About the controllability of $y_t - \varepsilon y_{xx} + M y_x = 0$ w.r.t. ε

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Nancy - IECN - November 7th 2017

Let $T > 0$, $M \in \mathbb{R}$, $\varepsilon > 0$ and $Q_T := (0,1) \times (0,T)$.

$$\begin{cases} L_\varepsilon y^\varepsilon := y_t^\varepsilon - \varepsilon y_{xx}^\varepsilon + M y_x^\varepsilon = 0 & Q_T, \\ y^\varepsilon(0,\cdot) = v^\varepsilon(t), \ y^\varepsilon(1,\cdot) = 0 & (0,T), \\ y^\varepsilon(\cdot,0) = y_0^\varepsilon & (0,1), \end{cases} \tag{1}$$

- Well-posedness:

$$\forall y_0^\varepsilon \in H^{-1}(0,1), v^\varepsilon \in L^2(0,T), \quad \exists! \ y^\varepsilon \in L^2(Q_T) \cap \mathcal{C}([0,T]; H^{-1}(0,1))$$

- Null control property: From (Russel'78),

$$\forall T > 0, y_0 \in H^{-1}(0,1), \exists v^\varepsilon \in L^2(0,T) \quad \text{such that}$$

$$y^\varepsilon(\cdot,T) = 0 \quad \text{in } H^{-1}(0,1).$$

- Main concern: Behavior of the controls v^ε as $\varepsilon \to 0$
  - Controllability of conservation law system;
  - Toy model for fluids when Navier-Stokes $\to$ Euler.

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• We note the non empty set of null controls by

$$\mathcal{C}(y_0, T, \varepsilon, M) := \{v \in L^2(0,T); y = y(v) \text{ solves } (5) \text{ and satisfies } y(\cdot, T) = 0\}$$

For any $\varepsilon > 0$, we define the cost of control by the following quantity :

$$K(\varepsilon, T, M) := \sup_{\|y_0\|_{L^2(0,1)}=1} \left\{ \min_{u \in \mathcal{C}(y_0,T,\varepsilon,M)} \|u\|_{L^2(0,T)} \right\}.$$

• We denote by T_M the minimal time for which the cost $K(\varepsilon, T, M)$ is uniformly bounded with respect to ε. In other words, (5) is uniformly controllable with respect to ε if and only if $T \geq T_M$.

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Remark

Let v^ε the control of minimal L^2-norm for the initial data y_0^ε: then

$$\|v^\varepsilon\|_{L^2(0,T)} \leq K(\varepsilon, T, M)\|y_0^\varepsilon\|_{L^2(0,1)}$$

Thus, $K(\varepsilon, T, M)$ is the norm of the (linear) operator $y_0^\varepsilon \to v_{HUM}$ where v_{HUM} is the control of minimal L^2-norm.

Remark

By duality, the controllability property is related to the existence of $C > 0$ such that

$$\|\varphi(\cdot,0)\|_{L^2(0,1)} \leq C\|\varepsilon\varphi_x(0,\cdot)\|_{L^2(0,T)}, \quad \forall \varphi_T \in H_0^1(0,1) \cap H^2(0,1) \tag{2}$$

The quantity

$$C_{obs}(\varepsilon, T, M) = \sup_{\varphi_T \in H_0^1(0,1)} \frac{\|\varphi(\cdot,0)\|_{L^2(0,1)}}{\|\varepsilon\varphi_x(0,\cdot)\|_{L^2(0,T)}}.$$

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The case $M = 0$

$$\begin{cases} y_t^\varepsilon - \varepsilon y_{xx}^\varepsilon = 0 & Q_T, \\ y^\varepsilon(0,\cdot) = v^\varepsilon(t),\ y^\varepsilon(1,\cdot) = 0 & (0,T), \\ y^\varepsilon(\cdot,0) = y_0^\varepsilon & (0,1), \end{cases}$$

Making the change of variable $\tilde{t} = \varepsilon t$, we easily have

Lemma

$\forall T > 0, \varepsilon > 0, \quad K(\varepsilon, T, 0) = \varepsilon^{-1/2} K(1, \varepsilon T, 0)$

Theorem (Miller, Lissy, Tenebaum-Tucsnak, ...)

$$K(1,T,0) \sim_{T\to 0^+} e^{\frac{\kappa}{T}}, \quad \kappa \in (1/2, 3/4)$$

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Theorem (Coron- Guerrero)

Let $T > 0$, $M \in \mathbb{R}^\star$, $y_0 \in L^2(0,1)$. Let $(v^\varepsilon)_{(\varepsilon)}$ be a sequence of functions in $L^2(0,T)$ such that for some $v \in L^2(0,T)$

$$v^\varepsilon \rightharpoonup v \quad in \quad L^2(0,T), \quad \varepsilon \to 0^+.$$

For $\varepsilon > 0$, let us denote by $y^\varepsilon \in C([0,T]; H^{-1}(0,1))$ the weak solution of

$$\begin{cases} y_t^\varepsilon - \varepsilon y_{xx}^\varepsilon + M y_x^\varepsilon = 0 & Q_T, \\ y^\varepsilon(0,\cdot) = v^\varepsilon(t), \; y^\varepsilon(1,\cdot) = 0 & (0,T), \\ y^\varepsilon(\cdot,0) = y_0 & (0,1), \end{cases} \tag{3}$$

Let $y \in C([0,T]; L^2(0,1))$ be the weak solution of

$$\begin{cases} y_t + M y_x = 0 & Q_T, \\ y(0,\cdot) = v(t) \quad if \quad M > 0 & (0,T), \\ y(1,\cdot) = 0 \quad if \quad M < 0 & (0,T), \\ y(\cdot,0) = y_0 & (0,1), \end{cases} \tag{4}$$

Then, $y^\varepsilon \rightharpoonup y \quad in \quad L^2(Q_T) \quad as \quad \varepsilon \to 0^+$.

Theorem

If $T < \frac{1}{|M|}$, $\lim_{\varepsilon\to 0} K(\varepsilon, T, M) \to \infty$. Consequently, $T_M \geq \frac{1}{|M|}$.

