Control of PDEs involving boundary layers phenomena

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in collaboration with Youcef Amirat and Carlos Castro

Let $\varepsilon > 0$. Consider the partial differential equation

$$\begin{cases} Y^{\varepsilon}_{t(t)} + (A + \varepsilon B) Y^{\varepsilon} = V^{\varepsilon}, t > 0, \\ Y^{\varepsilon}(0) = Y_0 \end{cases} \quad (1)$$

where B is an operator with higher order than the operator A.

Assume that for any $\varepsilon > 0$, system (1) is exactly controllable at time $T > 0$. The following issues arise :

- Behavior of controls V^{ε} as $\varepsilon \to 0$?
- In case of convergence of V^{ε}, rate of convergence of V^{ε}? Asymptotic expansion with respect to ε of V^{ε}?
- Behavior of the cost of control with respect to ε ?
- Minimal uniform time of controllability with respect to ε ?

The topic is not trivial, since in particular, boundary or internal (thin) layers may occur as ε goes to zero, i.e. Y^{ε} may exhibit locally singular behavior.

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Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^3 and Γ a subset of $\partial\Omega$ and $T > 0$. For any $\varepsilon > 0$, we consider the following linear equation of Petrowsky type

$$\begin{cases} y^\varepsilon_{tt} - \Delta y^\varepsilon + \varepsilon\Delta^2 y^\varepsilon = 0, & \text{in } Q_T := \Omega \times (0,T), \\ y^\varepsilon = 0, \quad \partial_\nu y^\varepsilon = v^\varepsilon 1_{\Gamma_T}, & \text{on } \Sigma_T := \partial\Omega \times (0,T), \\ (y^\varepsilon(\cdot,0), y^\varepsilon_t(\cdot,0)) = (y_0, y_1), & \text{on } \Omega. \end{cases} \tag{2}$$

Here, v^ε is a control function in $L^2(\Gamma_T)$, where Γ_T is a subset of Σ_T.This system models the dynamic of linear isotropic plates occupying the domain $\Omega\times]-\varepsilon,\varepsilon[$. $y^\varepsilon = y^\varepsilon(x,t)$ is the transversal displacement of the plate at point $x \in \Omega$ and time $t \in (0,T)$. y_0 denotes the initial position and y_1 the initial velocity assumed in $L^2(\Omega)$ and $H^{-2}(\Omega)$ respectively.

Well-posedness - $\forall \varepsilon > 0$, $v^\varepsilon \in L^2(\Omega)$, $(y_0, y_1) \in L^2(\Omega) \times H^{-2}(\Omega)$, $\exists! y^\varepsilon \in C^0([0,T], L^2(\Omega)) \cap C^1([0,T], H^{-2}(\Omega))$ with the following estimate:

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for some constant $c_\varepsilon > 0$.

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Main example of the talk : The Rayleigh plate model

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Controllability problem : For any final time $T > 0$, for any $(y_0, y_1) \in L^2(\Omega) \times H^{-2}(\Omega)$, find a control function $v^\varepsilon \in L^2(\Gamma_T)$ such that the corresponding solution to (27) satisfies

$$(y^\varepsilon(\cdot, T), y_t^\varepsilon(\cdot, T)) = (0, 0) \quad \text{in } L^2(\Omega) \times H^{-2}(\Omega). \tag{4}$$

For any $\varepsilon > 0$, this controllability property is proved in [Lions'86] assuming that the triplet (Ω, Γ, T) satisfies the usual geometric control condition for hyperbolic situations.

As is usual, the proof relies on an appropriate observability inequality for the adjoint problem: $\exists C > 0$ independent of ε s.t.

$$\|\varphi_0^\varepsilon\|^2_{H_0^1(\Omega)} + \|\varphi_1^\varepsilon\|^2_{L^2(\Omega)} + \varepsilon\|\Delta\varphi_0^\varepsilon\|^2_{L^2(\Omega)} \leq C \int_0^T\int_\Gamma \varepsilon|\Delta\varphi^\varepsilon|^2, \; \forall(\varphi_0^\varepsilon, \varphi_1^\varepsilon) \in H_0^2(\Omega) \times L^2(\Omega) \tag{5}$$

where φ^ε solves the corresponding homogeneous adjoint associated to the initial condition $(\varphi_0^\varepsilon, \varphi_1^\varepsilon)$,

$$\begin{cases} \varphi_{tt}^\varepsilon - \Delta\varphi^\varepsilon + \varepsilon\Delta^2\varphi^\varepsilon = 0, & \text{in } Q_T := \Omega \times (0, T), \\ \varphi^\varepsilon = \partial_\nu\varphi^\varepsilon = 0, & \text{on } \Sigma_T := \partial\Omega \times (0, T), \\ (\varphi^\varepsilon(\cdot, 0), \varphi_t^\varepsilon(\cdot, 0)) = (\varphi_0^\varepsilon, \varphi_1^\varepsilon), & \text{on } \Omega. \end{cases} \tag{6}$$

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Since the physical parameter ε is small with respect to one, the issue of the asymptotic behavior of elements of $\mathcal{C}$ as ε is smaller and smaller arise naturally. It turns out that the system (27) is not uniformly controllable with respect to ε. The following result is [Lions' 86], assuming additional regularity on the initial velocity.

Theorem (Lions'86)

Assume that the initial condition (y_0, y_1) belongs to $L^2(\Omega) \times H^{-1}(\Omega)$. Assume that the triplet (Ω, Γ, T) satisfies the geometric control condition. For any $\varepsilon > 0$, let v^ε be the control of minimal $L^2(\Gamma_T)$ norm for y^ε solution of (27). Then, one has

$$
\begin{aligned}
&-\sqrt{\varepsilon} v^\varepsilon \to v \quad in \quad L^2(\Gamma_T), \quad as \quad \varepsilon \to 0,\\
&y^\varepsilon \to y \quad in \quad L^\infty(0, T; L^2(\Omega)) - weak\text{-}star, \quad as \quad \varepsilon \to 0
\end{aligned}
\tag{7}
$$

where v is the control of minimal $L^2(\Gamma_T)$-norm for y, solution in $C^0([0, T]; L^2(\Omega)) \times C^1([0, T]; H^{-1}(\Omega))$ of the following system :

$$
\begin{cases}
y_{tt} - \Delta y = 0, & in \ Q_T = \Omega \times (0, T),\\
y = v 1_{\Gamma_T}, & on \ \Sigma_T = \partial\Omega \times (0, T),\\
(y(\cdot, 0), y_t(\cdot, 0)) = (y_0, y_1), & in \ \Omega.
\end{cases}
\tag{8}
$$

The degeneracy observed by Lions[1] comes from the following lemma

Lemma

$\Omega \in C^3$. Let $a_0 \in H^1(\Omega)$ and, for any $\varepsilon > 0$, a_ε the solution in $H^2(\Omega)$ of

$$\begin{cases} -\varepsilon \Delta a_\varepsilon + a_\varepsilon = a_0, & \Omega, \\ a_\varepsilon = 0, & \partial\Omega \end{cases} \tag{9}$$

satisfies

$$-\sqrt{\varepsilon}\frac{\partial a_\varepsilon}{\partial \nu} \rightarrow a_0 \quad in \quad L^2(\partial\Omega)$$

[1] J.-L. Lions, *Exact controllability and singular perturbations*, in Wave motion: theory, modelling, and computation (Berkeley, Calif., 1986), vol. 7 of Math. Sci. Res. Inst. Publ., Springer, New York, 1987, pp. 217–247.

