geqslant c_0, \qquad (20)$$

then from (19) one has, for ε small enough,

$$\|u_x(\cdot, 0)\|_{L^2(0,\tau_0)} \geqslant c_0\varepsilon^2.$$

This implies (14) with $T_0 = \tau_0$ since

$$\int_0^L |u(T, x)|^2 \, dx + \int_0^T |u_x(s, 0)|^2 \, ds = \int_0^L |u(0, x)|^2 \, dx.$$

Recall that

$$\begin{cases} \widetilde{u}_{2,t} + \widetilde{u}_{2,x} + \widetilde{u}_{2,xxx} + \widetilde{u}_{1,x}\widetilde{u}_1 = 0 & (0,+\infty)\times(0,L), \\ \widetilde{u}_2(\cdot,0) = \widetilde{u}_2(\cdot,L) = \widetilde{u}_{2,x}(\cdot,L) = 0 & (0,+\infty), \\ \widetilde{u}_2(0,\cdot) = 0 & (0,L). \end{cases}$$

To establish (20), we construct a special solution W of

$$\begin{cases} W_t + W_x + W_{xxx} + \widetilde{u}_{1,x}\widetilde{u}_1 = 0 & (0,+\infty)\times(0,L), \\ W(\cdot,0) = W(\cdot,L) = W_x(\cdot,L) = 0 & (0,+\infty), \end{cases} \tag{21}$$

via a separation-of-variable process (some details are given later !). To this end, we recall that

$$\mathcal{M}_L = \text{span}\Big\{\big\{\Re(\psi_m(x)); 1 \leqslant m \leqslant n_L\big\} \cup \big\{\Im(\psi_m(x)); 1 \leqslant m \leqslant n_L\big\}\Big\}.$$

where

$$\begin{cases} \psi_m(x) = \sum_{j=1}^{3}(\eta_{j+1,m} - \eta_{j,m})e^{\eta_{j+2,m}x} & [0,L], \\ \Psi_m(t,x) = e^{-itp_m}\psi_m(x) & \mathbb{R}\times[0,L], \end{cases}$$

Ideas of the proof - Slide 5

Moreover, we can prove for such a solution W that

$$W \text{ is bounded by } \|\widetilde{u}_1(0,\cdot)\|_{L^2(0,L)} \text{ up to a positive constant,}$$
$$\text{and } W_x(\cdot,0) \text{ is a non-trivial quasi-periodic function.} \quad (22)$$

It is in the proof of the existence of W and the second fact of (22) that the stated conditions are required. Thus, for large τ,

$$\|W_x(\cdot,0)\|_{L^2(\tau,2\tau)} \geqslant C \text{ by the theory of quasi-periodic functions!!}$$

Note that, for all $\delta > 0$, $\exists T_\delta > 0$ s.t. it holds, for $\tau \geqslant T_\delta$,

$$\|y_x(\cdot,0)\|_{L^2(\tau,2\tau)} \leqslant \delta\|y_0\|_{L^2(0,L)}, \quad (23)$$

for all solution y of the linearized KdV system (which in particular can be applied to $\widetilde{u}_2 - W$). This yields that $\partial_x W$ is a good approximation for $\partial_x \widetilde{u}_2$ for time sufficiently large (we do not say that W is a good approximation for $\widetilde{u}_2$ for time sufficiently large!).
Combining (22) and (23), we can derive (20).

On the construction of W

By the definition of $\mathcal{M}_L$, there exists $(\alpha_m)_{m=1}^{n_L} \subset \mathbb{C}$ such that

$$\frac{1}{\varepsilon} u_0 = \Re \left\{ \sum_{m=1}^{n_L} \alpha_m \Psi_m(0,x) \right\}. \tag{24}$$

The function $\widetilde{u}_1$ defined by (16) is then given by

$$\widetilde{u}_1(t,x) = \Re \left\{ \sum_{m=1}^{n_L} \alpha_m \Psi_m(t,x) \right\} = \Re \left\{ \sum_{m=1}^{n_L} \alpha_m e^{-ip_m t} \psi_m(x) \right\}.$$