PROOF. Assume that $K(\varepsilon, T, M) \not\to +\infty$. There exists $(\varepsilon_n)_{(n\in\mathbb{N})}$ positive tending to 0 such that $(K(\varepsilon_n, T, M))_{(n\in\mathbb{N})}$ is bounded.

Let v^{ε_n} the optimal control driving y_0 to 0 at time T and y^{ε_n} the corresponding solution. Let $T_0 \in (T, 1/|M|)$. We extend y^{ε_n} and v^{ε_n} by 0 on (T, T_0). From the inequality

$$\|v^{\varepsilon_n}\|_{L^2(0,T_0)} = \|v^{\varepsilon_n}\|_{L^2(0,T)} \leq K(\varepsilon^n, T, M)\|y_0\|_{L^2(0,1)},$$

we deduce that $(v^{\varepsilon_n})_{(n\in\mathbb{N})}$ is bounded in $L^2(0, T_0)$, so we extract a subsequence $(v^{\varepsilon_n})_{(n\in\mathbb{N})}$ such that $v^{\varepsilon_n} \rightharpoonup v$ in $L^2(0, T_0)$. We deduce that $y^{\varepsilon_n} \rightharpoonup y$ in $L^2(Q_{T_0})$ solution of the transport equation. Necessarily, $y \equiv 0$ on $(0,1) \times (T, T_0)$. Contradiction.

Lemma

The uncontrolled solution ($v^\varepsilon = 0$) satisfies

$$\|y^\varepsilon(\cdot, t)\|_{L^2(0,1)} \leq \|y^\varepsilon(\cdot, 0)\|_{L^2(0,1)} e^{-\frac{M^2}{4\varepsilon}(t-\frac{1}{M})^2}, \quad \forall t > \frac{1}{M}$$

PROOF. Let $\alpha > 0$. We check $z^\varepsilon(x,t) = e^{\frac{-M\alpha x}{2\varepsilon}} y^\varepsilon(x,t)$ solves

$$\begin{cases} z^\varepsilon_t - \varepsilon z^\varepsilon_{xx} + M(1-\alpha) z^\varepsilon_x - \frac{M^2}{4\varepsilon}(\alpha^2 - 2\alpha) z^\varepsilon = 0 & \text{in} \quad Q_T, \\ z^\varepsilon(0,\cdot) = z^\varepsilon(1,\cdot) = 0 & \text{on} \quad (0,T), \\ z^\varepsilon(\cdot,0) = e^{\frac{-M\alpha x}{2\varepsilon}} y^\varepsilon_0 & \text{in} \quad (0,L), \end{cases} \tag{5}$$

Consequently

$$\|e^{-\frac{M\alpha x}{2\varepsilon}} y^\varepsilon(\cdot,t)\|_{L^2(0,1)} \leq \|e^{-\frac{M\alpha x}{2\varepsilon}} y^\varepsilon(\cdot,0)\|_{L^2(0,1)} e^{\frac{M^2}{4\varepsilon}(\alpha^2 - 2\alpha)t}$$

$$\begin{aligned} \|y^\varepsilon(\cdot,t)\|_{L^2(0,1)} &\leq \|e^{+\frac{M\alpha x}{2\varepsilon}}\|_{L^\infty(0,1)} \|e^{-\frac{M\alpha x}{2\varepsilon}} y^\varepsilon(\cdot,t)\|_{L^2(0,1)} \\ &\leq \|e^{+\frac{M\alpha x}{2\varepsilon}}\|_{L^\infty(0,1)} \|e^{-\frac{M\alpha x}{2\varepsilon}} y^\varepsilon(\cdot,0)\|_{L^2(0,1)} e^{\frac{M^2}{4\varepsilon}(\alpha^2 - 2\alpha)t} \\ &\leq \|y^\varepsilon(\cdot,0)\|_{L^2(0,1)} e^{\frac{M\alpha}{2\varepsilon}\left(1 - Mt + \frac{M\alpha}{2}\right)} \end{aligned}$$

and the result with $\alpha = t - \frac{1}{M} > 0$.

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Lower bounds for T_M

We expect $T_M = \frac{L}{|M|}$ and that $\lim_{\varepsilon\to 0} K(\varepsilon, T, M) = 0^+$ because the transport eq. is null controlled at time $T \geq \frac{1}{|M|}$ with $v \equiv 0$! But,

Theorem (Coron-Guerrero'2006)

- *If $M > 0$, then $K(\varepsilon, T, M) \geq Ce^{c/\varepsilon}$, $c, C > 0$, when $\varepsilon \to 0$ for $T < \frac{L}{M}$.*
- *If $M < 0$, then $K(\varepsilon, T, M) \geq Ce^{c/\varepsilon}$, $c, C > 0$, when $\varepsilon \to 0$ for $T < 2\frac{L}{|M|}$.*

More precisely, the lower bound are obtained using specific initial condition:

$$y_0(x) = K_\varepsilon e^{-\frac{Mx}{2\varepsilon}} \sin(\pi x), \quad (K_\varepsilon = \mathcal{O}(\varepsilon^{-3/2}) \quad \text{s.t.} \quad \|y_0\|_{L^2(0,1)} = 1)$$

leading, for $M > 0$, to

$$K(\varepsilon, T, M) \geq C_1 \frac{\varepsilon^{-3/2} T^{-1/2} M^2}{1 + M^3 \varepsilon^{-3}} \exp\left(\frac{M}{2\varepsilon}(1 - TM) - \pi^2 \varepsilon T\right)$$

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Theorem (Coron-Guerrero'2006)

- *If* $M > 0$, *then* $K(\varepsilon, T, M) \leq Ce^{-c/\varepsilon}$ *when* $\varepsilon \to 0$ *for* $T \geq \frac{4.3}{M}$.
- *If* $M < 0$, *then* $K(\varepsilon, T, M) \leq Ce^{-c/\varepsilon}$ *when* $\varepsilon \to 0$ *for* $T \geq \frac{57.2}{|M|}$.