Main example of the talk : The Rayleigh plate model

The control of minimal L^2-norm is given by $v^\varepsilon = \Delta\varphi^\varepsilon 1_{\Gamma_T}$ where φ^ε solves the adjoint problem

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with initial condition minimizing the conjugate functional $J^\star_\varepsilon : H^2_0(\Omega) \times L^2(\Omega) \to \mathbb{R}$ defined by

$$J^\star_\varepsilon(\varphi^\varepsilon_0, \varphi^\varepsilon_1) = \frac{\varepsilon}{2}\|\Delta\varphi^\varepsilon\|^2_{L^2(\Gamma_T)} - (y_0, \varphi^\varepsilon_1)_{L^2(\Omega),L^2(\Omega)} + (y_1, \varphi^\varepsilon_0)_{H^{-2}(\Omega),H^2(\Omega)}.$$

• $\sqrt{\varepsilon}\Delta\varphi^\varepsilon$ is bounded in $L^2(\Gamma_T)$ but $\Delta\varphi^\varepsilon$ is not bounded; this is due to the boundary layer of length $\mathcal{O}(\sqrt{\varepsilon})$;

• The conjugate functional is not uniformly coercive for the norm $H^2_0(\Omega) \times L^2(\Omega)$ with respect to ε. (the minimization of $J^\star_\varepsilon$ is ill-conditionned)

• Uniform (w.r.t. ε) gap of the spectrum of the unbounded operator $A^\varepsilon_0 : D(A^\varepsilon_0) \cup L^2(\Omega) \to L^2(\Omega)$ defined by $A^\varepsilon_0 = -\Delta + \varepsilon\Delta^2$ with $D(A^\varepsilon_0) = (H^4 \cap H^2_0)(\Omega)$.

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$\Omega = (0,1)$. We consider a positive smooth weight function $\eta \geq 0$ with compact support in $(0,T)$, i.e. $\eta \in C_0^\infty(0,T)$, and such that $\eta(t) > \eta_0 > 0$ in a subinterval $[\delta, T-\delta] \subset (0,T)$ with δ such that $T - 2\delta > 2$. The optimality system associated to the null control which minimizes

$$\int_0^T \eta^{-1}(t)|v^\varepsilon|^2 dt.$$

is given by

$$\begin{cases} y^\varepsilon_{tt} + \varepsilon y^\varepsilon_{xxxx} - y^\varepsilon_{xx} = 0, & \text{in } Q_T, \\ y^\varepsilon(0,\cdot) = y^\varepsilon(1,\cdot) = y^\varepsilon_x(0,\cdot) = 0, \quad y^\varepsilon_x(1,\cdot) = v^\varepsilon = \eta\varphi^\varepsilon_{xx}(1,\cdot) & \text{in } (0,T), \\ (y^\varepsilon(\cdot,0), y^\varepsilon_t(\cdot,0)) = (y_0, y_1), & \text{in } (0,1), \\ y^\varepsilon(\cdot,T) = y^\varepsilon_t(\cdot,T) = 0, & \text{in } (0,1), \\ \varphi^\varepsilon_{tt} + \varepsilon\varphi^\varepsilon_{xxxx} - \varphi^\varepsilon_{xx} = 0, & \text{in } Q_T, \\ \varphi^\varepsilon(0,\cdot) = \varphi^\varepsilon(1,\cdot) = \varphi^\varepsilon_x(0,\cdot) = \varphi^\varepsilon_x(1,\cdot) = 0 & \text{in } (0,T), \\ \varphi^\varepsilon(\cdot,T) = \varphi^\varepsilon_0, \varphi^\varepsilon_t(\cdot,T) = \varphi^\varepsilon_1, & \text{in } (0,1). \end{cases} \tag{11}$$

J.-L. Lions *Perturbations singulières dans les problèmes aux limites et en contrôle optimal*. Lecture Notes in Mathematics. Springer 1973.

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The situation is tricky because

- y^ε exhibits a boundary layer of size $\mathcal{O}(\sqrt{\varepsilon})$ at $x = 0$ and $x = 1$ and an inner angular layer of size $\mathcal{O}(\varepsilon^{1/4})$ along the characteristics $\{(x,t) \in Q_T, x - t = 0\}$ and $\{(x,t) \in Q_T, x + t - 1 = 0\}$;
- φ^ε exhibits a boundary layer of size $\mathcal{O}(\sqrt{\varepsilon})$ at $x = 0$ and $x = 1$ and an inner angular layer of size $(\mathcal{O}(\varepsilon^{1/4}))$ along characteristics parallel to $\{(x,t) \in Q_T, x - t = 0\}$ and $\{(x,t) \in Q_T, x + t - 1 = 0\}$;

$$\begin{cases} y^\varepsilon_{tt} + \varepsilon y^\varepsilon_{xxxx} - y^\varepsilon_{xx} = 0, & \text{in } Q_T, \\ y^\varepsilon(0,\cdot) = y^\varepsilon(1,\cdot) = y^\varepsilon_x(0,\cdot) = 0, \quad y^\varepsilon_x(1,\cdot) = v^\varepsilon = \eta\varphi^\varepsilon_{xx}(1,\cdot) & \text{in } (0,T), \\ (y^\varepsilon(\cdot,0), y^\varepsilon_t(\cdot,0)) = (y_0, y_1), & \text{in } (0,1), \\ y^\varepsilon(\cdot,T) = y^\varepsilon_t(\cdot,T) = 0, & \text{in } (0,1), \end{cases} \tag{13}$$

- One can avoid the angular layer, imposing compatibilities conditions at the points $\partial\Omega \times \{t = 0\}$ between the initial data and the boundary conditions.
- The boundary layer of size $\mathcal{O}(\sqrt{\varepsilon})$) are unavoidable ! One can use the matched asymptotic expansion method which requires however regularity !

Asymptotic analysis of the controllability problem - The one dimensional case - Difficulties

$$\begin{cases} y^\varepsilon_{tt} + \varepsilon y^\varepsilon_{xxxx} - y^\varepsilon_{xx} = 0, & \text{in } Q_T, \\ y^\varepsilon(0,\cdot) = y^\varepsilon(1,\cdot) = y^\varepsilon_x(0,\cdot) = 0, \quad y^\varepsilon_x(1,\cdot) = v^\varepsilon = \eta\varphi^\varepsilon_{xx}(1,\cdot) & \text{in } (0,T), \\ (y^\varepsilon(\cdot,0), y^\varepsilon_t(\cdot,0)) = (y_0, y_1), & \text{in } (0,1), \\ y^\varepsilon(\cdot,T) = y^\varepsilon_t(\cdot,T) = 0, & \text{in } (0,1), \end{cases} \tag{13}$$

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Regularity property of the weighted control

Let $X = L^2(0,1) \times H^{-2}(0,1)$ and $X^* = H_0^2(0,1) \times L^2(0,1)$ its dual, with duality product given by

$$< (y_0, y_1), (\varphi_0, \varphi_1) >_{X,X^*} = \int_\Omega y_0 \varphi_1 \, dx - (y_1, \varphi_0)_{H^{-2}, H_0^2}, \tag{14}$$

where $(\cdot,\cdot)_{H^{-2},H_0^2}$ represents the usual duality product.