Since $\Re f(t,x) \Re f_x(t,x) = \frac{1}{8}\Big((f(t,x)^2)_x + \big(\bar{f}(t,x)^2\big)_x + 2(|f(t,x)|^2)_x \Big)$, one has

$$\begin{aligned}
\widetilde{u}_{1,x}(t,x)\widetilde{u}_1(t,x) =& \frac{1}{8} \sum_{m_1=1}^{n_L} \sum_{m_2=1}^{n_L} \left(\alpha_{m_1} \alpha_{m_2} e^{-i(p_{m_1}+p_{m_2})t} \psi_{m_1}(x) \psi_{m_2}(x) \right)_x \\
&+ \frac{1}{8} \sum_{m_1=1}^{n_L} \sum_{m_2=1}^{n_L} \left(\overline{\alpha_{m_1} \alpha_{m_2} e^{-i(p_{m_1}+p_{m_2})t} \psi_{m_1}(x) \psi_{m_2}(x)} \right)_x \\
&+ \frac{1}{4} \sum_{m_1=1}^{n_L} \sum_{m_2=1}^{n_L} \left(\alpha_{m_1} \bar{\alpha}_{m_2} e^{-i(p_{m_1}-p_{m_2})t} \psi_{m_1}(x) \bar{\psi}_{m_2}(x) \right)_x .
\end{aligned}$$

We search W of the form

$$W = \frac{1}{8}\sum_{m_1=1}^{n_L}\sum_{m_2=1}^{n_L}\left(\alpha_{m_1}\alpha_{m_2}e^{-i(p_{m_1}+p_{m_2})t}\varphi_{m_1,m_2}(x)\right)_x + \frac{1}{8}\sum_{m_1=1}^{n_L}\sum_{m_2=1}^{n_L}\left(\overline{\alpha_{m_1}\alpha_{m_2}e^{-i(p_{m_1}+p_{m_2})t}\varphi_{m_1,m_2}(x)(x)}\right)_x + \frac{1}{4}\sum_{m_1=1}^{n_L}\sum_{m_2=1}^{n_L}\left(\alpha_{m_1}\bar{\alpha}_{m_2}e^{-i(p_{m_1}-p_{m_2})t}\phi_{m_1,m_2}(x)\right)_x.$$

Here

$$-i(p_{m_1}+p_{m_2})\varphi_{m_1,m_2}(x) + {\varphi_{m_1,m_2}}'(x) + {\varphi_{m_1,m_2}}'''(x) + \left(\psi_{m_1}\psi_{m_2}\right)'(x) = 0 \text{ in } (0,L),$$

$$\varphi_{m_1,m_2}(0) = \varphi_{m_1,m_2}(L) = \varphi'_{m_1,m_2}(L) = 0,$$

$$-i(p_{m_1}-p_{m_2})\phi_{m_1,m_2}(x) + {\phi_{m_1,m_2}}'(x) + {\phi_{m_1,m_2}}'''(x) + \left(\psi_{m_1}\bar{\psi}_{m_2}\right)'(x) = 0 \text{ in } (0,L),$$

$$\phi_{m_1,m_2}(0) = \phi_{m_1,m_2}(L) = \phi'_{m_1,m_2}(L) = 0.$$

One can solve ψ_{m_1,m_2} and φ_{m_1,m_2} explicitly in principle by noting

Lemma

Let $L \in \mathbb{N}$ *and* $1 \leqslant m_1, m_2 \leqslant n_L$. *We have, in* $[0, L]$,

$$\left(\psi_{m_1}\psi_{m_2}\right)'(x) = \sum_{j=1}^{3}\sum_{k=1}^{3}(\eta_{j+1,m_1} - \eta_{j,m_1})(\eta_{k+1,m_2} - \eta_{k,m_2})$$
$$(\eta_{j+2,m_1} + \eta_{k+2,m_2})e^{(\eta_{j+2,m_1}+\eta_{k+2,m_2})x},$$

and

$$\left(\psi_{m_1}\bar{\psi}_{m_2}\right)'(x) = \sum_{j=1}^{3}\sum_{k=1}^{3}(\eta_{j+1,m_1} - \eta_{j,m_1})(\bar{\eta}_{k+1,m_2} - \bar{\eta}_{k,m_2})$$
$$(\eta_{j+2,m_1} + \bar{\eta}_{k+2,m_2})e^{(\eta_{j+2,m_1}+\bar{\eta}_{k+2,m_2})x}.$$