Estimates for T_M

Theorem (Coron-Guerrero'2006)

$$T_M \in [1, 4.3]\frac{1}{M} \quad \textit{if} \quad M > 0, \qquad [2, 57.2]\frac{1}{|M|} \quad \textit{if} \quad M < 0.$$

Theorem (Glass'2009)

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Theorem (Lissy'2015)

$$T_M \in [1, 2\sqrt{3}]\frac{1}{M} \quad \textit{if} \quad M > 0, \qquad [2\sqrt{2}, 2(1+\sqrt{3})]\frac{1}{|M|} \quad \textit{if} \quad M < 0.$$

($2\sqrt{3} \approx 3.46$)

Theorem (Darde-Ervedoza'2017)

$$T_M \in [1, K]\frac{1}{M} \quad \textit{if} \quad M > 0, K \approx 3.34$$

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$$T_M \in [1, 4.3]\frac{1}{M} \quad \textit{if} \quad M > 0, \qquad [2, 57.2]\frac{1}{|M|} \quad \textit{if} \quad M < 0.$$

Theorem (Glass'2009)

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Theorem (Lissy'2015)

$$T_M \in [1, 2\sqrt{3}]\frac{1}{M} \quad \textit{if} \quad M > 0, \qquad [2\sqrt{2}, 2(1+\sqrt{3})]\frac{1}{|M|} \quad \textit{if} \quad M < 0.$$

$(2\sqrt{3} \approx 3.46)$

Theorem (Darde-Ervedoza'2017)

$$T_M \in [1, K]\frac{1}{M} \quad \textit{if} \quad M > 0, K \approx 3.34$$

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Objective:

estimate the uniform minimal control time T_M !!??

We can try the following two approaches :

- Numerical estimation of $K(\varepsilon, T, M)$ with respect to ε and $T \geq \frac{1}{M}$ (for $M > 0$ and $M < 0$)
- Asymptotic analysis with respect to the parameter ε of the corresponding optimality system.

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Attempt 1 : Numerical estimation of $K(\varepsilon, T, M)$

$$K^2(\varepsilon, T, M) = \sup_{y_0 \in L^2(0,1)} \frac{(\mathcal{A}_\varepsilon y_0, y_0)_{L^2(0,1)}}{(y_0, y_0)_{L^2(0,1)}}$$

where $\mathcal{A}_\varepsilon : L^2(0,1) \to L^2(0,1)$ is the control operator defined by $\mathcal{A}_\varepsilon y_0 := -\hat{\varphi}(0)$ where $\hat{\varphi}$ solves the adjoint system

$$\begin{cases} -\varphi_t - \varepsilon\varphi_{xx} - M\varphi_x = 0 & \text{in} \quad Q_T, \\ \varphi(0,\cdot) = \varphi(1,\cdot) = 0 & \text{on} \quad (0,T), \\ \varphi(\cdot,T) = \varphi_T & \text{in} \quad (0,1), \end{cases} \tag{6}$$

associated to the initial condition $\varphi_T \in H_0^1(0,1)$, solution of the extremal problem

$$\inf_{\varphi_T \in H_0^1(0,L)} J^\star(\varphi_T) := \frac{1}{2}\int_0^T (\varepsilon\varphi_x(0,\cdot))^2 dt + (y_0, \varphi(\cdot,0))_{L^2(0,T)}.$$

REFORMULATION - $K(\varepsilon, T, M)$ is solution of the generalized eigenvalue problem :

$$\sup\left\{\sqrt{\lambda} \in \mathbb{R} : \exists\, y_0 \in L^2(0,1), y_0 \neq 0, \text{ s.t. } \mathcal{A}_\varepsilon y_0 = \lambda y_0 \quad \text{in} \quad L^2(0,1)\right\}.$$

The generalized eigenvalue problem by the power iterated method

In order to get the largest eigenvalue of the operator $\mathcal{A}_\varepsilon$, we may employ the power iterate method (Chatelain'89):

$$\begin{cases} y_0^0 \in L^2(0,1) \quad \text{given such that} \quad \|y_0^0\|_{L^2(0,1)} = 1, \\ \tilde{y}_0^{k+1} = \mathcal{A}_\varepsilon y_0^k, \quad k \geq 0, \\ y_0^{k+1} = \dfrac{\tilde{y}_0^{k+1}}{\|\tilde{y}_0^{k+1}\|_{L^2(0,1)}}, \quad k \geq 0. \end{cases} \tag{7}$$

The real sequence $\{\|\tilde{y}_0^k\|_{L^2(0,1)}\}_{k>0}$ converges to the eigenvalue with largest module of the operator $\mathcal{A}_\varepsilon$:

$$\sqrt{\|\tilde{y}_0^k\|_{L^2(0,1)}} \to K(\varepsilon, T, M) \quad \text{as} \quad k \to \infty. \tag{8}$$

The L^2 sequence $\{y_0^k\}_k$ then converges toward the corresponding eigenvector.

The first step requires to compute the image of $\mathcal{A}_\varepsilon$: this is done by determining the control of minimal L^2 norm by minimizing $J^\star$ with y_0^k as initial condition for (5).

Computation of the control of minimal L^2-norm

For a fixed initial data $y^0 \in L^2(0,1)$ and ε small, the numerical approximation of controls of minimal L^2-norm is a serious challenge :

- the minimization of $J^\star$ is ill-posed : the infimum φ_T lives in a huge dual space !!! this implies that the minimizer φ_T is highly oscillating at time T leading to highly oscillation of the control $\varepsilon\varphi_{,x}$.
- Tychonoff like regularization

$$\inf_{\varphi_T \in H_0^1(0,1)} J_\beta^\star(\varphi_T) := J^\star(\varphi_T) + \beta\|\varphi_T\|_{H_0^1(0,1)} \longrightarrow \|y^\varepsilon(\cdot,T)\|_{H^{-1}(0,1)} \leq \beta \quad (9)$$

  is meaningless here for $T > 1/|M|$ because the uncontrolled solution $y^\varepsilon(\cdot,T)$ goes to zero with ε.
- Boundary layers occurs for y^ε and φ^ε on the boundary and requires fine discretization.

We use the variational approach developed in [Fernandez-Cara-Munch, 2013], [De Souza-Munch, 2015] leading to convergent approximation with respect to the discretization parameter (ε being fixed).