Definition

For any $(y_0, y_1) \in X$ we define the minimal L^2-weighted control $v^\varepsilon(t)$ associated to (27) as the function

$$v^\varepsilon(t) = \eta(t)\varphi^\varepsilon_{xx}(1,t) \in L^2(0,T) \tag{15}$$

where φ^ε is the solution of the adjoint system with initial data $(\varphi^\varepsilon_0, \varphi^\varepsilon_1)$, the minimizer of

$$J^\varepsilon(\varphi_0, \varphi_1) = \frac{\varepsilon}{2} \int_{\Sigma_0} \eta(t) |\varphi^\varepsilon_{xx}(1,t)|^2 \, dt - < (y_0, y_1), (\varphi_0, \varphi_1) >_{X,X^*}, \tag{16}$$

in $(\varphi_0, \varphi_1) \in X^*$.

Fondamental property: If the initial data (y_0, y_1) are smooth, then the same is true for $(\varphi_0^\varepsilon, \varphi_1^\varepsilon)$, the minimizer of $J_\star^\varepsilon$ and so for the weighted control.

Let $A_0^\varepsilon : D(A_0^\varepsilon) \subset L^2(0,1) \to L^2(0,1)$ be the unbounded operator $A_0^\varepsilon = -\partial_{xx}^2 + \varepsilon \partial_{xxxx}^4$ with domain $D(A_0^\varepsilon) = H^4 \cap H_0^2(0,1)$. A_0^ε is a dissipative self-adjoint operator.

We also define the unbounded skew-adjoint operator on $X = L^2(0,1) \times H^{-2}(0,1)$,

$$A^\varepsilon = \begin{pmatrix} 0 & I \\ -A_0^\varepsilon & 0 \end{pmatrix}, \qquad D(A^\varepsilon) = H_0^2(0,1) \times L^2(0,1).$$

Associated to A^ε we consider the usual scale of Hilbert spaces $X_\alpha = D((A^\varepsilon)^\alpha)$, $\alpha > 0$. Note that if we use the duality product (14) then

$$(A^\varepsilon)^* : D((A^\varepsilon)^*) \subset X^* \to X^*,$$

is given by

$$(A^\varepsilon)^* = \begin{pmatrix} 0 & -I \\ A_0^\varepsilon & 0 \end{pmatrix}, \qquad D((A^\varepsilon)^*) = X_1^* = X.$$

In general, $X_\alpha^* = D(((A^\varepsilon)^*)^\alpha) = D((A^\varepsilon)^{\alpha+1})$.

The following result is a direct consequence of the results of [Dehman-Lebeau 2009], [Ervedoza-Zuazua 2010]:

Theorem

Given any $(y_0, y_1) \in X = L^2(0,1) \times H^{-2}(0,1)$*, there exists a unique weighted control* v^ε *of system (27) satisfying* (15)*. This control is the one that minimizes the norm*

$$\int_0^T \eta^{-1} |v^\varepsilon|^2 dt.$$

Furthermore, if $(y_0, y_1) \in D((A^\varepsilon)^\alpha)$ *for some* $\alpha > 0$*, then the control* v^ε *satisfies*

$$v^\varepsilon \in H_0^\alpha(0,T) \bigcap_{k=0}^{[\alpha]} C^k([0,T]),$$

with the estimate $\|v^\varepsilon\|_{H_0^\alpha(0,T)} \leq C\|(y_0, y_1)\|_{X_\alpha}$ *and the corresponding* $(\psi_0^{T,\varepsilon}, \psi_1^{T,\varepsilon}) \in X_\alpha^* = X_{\alpha+1}$*. In particular, the controlled solution* y *belongs to*

$$(y, y') \in C^\alpha([0,T]; X_0) \bigcap_{k=0}^{[\alpha]} C^k([0,T]; X_{\alpha-k}).$$

Definition

Let v^0 is the null control for u solution of

$$\begin{cases} u_{tt} - u_{xx} = 0, & \text{in } Q_T, \\ u(0,\cdot) = 0, u(1,\cdot) = v^0 & \text{in } (0,T), \\ (u(\cdot,0), u_t(\cdot,0)) = -(y_0, y_1), & \text{in } \Omega. \end{cases} \tag{17}$$

which minimizes $v \to \int_0^T \eta^{-1}(t)|v|^2 dt$.

Theorem (Castro, Munch 2019)

Assume that $(y_0, y_1) \in Z_4 \subset H^4 \times H_0^2$ and $T > 2$. Let $\varepsilon > 0$ and v^ε be the control of minimal L^2-weighted norm for (27) associated to the data (y_0, y_1). Then, there exists a constant $C > 0$ such that

$$\left\| \varepsilon^{1/2} v^\varepsilon - v^0 \right\|_{L^2(0,T)} \leq C\varepsilon^{1/4}.$$

We recover and refine the weak convergence results due to Lions (1986).

Rk. $\|e^{-x/\sqrt{\varepsilon}}\|_{L^2(0,1)} = \|e^{-(1-x)/\sqrt{\varepsilon}}\|_{L^2(0,1)} = \mathcal{O}(\varepsilon^{1/4})$

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Definition

Let φ^a be the solution of

$$\begin{cases} \varphi^a_{tt} - \varphi^a_{xx} = 0, & \text{in } Q_T, \\ \varphi^a(0,\cdot) = -\varphi^0_x(0,\cdot), \varphi^a(1,\cdot) = \varphi^0_x(1,\cdot) & \text{in } (0,T), \\ (\varphi^a(\cdot,0), \varphi^a_t(\cdot,0)) = (0,0), & \text{in } \Omega. \end{cases} \quad (18)$$

and depends on v^0 (through φ^0, optimal adjoint solution).

Let $v^1 = -\eta(t)\varphi^a_x(1,\cdot) - v$ where v is the null control for u solution of

$$\begin{cases} u_{tt} - u_{xx} = 0, & \text{in } Q_T, \\ u(0,\cdot) = -y^0_x(0,\cdot), \quad u(1,\cdot) = v(t) + y^0_x(1,\cdot) + \eta(t)\varphi^a_x(1,\cdot) & \text{in } (0,T), \\ (u(\cdot,0), u_t(\cdot,0)) = (0,0), & \text{in } \Omega. \end{cases} \quad (19)$$

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which minimizes $v \to \int_0^T \eta^{-1}(t)|v|^2 dt$.

Rate of convergence - First and second terms

Theorem (Castro, Munch, 2019)

Assume that $(y_0, y_1) \in Z_5 \subset (H^5 \times H_0^3)(\Omega)$.
Consider v^j, $0 \leq j \leq 1$, *the controls obtained previously. Let* $\varepsilon > 0$ *and* v^ε *be the control of minimal* L^2*-weighted norm for (27) associated to the data* (y_0, y_1). *Then, there exists a constant* $C > 0$ *such that*

$$\left\| \varepsilon^{1/2} v^\varepsilon - (v^0 + \sqrt{\varepsilon} v^1) \right\|_{L^2(0,T)} \leq C \varepsilon^{3/4}.$$

Definition

Let φ^a be the solution of

$$\begin{cases} \varphi^a_{tt} - \varphi^a_{xx} = -\varphi^0_{xxxx}, & \text{in } Q_T, \\ \varphi^a(0,\cdot) = -\varphi^1_x(0,\cdot), \varphi^a(1,\cdot) = \varphi^1_x(1,\cdot) & \text{in } (0,T), \\ (\varphi^a(\cdot,0), \varphi^a_t(\cdot,0)) = (0,0), & \text{in } \Omega. \end{cases} \tag{20}$$

and depends on v^1 (through φ^1, optimal adjoint solution) and v^0 (through y^0).