More facts on $\varphi_{m,m}$

$$-i(p_{m_1}+p_{m_2})\varphi_{m_1,m_2}(x)+{\varphi_{m_1,m_2}}'(x)+{\varphi_{m_1,m_2}}'''(x) + \left(\psi_{m_1}\psi_{m_2}\right)'(x)=0 \text{ in } (0,L),$$

$$\varphi_{m_1,m_2}(0)=\varphi_{m_1,m_2}(L)=\varphi'_{m_1,m_2}(L)=0.$$

Definition

For $z\in\mathbb{C}$, let $\lambda_j=\lambda_j(z)$ $(1\leqslant j\leqslant 3)$ be the roots of the equation

$$\lambda^3+\lambda-iz=0, \tag{25}$$

and set

$$Q(z)=\begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_1 L} & e^{\lambda_2 L} & e^{\lambda_3 L} \\ \lambda_1 e^{\lambda_1 L} & \lambda_2 e^{\lambda_2 L} & \lambda_3 e^{\lambda_3 L} \end{pmatrix}. \tag{26}$$

Recall that

$$\eta_{1,m}=-\frac{2\pi i(2k_m+l_m)}{3L},\ \eta_{2,m}=\eta_{1,m}+\frac{2\pi i}{L}k_m,\ \eta_{3,m}=\eta_{2,m}+\frac{2\pi i}{L}l_m.$$

Lemma

Let $L \in \mathcal{N}$ *and* $1 \leqslant m \leqslant n_L$ *with* $k_m \neq l_m$. *Let* $\lambda_j = \lambda_j(2p_m)$ *and* $\mathcal{Q} = Q(2ip_m)$. *Set*

$$D = D_{m,m} = \sum_{j=1}^{3}\sum_{k=1}^{3} \frac{(\eta_{j+1,m} - \eta_{j,m})(\eta_{k+1,m} - \eta_{k,m})}{3\eta_{j+2,m}\eta_{k+2,m}},$$

$$\chi_{m,m}(x) = -\sum_{j=1}^{3}\sum_{k=1}^{3} \frac{(\eta_{j+1,m} - \eta_{j,m})(\eta_{k+1,m} - \eta_{k,m})}{3\eta_{j+2,m}\eta_{k+2,m}} \times e^{(\eta_{j+2,m}+\eta_{k+2,m})x} \text{ in } [0, L].$$

Then

$$\varphi_{m,m}(x) = \chi_{m,m}(x) + \sum_{j=1}^{3} a_j e^{\lambda_j x}, \tag{27}$$

$$\mathcal{Q}(a_1, a_2, a_3)^T = D(1, e^{(\eta_{1,m}+\eta_{1,m})L}, 0)^T. \tag{28}$$

Lemma

Assume that $k_m \neq l_m$. *We have*

$$D_{m,m} \neq 0$$

and

$$\chi'_{m,m}(0) = 0.$$

The proof is simple and based on the information :

$$\eta_{1,m} = -\frac{2\pi i(2k_m + l_m)}{3L},\ \eta_{2,m} = \eta_{1,m} + \frac{2\pi i}{L}k_m,\ \eta_{3,m} = \eta_{2,m} + \frac{2\pi i}{L}l_m.$$