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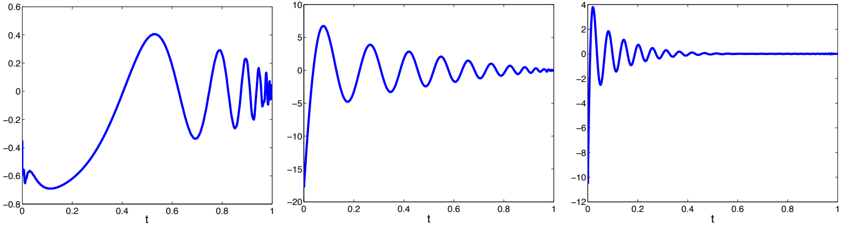
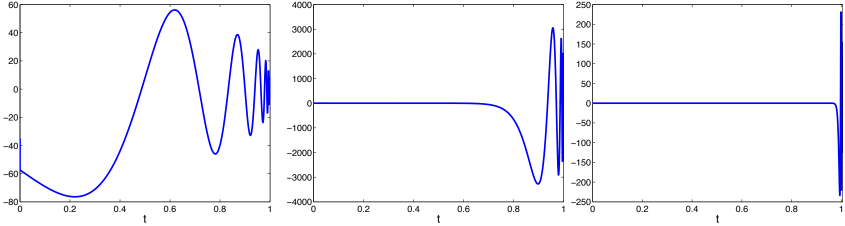
$$y_0(x) = \sin(\pi x); \quad T = 1; \quad M = 1$$



Control of minimal $L^2(0,T)$-norm $v^\varepsilon(t) \in [0,T]$ for $\varepsilon = 10^{-1}, 10^{-2}$ and 10^{-3}

$$y_0(x) = \sin(\pi x); \quad T = 1; \quad M = -1$$



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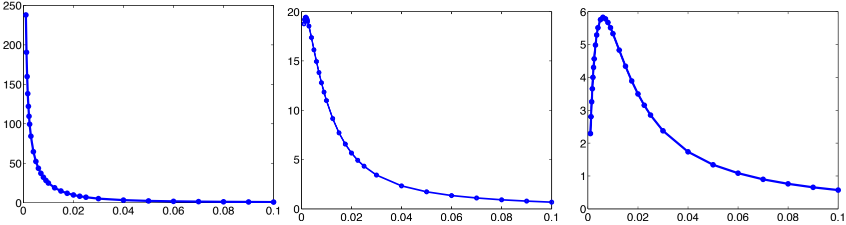


Cost of control w.r.t. ε for $T = 0.95\frac{1}{M}$, $T = \frac{1}{M}$ and $T = 1.05\frac{1}{M}$

In agreement with Coron-Guerrero'2006,

$$K(\varepsilon, T, M) \geq C_1 \frac{\varepsilon^{-3/2} T^{-1/2} M^2}{1 + M^3 \varepsilon^{-3}} \exp\left(\frac{M}{2\varepsilon}(1 - TM) - \pi^2 \varepsilon T\right) \qquad (10)$$

Corresponding worst initial condition

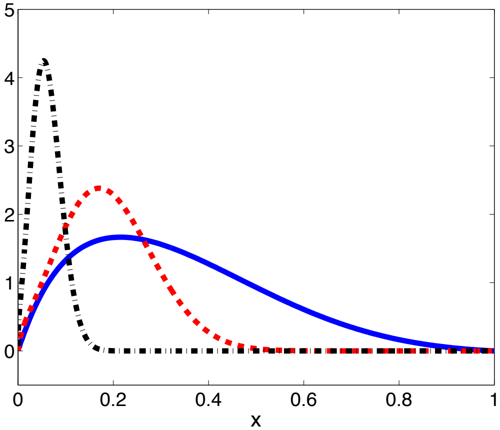
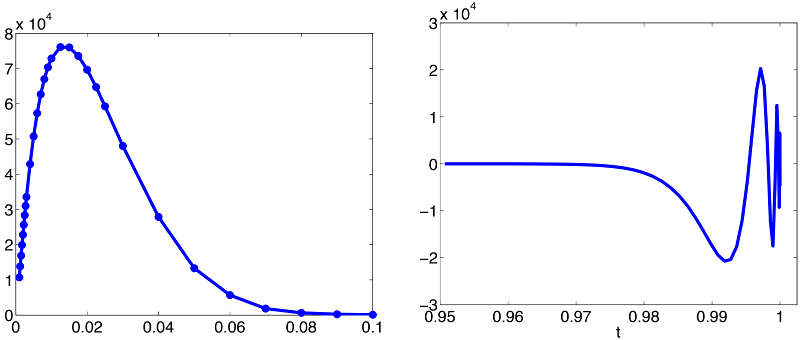


Figure: $T = 1$ - $M = 1$ - The optimal initial condition y_0 in $(0, 1)$ for $\varepsilon = 10^{-1}$ (full line), $\varepsilon = 10^{-2}$ (dashed line) and $\varepsilon = 10^{-3}$ (dashed-dotted line).

$\Longrightarrow y_0$ is close to $e^{-\frac{Mx}{2\varepsilon}} \sin(\pi x)/\|e^{-\frac{Mx}{2\varepsilon}} \sin(\pi x)\|_{L^2(0,1)}$



Left: Cost of control w.r.t. ε for $T = \frac{1}{|M|}$; **Right**: Corresponding control v^{ε} in the neighborhood of T for $\varepsilon = 10^{-3}$

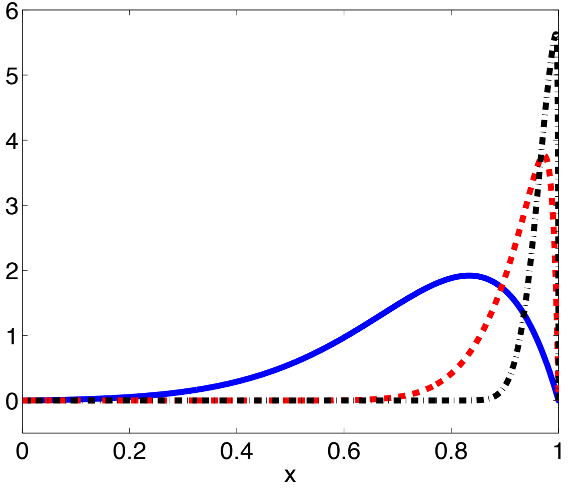


Figure: $T = 1$ - $M = -1$ - The optimal initial condition y_0 in $(0, 1)$ for $\varepsilon = 10^{-1}$ (full line), $\varepsilon = 10^{-2}$ (dashed line) and $\varepsilon = 10^{-3}$ (dashed-dotted line).