Let $v^2 = -\eta(t)\varphi^a_x(1,\cdot) - v$ where v is the null control for u solution of

$$\begin{cases} u_{tt} - u_{xx} = -y^0_{xxxx}, & \text{in } Q_T, \\ u(0,\cdot) = -y^1_x(0,\cdot), \quad u(1,\cdot) = v(t) + y^1_x(1,\cdot) + \frac{1}{2}y^0_{tt}(1,\cdot) + \eta(t)\varphi^a_x(1,\cdot) & \text{in } (0,T), \\ (u(\cdot,0), u_t(\cdot,0)) = (0,0), & \text{in } \Omega. \end{cases} \tag{21}$$

which minimizes $v \to \int_0^T \eta^{-1}(t)|v|^2 dt$.

Theorem (Castro, Munch, 2019)

Assume that $(y_0, y_1) \in Z_6 \subset (H^6 \times H_0^4)(\Omega)$ *and* $T > 2$.
Consider v^j, $0 \leq j \leq 2$, *the controls obtained previously. Let* $\varepsilon > 0$ *and* v^ε *be the control of minimal* L^2*-weighted norm for (27) associated to the data* (y_0, y_1)*. Then, there exists a constant* $C > 0$ *such that*

$$\left\| \varepsilon^{1/2} v^\varepsilon - (v^0 + \sqrt{\varepsilon} v^1 + \varepsilon v^2) \right\|_{L^2(0,T)} \leq C \varepsilon^{5/4}.$$

Sketch of the proof (1)

Very technical proof ! (see details at arxiv.org/abs/1907.04118)

- Matched asymptotic expansion method on the direct problem: Assuming $v^\varepsilon = \varepsilon^{-1/2}v^0 + v^1 + \sqrt{\varepsilon}v^2$, we explicitly construct an approximation $\tilde{y}^\varepsilon$ of the solution y^ε of the form

$$\tilde{y}^\varepsilon(x,t) = \sum_{j=0}^{2} \varepsilon^{j/2}\left[y^j(x,t) - y^j(0,t)e^{-z} - \left(y^j(1,t) + \frac{w}{2}y_{tt}^{j-2}(1,t)\right)e^{-w}\right]. \tag{22}$$

with $z = x/\sqrt{\varepsilon}$ and $w = (1-x)/\sqrt{\varepsilon}$

- Matched asymptotic expansion method on the adjoint problem: Assuming the initial condition of the adjoint problem of the form

$$\begin{aligned} \tilde{\varphi}^\varepsilon(x,0) &= \sum_{k=0}^{2} \varepsilon^{k/2}\left[\varphi_0^k(x) - \varphi_0^k(0)e^{-x/\varepsilon^{1/2}} - \varphi_0^k(1)e^{-(1-x)/\varepsilon^{1/2}}\right] \\ \tilde{\varphi}_t^\varepsilon(x,0) &= \sum_{k=0}^{2} \varepsilon^{k/2}\left[\varphi_1^k(x) - \varphi_1^k(0)e^{-x/\varepsilon^{1/2}} - \varphi_1^k(1)e^{-(1-x)/\varepsilon^{1/2}}\right], \end{aligned}$$

for some $(\varphi_0^k, \varphi_1^k)$ with $k = 0,1,2$, we explicitly construct an approximation

$$\tilde{\varphi}^\varepsilon(x,t) = \sum_{j=0}^{2} \varepsilon^{j/2}\left[\varphi^j(x,t) - \varphi^j(0,t)e^{-z} - \left(\varphi^j(1,t) + \frac{w}{2}\varphi_{tt}^{j-2}(1,t)\right)e^{-w}\right]. \tag{23}$$

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\tilde{\varphi}_t^\varepsilon(x,0) &= \sum_{k=0}^{2} \varepsilon^{k/2} \left[\varphi_1^k(x) - \varphi_1^k(0) e^{-x/\varepsilon^{1/2}} - \varphi_1^k(1) e^{-(1-x)/\varepsilon^{1/2}} \right],
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Sketch of the proof (2)

- A priori estimate for $y^\varepsilon - \tilde{y}^\varepsilon$ and $\varphi^\varepsilon - \tilde{\varphi}^\varepsilon$. The main tool is the following lemma

Lemma

Let ψ^ε be the solution of the system

$$\begin{cases} \psi^\varepsilon_{tt} + \varepsilon\psi^\varepsilon_{xxxx} - \psi^\varepsilon_{xx} = f, & \text{in } Q_T, \\ \psi^\varepsilon(0,\cdot) = g_1, \quad \psi^\varepsilon(1,\cdot) = g_2, & \text{in } \ (0,T), \\ \psi^\varepsilon_x(0,\cdot) = h_1, \quad \psi^\varepsilon_x(1,\cdot) = h_2, & \text{in } \ (0,T), \\ \psi^\varepsilon(\cdot,0) = \psi_0, \quad \psi^\varepsilon_t(\cdot,0) = \psi_1, & \text{in } \ \Omega, \end{cases} \tag{24}$$

where $f \in L^1(0,T;L^2)$, $g_1, g_2, h_1, h_2 \in H^2(0,T)$ and $(\psi_0,\psi_1) \in H^2 \times L^2$ satisfying the compatibility conditions

$$\psi^\varepsilon_0(0) = g_1(0), \quad \psi^\varepsilon_0(1) = g_2(0), \quad \psi^\varepsilon_{0,x}(0) = h_1(0), \quad \psi^\varepsilon_{0,x}(1) = h_2(0). \tag{25}$$

Then, there exists a constant $C > 0$ such that

$$\begin{aligned} &\|\psi^\varepsilon\|_{L^\infty(0,T;H^1)} + \|\psi^\varepsilon_t\|_{L^\infty(0,T;L^2)} + \varepsilon^{1/2}\|\psi^\varepsilon_{xx}\|_{L^\infty(0,T;L^2)} \le C\, F(f,g_1,g_2,h_1,h_2,\psi_0,\psi_1), \\ &\|\varepsilon^{1/2}\psi^\varepsilon_{xx}(0,\cdot) + \psi^\varepsilon_x(0,\cdot)\|_{L^2(0,T)} \le C\, F(f,g_1,g_2,h_1,h_2,\psi_0,\psi_1), \\ &\|\varepsilon^{1/2}\psi^\varepsilon_{xx}(1,\cdot) - \psi^\varepsilon_x(1,\cdot)\|_{L^2(0,T)} \le C\, F(f,g_1,g_2,h_1,h_2,\psi_0,\psi_1), \\ &\text{where} \\ &F(f,g_1,g_2,h_1,h_2,\psi_0,\psi_1) = \|f\|_{L^1(0,T;L^2)} + \|g_1\|_{H^2(0,T)} + \|g_2\|_{H^2(0,T)} \\ &\qquad + \varepsilon^{1/2}(\|h_1\|_{H^2(0,T)} + \|h_2\|_{H^2(0,T)}) + \|(\psi_0,\psi_1)\|_{H^1\times L^2} + \varepsilon^{1/2}\|\psi_{0,xx}\|_{L^2}. \end{aligned} \tag{26}$$