In particular, we have

$$e^{\eta_{1,m}L} = e^{\eta_{2,m}L} = e^{\eta_{3,m}L}. \quad \square$$

Proposition

Let $L \in \mathcal{N}$ *and* $1 \leqslant m \leqslant n_L$ *with* $k_m \neq l_m$. *If* $s_m \neq 0$ *then*

$$\varphi'_{m,m}(0) \neq 0.$$

Recall that

$$L = 2\pi\sqrt{\frac{k_m^2 + k_m l_m + l_m^2}{3}}.$$

$$p_m = p(k_m, l_m) = \frac{(2k_m + l_m)(k_m - l_m)(2l_m + k_m)}{3\sqrt{3}(k_m^2 + k_m l_m + l_m^2)^{3/2}},$$

$$s_m = s(k_m, l_m) := \sum_{j=1}^{3} \sigma_{j,m}(\sigma_{j+2,m} - \sigma_{j+1,m}) \left(e^{\frac{4\pi i(k_m - l_m)}{3}} e^{2\pi i \sigma_{j,m}} + e^{-2\pi i \sigma_{j,m}}\right).$$

where, for $k_m \neq l_m$, $\sigma_{j,m}$ $(1 \leqslant j \leqslant 3)$ are the solutions of

$$\sigma^3 - 3(k_m^2 + k_m l_m + l_m^2)\sigma + 2(2k_m + l_m)(2l_m + k_m)(k_m - l_m) = 0.$$

A result related to quasi-periodic functions

Lemma

Let $\ell \in \mathbb{N}_*$, $a_j \in \mathbb{C}$, $q_j > 0$ *for* $1 \leqslant j \leqslant \ell$, *and* $M_{j_1,j_2}, N_{j_1,j_2} \in \mathbb{C}$ *with* $1 \leqslant j_1, j_2 \leqslant \ell$. *Assume that*

$$\begin{cases} q_{j_1} \neq q_{j_2} \text{ for } 1 \leqslant j_1 \neq j_2 \leqslant \ell, \\ M_{j,j} \neq 0 \text{ for } 1 \leqslant j \leqslant \ell. \end{cases}$$

Set, for $t \in \mathbb{R}$,

$$g(t) = \sum_{j_1=1}^{\ell} \sum_{j_2=1}^{\ell} \Big(a_{j_1} a_{j_2} M_{j_1,j_2} e^{-i(q_{j_1}+q_{j_2})t} + \bar{a}_{j_1} \bar{a}_{j_2} \bar{M}_{j_1,j_2} e^{i(q_{j_1}+q_{j_2})t} + 2a_{j_1} \bar{a}_{j_2} N_{j_1,j_2} e^{-i(q_{j_1}-q_{j_2})} \Big).$$

Given $0 < \gamma_1 < \gamma_2$, $\exists \gamma_0 > 0$ *and* $\tau_0 > 0$ *independent of* a_j *s.t. if* $\gamma_1 \leqslant \sum_{j=1}^{\ell} |a_j|^2 \leqslant \gamma_2$, *then*

$$\|g\|_{L^2(\tau,2\tau)} \geqslant \gamma_0 \text{ for all } \tau \geqslant \tau_0.$$

Proof : The proof involves the theory of quasiperiodic functions.

As a consequence, we obtain

Corollary

Assume that $k_m \neq l_m$ *and* $s_m \neq 0$ *for* $1 \leqslant m \leqslant n_L$*. We have*

$$\|W_x(\cdot, 0)\|_{L^2(\tau, 2\tau)} \geqslant C \text{ for all } \tau \geqslant \tau_0.$$

Note that

$$q_{j_1} \neq q_{j_2} \Leftrightarrow p_{m_1} \neq p_{m_2} : automatic$$

$$M_{j,j} \neq 0 \Leftrightarrow s_m \neq 0 : assumption$$

Part 5 : Conclusion and perspectives.

Conclusion and perspectives.

• We derive a condition for which the decay of the order $1/t^{1/2}$ holds. We also show that the decay cannot be faster than $\ln(2+t)/t$, so not faster than $1/t$ then.
• This condition can be checked numerically, and is based on the second approximation and the information of the second approximation is obtained approximately for sufficiently large time.
• Part of the analysis involves the theory of quasi-periodic functions.
• The approach presented is quite robust and can be applied to other problems.

Thank you for your attention !