Attempt 2 : Asymptotic analysis w.r.t. ε

We take $M > 0$.

Optimality system :

$$\begin{cases} L_\varepsilon y^\varepsilon = 0, \quad L_\varepsilon^\star \varphi^\varepsilon = 0, & (x,t) \in Q_T, \\ y^\varepsilon(\cdot,0) = y_0^\varepsilon, & x \in (0,1), \\ v^\varepsilon(t) = y^\varepsilon(0,t) = \varepsilon \varphi_x^\varepsilon(0,t), & t \in (0,T), \\ y^\varepsilon(1,t) = 0, . & t \in (0,T), \\ \varphi^\varepsilon(0,t) = \varphi^\varepsilon(1,t) = 0, & t \in (0,T), \\ -\beta \varphi_{xx}^\varepsilon(\cdot,T) + y^\varepsilon(\cdot,T) = 0, & x \in (0,1). \end{cases} \tag{11}$$

$\beta > 0$- Regularization parameter

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$\beta > 0$- Regularization parameter

$$\begin{cases} y_t^\varepsilon - \varepsilon y_{xx}^\varepsilon + M y_x^\varepsilon = 0, & (x,t) \in (0,1) \times (0,T), \\ y^\varepsilon(0,t) = v^\varepsilon(t), & t \in (0,T), \\ y^\varepsilon(1,t) = 0, & t \in (0,T), \\ y^\varepsilon(x,0) = y_0(x), & x \in (0,1), \end{cases} \tag{12}$$

y_0 and v^ε are given functions.

We assume that

$$v^\varepsilon = \sum_{k=0}^{m} \varepsilon^k v^k,$$

the functions $v^0, v^1, \cdots, v^m$ being known.

We construct an asymptotic approximation of the solution y^ε of (12) by using the matched asymptotic expansion method.

Direct problem - Asymptotic expansion

Let us consider two formal asymptotic expansions of y^ε:

– the outer expansion

$$\sum_{k=0}^{m} \varepsilon^k y^k(x,t), \quad (x,t) \in (0,T),$$

– the inner expansion

$$\sum_{k=0}^{m} \varepsilon^k Y^k(z,t), \quad z = \frac{1-x}{\varepsilon} \in (0,\varepsilon^{-1}),\ t \in (0,T).$$

Direct problem - Outer expansion

Putting $\sum_{k=0}^{m} \varepsilon^k y^k(x,t)$ into equation $(12)_1$, the identification of the powers of ε yields

$$\begin{aligned} \varepsilon^0 : &\quad y_t^0 + M y_x^0 = 0, \\ \varepsilon^k : &\quad y_t^k + M y_x^k = y_{xx}^{k-1}, \quad \text{for any } 1 \le k \le m. \end{aligned}$$

Taking the initial and boundary conditions into account we define y^0 and y^k $(1 \le k \le m)$ as functions satisfying the transport equations, respectively,

$$\begin{cases} y_t^0 + M y_x^0 = 0, & (x,t) \in Q_T, \\ y^0(0,t) = v^0(t), & t \in (0,T), \\ y^0(x,0) = y_0(x), & x \in (0,1), \end{cases} \tag{13}$$

and

$$\begin{cases} y_t^k + M y_x^k = y_{xx}^{k-1}, & (x,t) \in Q_T, \\ y^k(0,t) = v^k(t), & t \in (0,T), \\ y^k(x,0) = 0, & x \in (0,1). \end{cases} \tag{14}$$

Direct problem - Outer expansion

$$y^0(x,t) = \begin{cases} y_0(x - Mt) & x > Mt, \\ v^0\left(t - \dfrac{x}{M}\right), & x < Mt. \end{cases}$$

Using the method of characteristics we find that, for any $1 \leq k \leq m$,

$$y^k(x,t) = \begin{cases} \displaystyle\int_0^t y_{xx}^{k-1}(x + (s-t)M, s)ds, & x > Mt, \\ v^k\left(t - \dfrac{x}{M}\right) + \displaystyle\int_0^{x/M} y_{xx}^{k-1}(sM, t - \dfrac{x}{M} + s)ds, & x < Mt. \end{cases}$$

Remark

$$y^1(x,t) = \begin{cases} t\, y_0''(x - Mt), & x > Mt, \\ v^1\left(t - \dfrac{x}{M}\right) + \dfrac{x}{M^3}(v^0)''\left(t - \dfrac{x}{M}\right), & x < Mt, \end{cases}$$

$$y^2(x,t) = \begin{cases} \dfrac{t^2}{2} y_0^{(4)}(x - Mt), & x > Mt, \\ v^2\left(t - \dfrac{x}{M}\right) + \dfrac{x}{M^3}(v^1)''\left(t - \dfrac{x}{M}\right) \\ - \dfrac{2x}{M^5}(v^0)^{(3)}\left(t - \dfrac{x}{M}\right) + \dfrac{x^2}{2M^6}(v^0)^{(4)}\left(t - \dfrac{x}{M}\right), & x < Mt. \end{cases}$$

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Now we turn back to the construction of the inner expansion. Putting $\sum_{k=0}^{m} \varepsilon^k Y^k(z,t)$ into equation $(12)_1$, the identification of the powers of ε yields

$$\begin{aligned}
\varepsilon^{-1}: &\quad Y^0_{zz}(z,t) + MY^0_z(z,t) = 0,\\
\varepsilon^{k-1}: &\quad Y^k_{zz}(z,t) + MY^k_z(z,t) = Y^{k-1}_t(z,t), \quad \text{for any } 1 \le k \le m.
\end{aligned}$$

We impose that $Y^k(0,t) = 0$ for any $0 \le k \le m$ and use the asymptotic matching conditions

$$\begin{aligned}
&Y^0(z,t) \sim y^0(1,t), \quad \text{as } z \to +\infty,\\
&Y^1(z,t) \sim y^1(1,t) - y^0_x(1,t)z, \quad \text{as } z \to +\infty,\\
&Y^2(z,t) \sim y^2(1,t) - y^1_x(1,t)z + \frac{1}{2} y^0_{xx}(1,t)z^2, \quad \text{as } z \to +\infty,\\
&\dots\\
&Y^m(z,t) \sim y^m(1,t) - y^{m-1}_x(1,t)z + \frac{1}{2} y^{m-2}_{xx}(1,t)z^2 + \cdots + \frac{1}{m!}(y^0)^{(m)}_x(1,t)(-z)^m,\\
&\text{as } z \to +\infty.
\end{aligned}$$