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$$\begin{aligned} F(f,g_1,g_2,h_1,h_2,\psi_0,\psi_1) &= \|f\|_{L^1(0,T;L^2)} + \|g_1\|_{H^2(0,T)} + \|g_2\|_{H^2(0,T)} \\ &+ \varepsilon^{1/2}(\|h_1\|_{H^2(0,T)} + \|h_2\|_{H^2(0,T)}) + \|(\psi_0,\psi_1)\|_{H^1\times L^2} + \varepsilon^{1/2}\|\psi_{0,xx}\|_{L^2}. \end{aligned}$$

Sketch of the proof (2)

- A priori estimate for $y^\varepsilon - \tilde{y}^\varepsilon$ and $\varphi^\varepsilon - \tilde{\varphi}^\varepsilon$. The main tool is the following lemma

Lemma

Let ψ^ε be the solution of the system

$$\begin{cases} \psi^\varepsilon_{tt} + \varepsilon\psi^\varepsilon_{xxxx} - \psi^\varepsilon_{xx} = f, & \text{in } Q_T, \\ \psi^\varepsilon(0,\cdot) = g_1, \quad \psi^\varepsilon(1,\cdot) = g_2, & \text{in } (0,T), \\ \psi^\varepsilon_x(0,\cdot) = h_1, \quad \psi^\varepsilon_x(1,\cdot) = h_2, & \text{in } (0,T), \\ \psi^\varepsilon(\cdot,0) = \psi_0, \quad \psi^\varepsilon_t(\cdot,0) = \psi_1, & \text{in } \Omega, \end{cases} \tag{24}$$

where $f \in L^1(0,T;L^2)$, $g_1, g_2, h_1, h_2 \in H^2(0,T)$ and $(\psi_0,\psi_1) \in H^2 \times L^2$ satisfying the compatibility conditions

$$\psi^\varepsilon_0(0) = g_1(0), \quad \psi^\varepsilon_0(1) = g_2(0), \quad \psi^\varepsilon_{0,x}(0) = h_1(0), \quad \psi^\varepsilon_{0,x}(1) = h_2(0). \tag{25}$$

Then, there exists a constant $C > 0$ such that

$$\begin{aligned} &\|\psi^\varepsilon\|_{L^\infty(0,T;H^1)} + \|\psi^\varepsilon_t\|_{L^\infty(0,T;L^2)} + \varepsilon^{1/2}\|\psi^\varepsilon_{xx}\|_{L^\infty(0,T;L^2)} \leq C\, F(f,g_1,g_2,h_1,h_2,\psi_0,\psi_1), \\ &\|\varepsilon^{1/2}\psi^\varepsilon_{xx}(0,\cdot) + \psi^\varepsilon_x(0,\cdot)\|_{L^2(0,T)} \leq C\, F(f,g_1,g_2,h_1,h_2,\psi_0,\psi_1), \\ &\|\varepsilon^{1/2}\psi^\varepsilon_{xx}(1,\cdot) - \psi^\varepsilon_x(1,\cdot)\|_{L^2(0,T)} \leq C\, F(f,g_1,g_2,h_1,h_2,\psi_0,\psi_1), \end{aligned} \tag{26}$$

where

$$\begin{aligned} F(f,g_1,g_2,h_1,h_2,\psi_0,\psi_1) = {} & \|f\|_{L^1(0,T;L^2)} + \|g_1\|_{H^2(0,T)} + \|g_2\|_{H^2(0,T)} \\ & + \varepsilon^{1/2}(\|h_1\|_{H^2(0,T)} + \|h_2\|_{H^2(0,T)}) + \|(\psi_0,\psi_1)\|_{H^1\times L^2} + \varepsilon^{1/2}\|\psi_{0,xx}\|_{L^2}. \end{aligned}$$

- Substitution of the expansion $\tilde{y}^\varepsilon$ and $\tilde{\varphi}^\varepsilon$ in the optimality system :

$$\begin{cases} y^\varepsilon_{tt} + \varepsilon y^\varepsilon_{xxxx} - y^\varepsilon_{xx} = 0, & \text{in } Q_T, \\ y^\varepsilon(0,\cdot) = y^\varepsilon(1,\cdot) = y^\varepsilon_x(0,\cdot) = 0, \quad y^\varepsilon_x(1,\cdot) = v^\varepsilon = \eta\varphi^\varepsilon_{xx}(1,\cdot) & \text{in } (0,T), \\ (y^\varepsilon(\cdot,0), y^\varepsilon_t(\cdot,0)) = (y_0, y_1), & \text{in } (0,1), \\ y^\varepsilon(\cdot,T) = y^\varepsilon_t(\cdot,T) = 0, & \text{in } (0,1), \\ \varphi^\varepsilon_{tt} + \varepsilon\varphi^\varepsilon_{xxxx} - \varphi^\varepsilon_{xx} = 0, & \text{in } Q_T, \\ \varphi^\varepsilon(0,\cdot) = \varphi^\varepsilon(1,\cdot) = \varphi^\varepsilon_x(0,\cdot) = \varphi^\varepsilon_x(1,\cdot) = 0 & \text{in } (0,T), \\ \varphi^\varepsilon(\cdot,T) = \varphi^\varepsilon_0, \varphi^\varepsilon_t(\cdot,T) = \varphi^\varepsilon_1, & \text{in } (0,1). \end{cases} \tag{27}$$

In particular,

$$\sum_{j=0}^{2} \varepsilon^{j/2} v^j + \mathcal{O}(\varepsilon^{3/2}) = \eta(t)\varphi^\varepsilon_{xx}(1,t) = -\eta(t)\sum_{j=0}^{2} \varepsilon^{j/2}\varphi^j_x(1,t) + \mathcal{O}(\varepsilon^{3/2}),$$

leads to

$$v^j(t) = -\eta(t)\varphi^j_x(1,t), \quad j = 0,1,2. \tag{28}$$

- Consider now the error function $w^\varepsilon = y^\varepsilon - \tilde{y}^\varepsilon$ and $\zeta^\varepsilon = \varphi^\varepsilon - \tilde{\varphi}^\varepsilon$. They satisfy the coupled system

$$\left\{\begin{array}{ll} \zeta^\varepsilon_{tt} + \varepsilon\zeta^\varepsilon_{xxxx} - \zeta^\varepsilon_{xx} = 0, & \text{in} \quad Q_T, \\ \zeta^\varepsilon(0,t) = \zeta^\varepsilon(1,t) = 0, & t \in (0,T), \\ \zeta^\varepsilon_x(0,t) = \zeta^\varepsilon_x(1,t) = 0, & t \in (0,T), \\ \zeta^\varepsilon(x,0) = \varphi^\varepsilon_0 - \psi^\varepsilon_0, \quad \zeta^\varepsilon_t(x,0) = \varphi^\varepsilon_1 - \tilde{\phi}^\varepsilon_1, & x \in (0,1), \end{array}\right. \tag{29}$$

$$\left\{\begin{array}{ll} w^\varepsilon_{tt} + \varepsilon w^\varepsilon_{xxxx} - w^\varepsilon_{xx} = 0, & \text{in} \quad Q_T, \\ w^\varepsilon(0,t) = w^\varepsilon(1,t) = 0, & t \in (0,T), \\ w^\varepsilon_x(0,t) = 0, \quad w^\varepsilon_x(1,t) = \eta(t)\zeta^\varepsilon_{xx}(1,t), & t \in (0,T), \\ w^\varepsilon(x,0) = 0, \quad w^\varepsilon_t(x,0) = 0, & x \in (0,1), \\ w^\varepsilon(x,T) = -g^\varepsilon_0, \quad w^\varepsilon_t(x,T) = -g^\varepsilon_1. & \end{array}\right. \tag{30}$$