Lemma

$$Y^0(z,t) = y^0(1,t)\left(1 - e^{-Mz}\right), \quad (z,t) \in (0,+\infty) \times (0,T).$$

For any $1 \leq k \leq m$*, the solution of reads*

$$Y^k(z,t) = Q^k(z,t) + e^{-Mz}P^k(z,t), \quad (z,t) \in (0,+\infty) \times (0,T), \tag{15}$$

where

$$P^k(z,t) = -\sum_{i=0}^{k} \frac{1}{i!}\frac{\partial^i y^{k-i}}{\partial x^i}(1,t)z^i, \quad Q^k(z,t) = \sum_{i=0}^{k} \frac{(-1)^i}{i!}\frac{\partial^i y^{k-i}}{\partial x^i}(1,t)z^i.$$

Asymptotic approximation

Let $\mathcal{X} : \mathbb{R} \to [0,1]$ denote a C^2 cut-off function satisfying

$$\mathcal{X}(s) = \begin{cases} 1, & s \geq 2, \\ 0, & s \leq 1, \end{cases}$$

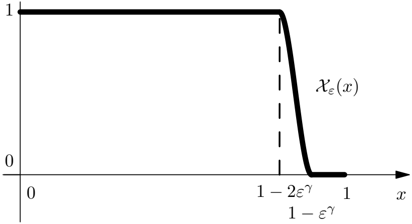


Figure: The function $\mathcal{X}_\varepsilon$.

We define, for $\gamma \in (0,1)$, the function $\mathcal{X}_\varepsilon(x) = \mathcal{X}\left(\frac{1-x}{\varepsilon^\gamma}\right)$, then introduce the function

$$w_m^\varepsilon(x,t) = \mathcal{X}_\varepsilon(x) \sum_{k=0}^{m} \varepsilon^k y^k(x,t) + (1 - \mathcal{X}_\varepsilon(x)) \sum_{k=0}^{m} \varepsilon^k Y^k\left(\frac{1-x}{\varepsilon}, t\right), \qquad (16)$$

defined to be an asymptotic approximation at order m of the solution y^ε of (12).

Lemma

(i) Assume that $y_0 \in C^{2m+1}[0,1]$, $v^0 \in C^{2m+1}[0,T]$ *and the following* C^{2m+1}*-matching conditions are satisfied*

$$M^p (y_0)^{(p)}(0) + (-1)^{p+1}(v^0)^{(p)}(0) = 0, \quad 0 \le p \le 2m+1. \tag{17}$$

Then the function y^0 *belongs to* $C^{2m+1}([0,1]\times[0,T])$.
(ii) Additionally, assume that $v^k \in C^{2(m-k)+1}[0,T]$, *and the following* $C^{2(m-k)+1}$*-matching conditions are satisfied, respectively,*

$$(v^k)^{(p)}(0) = \sum_{i+j=p-1} (-1)^i M^i \frac{\partial^{p+1} y^{k-1}}{\partial x^{i+2}\partial t^j}(0,0), \quad 0 \le p \le 2(m-k)+1. \tag{18}$$

Then the function y^k *belongs to* $C^{2(m-k)+1}([0,1]\times[0,T])$.

Compatibility conditions - example with $m = 2$

Lemma

(i) Assume that $y_0 \in C^5[0,1]$, $v^0 \in C^5[0,T]$ *and the following* C^5*-matching conditions are satisfied*

$$M^p (y_0)^{(p)}(0) + (-1)^{p+1}(v^0)^{(p)}(0) = 0, \quad 0 \le p \le 5. \tag{19}$$

Then the function y^0 *belongs to* $C^5([0,1] \times [0,T])$.
(ii) Additionally, assume that $v^1 \in C^3[0,T]$, $v^2 \in C^1[0,T]$ *and the following* C^3 *and* C^1*-matching conditions are satisfied, respectively,*

$$\begin{cases} v^1(0) = 0, \quad (v^1)^{(1)}(0) = M^{-2}(v^0)^{(2)}(0) = y_0^{(2)}(0), \\ (v^1)^{(2)}(0) = 2M^{-2}(v^0)^{(3)}(0) = -2My_0^{(3)}(0), \\ (v^1)^{(3)}(0) = 3M^{-2}(v^0)^{(4)}(0) = 3M^2 y_0^{(4)}(0), \end{cases} \tag{20}$$

$$v^2(0) = 0, \quad (v^2)^{(1)}(0) = 0. \tag{21}$$

Then the function y^1 *belongs to* $C^3([0,1] \times [0,T])$, *and the function* y^2 *belongs to* $C^1([0,1] \times [0,T])$.

Lemma

Let w_m^ε be the function defined by (16). *Assume the previous regularity and compatibility. Then there is a constant c_m independent of ε such that*

$$\|L_\varepsilon(w_m^\varepsilon)\|_{C([0,T];L^2(0,1))} \leq c_m\,\varepsilon^{\frac{(2m+1)\gamma}{2}}. \tag{22}$$

$L_\varepsilon(w_m^\varepsilon)(x,t) = \sum_{i=1}^5 J_\varepsilon^i(x,t)$,

$$J_\varepsilon^1(x,t) = -\varepsilon^{m+1} y_{xx}^m(x,t)\mathcal{X}_\varepsilon(x),$$

$$J_\varepsilon^2(x,t) = \varepsilon^m(1-\mathcal{X}_\varepsilon(x))Y_t^m\left(\frac{1-x}{\varepsilon},t\right),$$

$$J_\varepsilon^3(x,t) = M\mathcal{X}'\left(\frac{1-x}{\varepsilon^\gamma}\right)\varepsilon^{-\gamma}\left(\sum_{k=0}^m \varepsilon^k Y^k\left(\frac{1-x}{\varepsilon},t\right) - \sum_{k=0}^m \varepsilon^k y^k(x,t)\right),$$

$$J_\varepsilon^4(x,t) = \mathcal{X}''\left(\frac{1-x}{\varepsilon^\gamma}\right)\varepsilon^{1-2\gamma}\left(\sum_{k=0}^m \varepsilon^k Y^k\left(\frac{1-x}{\varepsilon},t\right) - \sum_{k=0}^m \varepsilon^k y^k(x,t)\right),$$

$$J_\varepsilon^5(x,t) = 2\mathcal{X}'\left(\frac{1-x}{\varepsilon^\gamma}\right)\varepsilon^{1-\gamma}\left(\varepsilon^{-1}\sum_{k=0}^m \varepsilon^k Y_z^k\left(\frac{1-x}{\varepsilon},t\right) + \sum_{k=0}^m \varepsilon^k y_x^k(x,t)\right).$$

Theorem

Let y^ε be the solution of problem (12) *and let w_m^ε be the function defined by* (16). *Assume that the assumptions of Lemma 6 hold true. Then there is a constant c_m independent of ε such that*

$$\|y^\varepsilon - w_m^\varepsilon\|_{C\left([0,T];L^2(0,1)\right)} \leq c_m \varepsilon^{\frac{2m+1}{2}\gamma}. \tag{23}$$

Introduce the function $z^\varepsilon = y^\varepsilon - w_m^\varepsilon$. It satisfies

$$\begin{cases} L_\varepsilon(z^\varepsilon) = -L_\varepsilon(w_m^\varepsilon), & (x,t) \in (0,1)\times(0,t), \\ z^\varepsilon(0,t) = z^\varepsilon(1,t) = 0, & t \in (0,T), \\ z^\varepsilon(x,0) = z_0^\varepsilon(x), & x \in (0,1), \end{cases} \tag{24}$$

Using the Gronwall Lemma it holds that

$$\|z^\varepsilon(\cdot,t)\|^2_{L^2(0,1)} \leq \left(\|L_\varepsilon(w_m^\varepsilon)\|^2_{L^2(Q_T)} + \|z_0^\varepsilon\|^2_{L^2(0,1)}\right) e^t, \quad \forall t \in (0,T]. \tag{25}$$

We have

$$z_0^\varepsilon(x) = (1-\mathcal{X}_\varepsilon(x))\left(y_0(x) - \sum_{k=0}^{m} \varepsilon^k Y^k\left(\frac{1-x}{\varepsilon}, 0\right)\right),$$

Convergence with respect to m

Theorem

Let, for any $m \in \mathbb{N}$, y_m^ε denote the solution of problem (12). We assume that the initial condition y_0 belongs to $C^\infty[0,1]$ and there are $c, b \in \mathbb{R}$ such that

$$\|y_0^{(m)}\|_{C[0,1]} \leq c\, b^m, \quad \forall m \in \mathbb{N}. \tag{26}$$

We assume that $(v^k)_{k\geq 0}$ is a sequence of polynomials of degree $\leq p-1$, $p \geq 1$, uniformly bounded in $C^{p-1}[0,T]$. We assume in addition that, for any $k \in \mathbb{N}$, for any $m \in \mathbb{N}$, the functions v^k and y_0 satisfy the matching conditions. Let w_m^ε be the function defined by (16). Then, there is $\varepsilon_0 > 0$ such that, for any fixed $0 < \varepsilon < \varepsilon_0$, we have

$$y_m^\varepsilon - w_m^\varepsilon \to 0 \quad \text{in } C([0,T], L^2(0,1)), \quad \text{as } m \to +\infty,$$

consequently

$$\lim_{m\to+\infty} w_m^\varepsilon(x,t) = \mathcal{X}_\varepsilon(x) \sum_{k=0}^{\infty} \varepsilon^k y^k(x,t) + (1-\mathcal{X}_\varepsilon(x)) \sum_{k=0}^{\infty} \varepsilon^k Y^k\left(\frac{1-x}{\varepsilon}, t\right)$$
$$= y^\varepsilon(x,t) \quad \text{a.e. in } (0,1)\times(0,T),$$

where y^ε is the solution of problem (12) with $y^\varepsilon(0,t) = \sum_{k=0}^\infty \varepsilon^k v^k(t)$, $t \in (0,T)$.

First approximate controllability result

Proposition

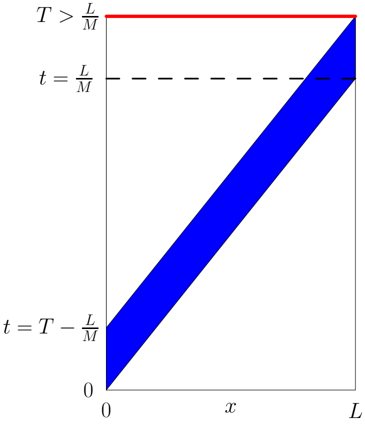
Let $m \in \mathbb{N}$, $T > \frac{1}{M}$ and $a \in]0, T - \frac{1}{M}[$. Assume regularity and matching conditions on the initial condition y_0 and functions v^k, $0 \leq k \leq m$. Assume moreover that

$$v^k(t) = 0, \quad 0 \leq k \leq m, \forall t \in [a, T].$$

Then, the solution y^ε of problem (12) satisfies the following property

$$\|y^\varepsilon(\cdot, T)\|_{L^2(0,1)} \leq c_m \, \varepsilon^{\frac{(2m+1)\gamma}{2}}, \quad \forall \gamma \in (0, 1)$$

for some constant $c_m > 0$ independent of ε. The function $v^\varepsilon \in C([0, T])$ defined by $v^\varepsilon := \sum_{k=0}^{m} \varepsilon^k v^k$ is an approximate null control for (5).



Remark

For $\varepsilon > 0$ small enough, we can not pass to the limit as $m \to \infty$ in order to get a null-controllability result.