Note that this is the optimality system for the unique minimal weighted L^2-norm that drives the initial state $(0,0)$ to the final state $(-g^\varepsilon_0, -g^\varepsilon_1)$. Therefore,

$$\|\eta(t)\zeta^\varepsilon_{xx}(1,\cdot)\|_{L^2(0,T)} = \|v^\varepsilon - \eta(t)\tilde{\varphi}^\varepsilon_{xx}(1,t)\|_{L^2(0,T)} \leq C\|(g^\varepsilon_0, g^\varepsilon_1)\|_{X_1}$$
$$= \varepsilon^{1/2}\|g^0_{0,xx}\|_{L^2} + \|(g^\varepsilon_0, g^\varepsilon_1)\|_{H^1\times L^2} = \mathcal{O}(\varepsilon^{n/2+1/4})$$

which allows to conclude.

The asymptotic expansion of v^ε is also relevant from an approximation viewpoint, since the expansion

$$v^\varepsilon = \frac{1}{\sqrt{\varepsilon}}(v^0 + \sqrt{\varepsilon}v^1 + \varepsilon v^2) + \mathcal{O}(\varepsilon^{3/4}) \qquad (\text{in} \quad L^2) \tag{31}$$

involves controls for wave equations which are simpler to approximate than v^ε.

Corollary (Convergent discrete approximation of v^ε)

Assume that for $k = 0, 1, 2$, $\{v_h^k\}_{(h>0)}$, is approximation of v^k, h being a discretization parameter, satisfying $\|v^k - v_h^k\|_{L^2(0,T)} = \mathcal{O}(h^\alpha)$ for some $\alpha > 0$.
Then the approximation

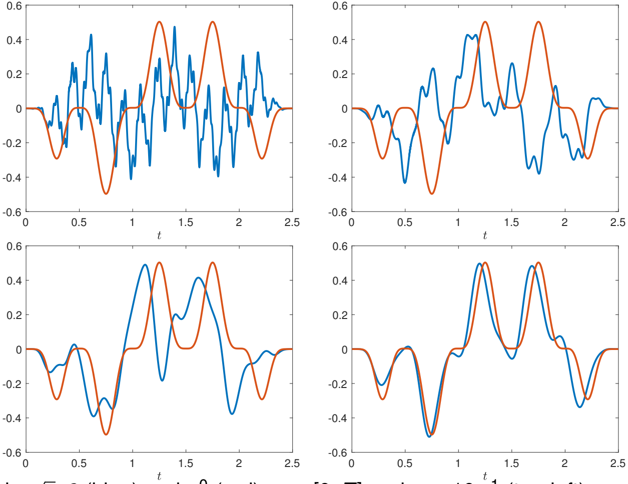
$$v_h^\varepsilon := \varepsilon^{-1/2}(v_h^0 + \sqrt{\varepsilon}v_h^1 + \varepsilon v_h^2)$$

satisfies the estimate

$$\|\sqrt{\varepsilon}(v^\varepsilon - v_h^\varepsilon)\|_{L^2(0,T)} = \mathcal{O}(\varepsilon^{3/4}) + \mathcal{O}(h^\alpha), \quad \forall \varepsilon > 0, h > 0.$$

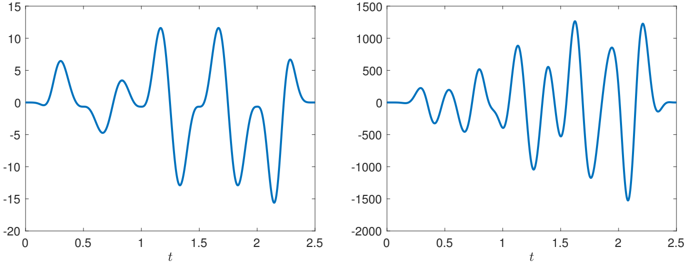
Numerical experiments

$T = 2.5$, $(y_0, y_1) = (\sin(2\pi x)^4, 0) \in Z^6$. $\eta(t) = \left((1 - e^{-40t})(1 - e^{-40(T-t)})\right)^3$.



Controls $\sqrt{\varepsilon} v^{\varepsilon}$ (blue) and v^0 (red) over $[0, T]$ and $\varepsilon = 10^{-1}$ (top-left), $\varepsilon = 10^{-2}$ (top-right), $\varepsilon = 10^{-3}$ (bottom-left) and $\varepsilon = 10^{-4}$ (bottom-right).

$T = 2.5$, $(y_0, y_1) = (\sin(2\pi x)^4, 0) \in Z^6$. $\eta(t) = \left((1 - e^{-40t})(1 - e^{-40(T-t)})\right)^3$.



Controls v^1 (left) and v^2 (right) over $[0, T]$.

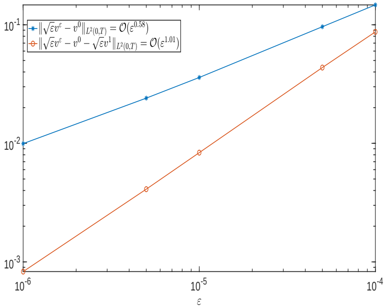
Numerical experiments

$T = 2.5$, $(y_0, y_1) = (\sin(2\pi x)^4, 0) \in Z^6$. $\eta(t) = \left((1 - e^{-40t})(1 - e^{-40(T-t)})\right)^3$, $t \in [0, T]$. $\|v^0\|_{L^2(0,T)} \approx 0.349834$

ε	$\sharp$ CG iterates	$\|\sqrt{\varepsilon} v_\varepsilon\|_{L^2(0,T)}$	E_0^ε	E_1^ε	E_2^ε
10^{-1}	5	0.2625	4.68×10^{-1}	4.12×10^{-1}	3.1×10^{-1}
10^{-2}	11	0.2965	4.28×10^{-1}	3.32×10^{-1}	2.1×10^{-1}
10^{-3}	24	0.3542	3.61×10^{-1}	2.82×10^{-1}	1.79×10^{-1}
10^{-4}	51	0.3510	1.47×10^{-1}	8.71×10^{-2}	6.21×10^{-2}
5×10^{-5}	90	0.3508	9.29×10^{-2}	4.35×10^{-2}	2.01×10^{-2}
10^{-5}	101	0.3499	3.59×10^{-2}	8.34×10^{-3}	2.37×10^{-3}
5×10^{-6}	171	0.3498	2.40×10^{-2}	4.30×10^{-3}	9.31×10^{-4}
10^{-6}	203	0.3498	9.95×10^{-3}	8.34×10^{-4}	1.13×10^{-4}

$$E_0^\varepsilon = \|\sqrt{\varepsilon} v^\varepsilon - v^0\|_{L^2(0,T)} = \mathcal{O}(\varepsilon^{0.58}),$$
$$E_1^\varepsilon = \|\sqrt{\varepsilon} v^\varepsilon - v^0 - \sqrt{\varepsilon} v^1\|_{L^2(0,T)} = \mathcal{O}(\varepsilon^{1.01}),$$
$$E_2^\varepsilon = \|\sqrt{\varepsilon} v^\varepsilon - v^0 - \sqrt{\varepsilon} v^1 - \varepsilon v^2\|_{L^2(0,T)} = \mathcal{O}(\varepsilon^{1.36}),$$

$T = 2.5$, $(y_0, y_1) = (\sin(2\pi x)^4, 0) \in Z^6$. $\eta(t) = \left((1 - e^{-40t})(1 - e^{-40(T-t)})\right)^3$.



Evolution of $\|\sqrt{\varepsilon}v^\varepsilon - v^0\|_{L^2(0,T)}$ and $\|\sqrt{\varepsilon}v^\varepsilon - v^0 - \sqrt{\varepsilon}v^1\|_{L^2(0,T)}$ with respect to ε.

The model of an elastic cylindric arch (considered in [AmmarKhodja-Geymonat-Munch, 2010]) of length one and constant curvature $c > 0$

$$\begin{cases} u^\varepsilon_{tt} - (u^\varepsilon_x + cv^\varepsilon)_x = 0, & \text{in } Q_T, \\ v^\varepsilon_{tt} + c(u^\varepsilon_x + cv^\varepsilon) + \varepsilon v^\varepsilon_{xxxx} = 0, & \text{in } Q_T, \\ u^\varepsilon(0,\cdot) = v^\varepsilon(0,\cdot) = v^\varepsilon_x(0,\cdot) = v^\varepsilon(1,\cdot) = 0, & \text{in } (0,T), \\ u^\varepsilon(1,\cdot) = f^\varepsilon, v^\varepsilon_x(1,\cdot) = g^\varepsilon & \text{in } (0,T), \\ (u^\varepsilon(\cdot,0), u^\varepsilon_t(\cdot,0)) = (u_0,u_1), (v^\varepsilon(\cdot,0), v^\varepsilon_t(\cdot,0)) = (v_0,v_1), & \text{in } (0,1). \end{cases} \quad (32)$$

u^ε and v^ε denote the tangential and normal displacement of the arch.

For any $T > T^\star(c,\varepsilon)$ and $(u_0,u_1) \in H^1_0(0,1) \times L^2(0,1)$, $(v_0,v_1) \in H^2_0(0,1) \times L^2(0,1)$, (32) is null controllable through the controls f^ε and g^ε.

v^ε exhibits a boundary layer which makes the control g^ε not uniformly bounded w.r.t. ε:

$$g^\varepsilon = \varepsilon^{-1/2} g^0 + g^1 + \varepsilon^{1/2} g^2 + \cdots, \qquad f^\varepsilon = f^0 + \varepsilon^{1/2} f^1 + \cdots \quad (33)$$

The underlying limit operator involves an essential spectrum (as $\varepsilon \to 0$) $\sigma_{ess}(A_M) = \{0\}$ so that (32) is not uniformly controllable with respect to the data, as $\varepsilon \to 0$.

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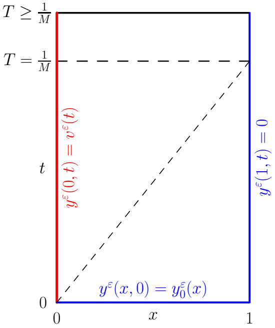

Let $T > 0$, $M \in \mathbb{R}^\star$, $\varepsilon > 0$ and $Q_T := (0,1) \times (0,T)$.

$$\begin{cases} L_\varepsilon y^\varepsilon := y_t^\varepsilon - \varepsilon y_{xx}^\varepsilon + M y_x^\varepsilon = 0, & \text{in} \quad Q_T, \\ y^\varepsilon(0,\cdot) = v^\varepsilon(t), \ y^\varepsilon(1,\cdot) = 0, & \text{in} \quad (0,T), \\ y^\varepsilon(\cdot,0) = y_0^\varepsilon, & \text{in} \quad (0,1). \end{cases} \tag{34}$$

- Well-posedness:

$$\forall y_0^\varepsilon \in H^{-1}(0,1), v^\varepsilon \in L^2(0,T), \quad \exists!\ y^\varepsilon \in L^2(Q_T) \cap C([0,T]; H^{-1}(0,1))$$

- Null controllability property: From [Fursikov'91],

$$\forall T > 0, y_0^\varepsilon \in H^{-1}(0,1), \exists v^\varepsilon \in L^2(0,T) \quad \text{s.t.} \quad y^\varepsilon(\cdot,T) = 0 \quad \text{in } H^{-1}(0,1)$$

- Main concern: Behavior of the controls v^ε as $\varepsilon \to 0$

- Remark: y^ε exhibits internal and boundary layers as $\varepsilon \to 0$ and make non trivial the analysis of the direct and control problems !

Behavior of the free solution

Lemma (Exponential decay of $\|y^\varepsilon(\cdot,T)\|_{L^2(0,T)}$ for $T > \frac{1}{|M|}$)

Let $\alpha \in [0,1)$. The free solution *(i.e. $v^\varepsilon = 0$) satisfies*

$$\|y^\varepsilon(\cdot,t)\|_{L^2(0,1)} \leq \|y_0^\varepsilon\|_{L^2(0,1)} e^{-\frac{M\alpha^2}{4\varepsilon(1-\alpha)}}, \quad \forall t \geq \frac{1}{|M|(1-\alpha)}.$$

Proposition (Amirat, Munch, 2019, Polynomial decay of $\|y^\varepsilon(\cdot,T)\|_{L^2(0,T)}$ for $T = \frac{1}{|M|}$)

Assume $M > 0$ and $v^\varepsilon \equiv 0$, $y_0^\varepsilon = y_0 \in H^3(0,1)$. For $\varepsilon > 0$ small enough, the free solution y^ε satisfies

$$\left\| y^\varepsilon\left(\cdot,\frac{1}{|M|}\right)\right\|_{L^2(0,1)} \leq c\left(|y_0(0)|\varepsilon^{1/4} + |y_0^{(1)}(0)|\varepsilon^{3/4} + |y_0^{(2)}(0)|\varepsilon^{5/4}\right) + \mathcal{O}(\varepsilon^{3/2}) \quad (35)$$

for some constant $c > 0$, independent of ε.

$\Longrightarrow$ For ε small enough, the cost of approximate controllability is zero.

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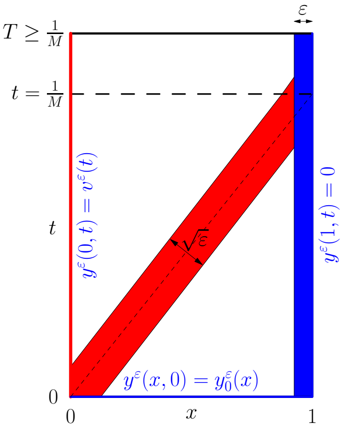


Singular layers zone for y^ε in the case $M > 0$.

Occurence of two interacting singular layers of different sizes !

Direct problem - Matched asymptotic expansion method - Case 2 - First order approximation (1)

Let

$$P^{\varepsilon} = P^0_{\varepsilon} + \sqrt{\varepsilon}P^{1/2}_{\varepsilon} + \varepsilon P^1_{\varepsilon} + \varepsilon^{3/2}P^{3/2}_{\varepsilon}$$

Theorem (Amirat, Munch, 19)

Assume $v \in H^3([0,T])$, $y_0 \in H^3([0,1])$. *Then* $\exists C > 0$ *independent of* ε *s.t.*

$$\Big\| y^{\varepsilon}(\cdot,t) - P^{\varepsilon}(\cdot,t) \Big\|_{L^2(0,1))} \leq C(\varepsilon^{3/2} + \varepsilon^{1/2} e^{-\frac{M^2}{2\varepsilon^{1/2}}t}) \quad \forall t \in [0,T]$$

and (assuming $y_0(1) = y_0'(1) = 0$*)*

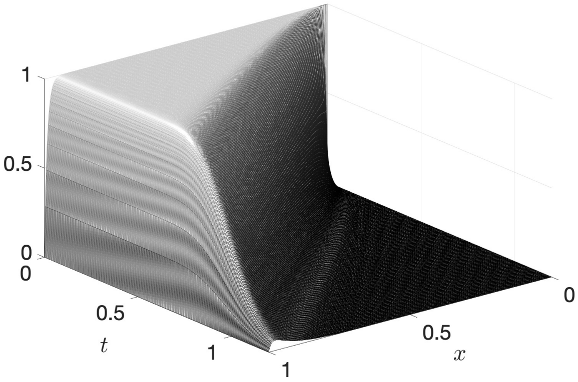
$$\|(y^{\varepsilon} - P^{\varepsilon})_x\|_{L^2(Q_T))} \leq C\varepsilon$$

As an illustration, we consider the simple case $v \equiv 0$ and $y_0 \equiv 1$ for which

$$\begin{cases} P^{\varepsilon}(x,t) = W^0_{\varepsilon}(w,t) - \Big(W^0_{\varepsilon}(M\tau,t) + \varepsilon^{1/2} z W^0_{\varepsilon,w}(M\tau,t) + \\ \qquad \varepsilon \frac{z^2}{2} W^0_{\varepsilon,ww}(M\tau,t) + \varepsilon^{3/2} \frac{z^3}{6} W^0_{\varepsilon,www}(M\tau,t) \Big) e^{-Mz}, \\ w = \frac{x - Mt}{\sqrt{\varepsilon}}, \quad M\tau = \frac{1 - Mt}{\sqrt{\varepsilon}}, \quad z = \frac{1-x}{\varepsilon}. \end{cases}$$

with

$$\begin{aligned} W^0_{\varepsilon}(w,t) = & \frac{y_0(0) - v(0)}{2} erf\left(\frac{w}{2\sqrt{t}}\right) + \frac{y_0(0) + v(0)}{2} \\ & + \frac{v(0) - y_0(0)}{2} e^{\frac{Mw}{\sqrt{\varepsilon}} + \frac{M^2 t}{\varepsilon}} erfc\left(\frac{w}{2\sqrt{t}} + \frac{M\sqrt{t}}{\sqrt{\varepsilon}}\right) \end{aligned} \tag{36}$$



P^{ε} in $(0,1)\times(0,1.2/M)$; $M=1$, $\varepsilon=10^{-2}$; $v\equiv 0$, $y_0\equiv 1$.

Asymptotic controllability property

With respect to the null controllability issue

- There is a kind of competition between the transport and the diffusion terms: as $\varepsilon \to 0$, the transport term becomes dominant, pushes the solution out of $(0,1)$ and makes $\|y^\varepsilon(\cdot, T)\|_2$ small for all $T \geq 1/|M|$. However, as $\varepsilon \to 0$, the diffusion term, which is the main tools to control to zero the solution, is small.

  Intuitively, one have to wait enough time, from $t = 1/|M|$, to control uniformly w.r.t. ε the remainder $y^\varepsilon(\cdot, 1/|M|)$.
- The negative case $M < 0$ is the "most singular" since then the transport term pushes the solution y^ε from the right to the left line $x = 0$ where the control acts. The control requires more "energy" to act on the whole spatial domain.

Theorem (Coron, Guerrero, Glass, Lissy, ...)

Define the cost of control as follows.

$$K(\varepsilon, T, M) := \sup_{\|y_0^\varepsilon\|_{L^2(0,1)}=1} \left\{ \min_{v \in C(y_0^\varepsilon, T, \varepsilon, M)} \|v\|_{L^2(0,T)} \right\}.$$

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Asymptotic controllability property

With respect to the null controllability issue

- There is a kind of competition between the transport and the diffusion terms: as $\varepsilon \to 0$, the transport term becomes dominant, pushes the solution out of $(0,1)$ and makes $\|y^\varepsilon(\cdot,T)\|_2$ small for all $T \geq 1/|M|$. However, as $\varepsilon \to 0$, the diffusion term, which is the main tools to control to zero the solution, is small.

  Intuitively, one have to wait enough time, from $t = 1/|M|$, to control uniformly w.r.t. ε the remainder $y^\varepsilon(\cdot, 1/|M|)$.
- The negative case $M < 0$ is the "most singular" since then the transport term pushes the solution y^ε from the right to the left line $x = 0$ where the control acts. The control requires more "energy" to act on the whole spatial domain.

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The optimality system associated to the control of minimal L^2-norm is

$$
\begin{cases}
y_t^\varepsilon - \varepsilon y_{xx}^\varepsilon + M y_x^\varepsilon = 0, & (x,t) \in Q_T, \\
\varphi_t^\varepsilon + \varepsilon \varphi_{xx}^\varepsilon + M \varphi_x^\varepsilon = 0, & (x,t) \in Q_T, \\
y^\varepsilon(\cdot, 0) = y_0^\varepsilon, & x \in (0,1), \\
v^\varepsilon(t) = y^\varepsilon(0,t) = \varepsilon \varphi_x^\varepsilon(0,t), & t \in (0,T), \\
y^\varepsilon(1,t) = 0, . & t \in (0,T), \\
\varphi^\varepsilon(0,t) = \varphi^\varepsilon(1,t) = 0, & t \in (0,T), \\
y^\varepsilon(\cdot, T) = 0, & x \in (0,1).
\end{cases}
$$

Main difficulty: whatever be the regularity of y_0^ε, the control of minimal $L^2(0,T)$-norm is only $L^2(0,T)$ (with weight, it can be $L^\infty(0,T)$) !?!!

A last example: dissipative wave equation

There are many others partial differential equations involving a small (singular) parameter. We mention the case of the dissipative wave equation (ω denotes an open nonempty subset of $(0,1)$)

$$\begin{cases} \varepsilon y^{\varepsilon}_{tt} + y^{\varepsilon}_{t} - y^{\varepsilon}_{xx} = v^{\varepsilon} 1_{\omega}, & \text{in } Q_T, \\ y^{\varepsilon}(0,t) = y^{\varepsilon}(1,t) = 0, & \text{in } (0,T), \\ (y^{\varepsilon}(\cdot,0), y^{\varepsilon}_{t}(\cdot,0)) = (y_0, y_1), & \text{in } (0,1) \end{cases}$$

controllable for any $\varepsilon > 0$ and for which one can find a sequence of controls $\{v^{\varepsilon}\}_{\varepsilon>0}$ which converges to a null control for the heat equation (we refer to [Lopez-Zuazua 2006] using spectral arguments).

As $\varepsilon \to 0$, an initial singular layer at $t = 0$ is developed ! Rate of convergence of v^{ε} ?

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THANK YOU FOR YOUR ATTENTION