Let $y_0(x) = 1$. The functions v^k, $k \geq 0$ defined as follows

$$v^0(t) = \mathcal{X}(t), \quad v^k(t) = 0, \quad k > 0 \tag{27}$$

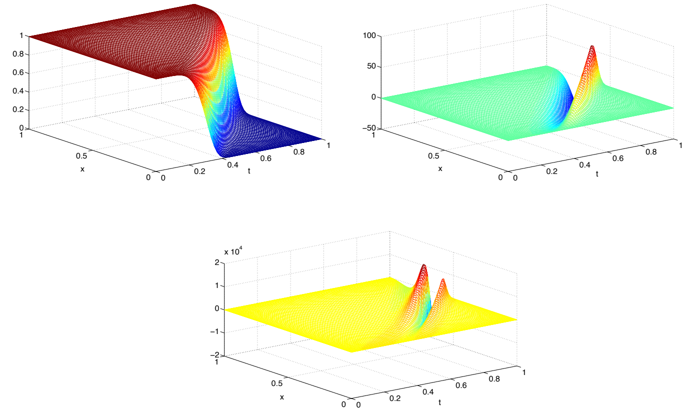
with $\mathcal{X} = \{f \in C^\infty[0,T], f(0) = 1, f(a) = 0, f^{(p)}(0) = f^{(p)}(a) = 0, p \in \mathbb{N}^\star\}$, $a \in [0,T]$ satisfy the matching conditions.

If $a \in]0, T - 1/M[$, then $\|w_m^\varepsilon(\cdot, T)\|_{L^2(0,T)} = 0 \ \forall m \in \mathbb{N}$.

If $\lim_{m\to\infty} c_m \varepsilon^{\frac{(2m+1)\gamma}{2}} = 0$, then the function $v^\varepsilon = \sum_{k=0}^{\infty} \varepsilon^k v^k = v^0$ is a null control for y^ε as time T. Contradiction.

$$y^0(x) = 1, \quad v^0(t) = \mathcal{X}(t), \quad v^k(t) = 0, \quad k > 0 \tag{28}$$

$\mathcal{X}(t) = e^{-kt^{2m+2}}$, with $k > 0$ so that $\mathcal{X}^p(0)) = 0$ for all $p \leq 2m+1$. $M = 1$, $k = 10^2$, $T = 1$ and $m = 2$. $Y_0(z,t) = (1 - e^{-z})$ independent of t while Y_1 and Y_2 are identically zero.



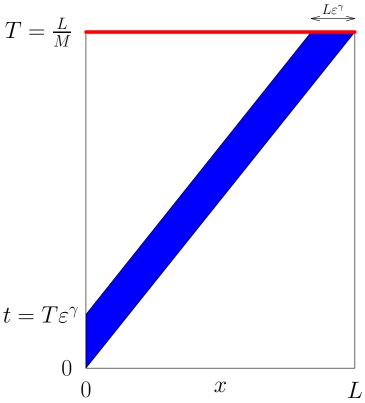
$$w_0^\varepsilon(x,T) = \mathcal{X}_\varepsilon(x) y^0(x,T) + (1-\mathcal{X}_\varepsilon(x))\, Y^0\left(\frac{1-x}{\varepsilon}, T\right), \quad (x,t)\in Q_T,$$
$$= \mathcal{X}_\varepsilon(x) v^0(T(1-x)) + (1-\mathcal{X}_\varepsilon(x))\, y_0(0)\left(1 - e^{-\frac{M(1-x)}{\varepsilon}}\right)$$

If $supp(v^0) \subset [0, T\varepsilon^\gamma]$,

$$\|w_0^\varepsilon(\cdot,T)\|_{L^2(0,1)} \le |y_0(0)|\, \varepsilon^{\gamma/2}$$

and

$$\begin{aligned} &\|y^\varepsilon(\cdot,T)\|_{L^2(0,1)} \\ &\quad\le \|w_0^\varepsilon(\cdot,T)\|_{L^2(0,1)} + \|(y^\varepsilon - w_0^\varepsilon)(\cdot,T)\|_{L^2(0,1)} \\ &\quad\le |y_0(0)|\, \varepsilon^{\gamma/2} + \varepsilon \|v^0\|_{L^2(0,\varepsilon^\gamma T)} \\ &\quad\le |y_0(0)|(c\varepsilon^{\tau(\gamma)} + \varepsilon^{\gamma/2}), \end{aligned}$$



Let us consider the case of the $C^\infty([0,1])$ initial data (with $L = 1$):

$$y_0^\varepsilon(x) := K_\varepsilon \sin(\pi x) e^{-\frac{Mx}{2\epsilon}}, \quad K_\varepsilon = \mathcal{O}(\varepsilon^{-3/2}). \tag{29}$$

Taking $m = 0$, the function $L_\varepsilon(w_0^\varepsilon)$ involves the term $-\varepsilon y^0_{xx}(x,t)\mathcal{X}_\varepsilon(x)$. For points in $C^- := \{(x,t) \in Q_T, x > Mt\}$, we obtain $y^0(x,t) = y_0^\varepsilon(x - Mt)$; this leads, to (writing that $\mathcal{X}_\varepsilon = 1$ on $(0, 1 - 2\varepsilon^\gamma)$)

$$\begin{aligned} \varepsilon \| y^0_{xx} \mathcal{X}_\varepsilon \|_{L^2(C^-)} &\geq \varepsilon K_\varepsilon \left(\int_0^{1-2\varepsilon^\gamma} \int_0^{x/M} \left((\sin(\pi(x - Mt)) e^{-\frac{M(x-Mt)}{2\varepsilon}})_{xx} \right)^2 dt dx \right)^{1/2} \\ &= \varepsilon K_\varepsilon \mathcal{O}(1) = \mathcal{O}(\varepsilon^{-1/2}). \end{aligned} \tag{30}$$

$$\begin{cases} y^\varepsilon(x,t) = e^{\frac{M\alpha}{2\varepsilon}\left(x - \frac{(2-\alpha)Mt}{2}\right)} z^\varepsilon(x,t), \\ L_\varepsilon y^\varepsilon(x,t) = e^{\frac{M\alpha}{2\varepsilon}\left(x - \frac{(2-\alpha)Mt}{2}\right)} \left(z_t^\varepsilon - \varepsilon z_{xx}^\varepsilon + M(1-\alpha) z_x^\varepsilon \right) \end{cases} \tag{31}$$

We then define the approximations

$$\begin{cases} z_m^\varepsilon(x,t) = \mathcal{X}_\varepsilon(x) \sum_{k=0}^{m} \varepsilon^k z^k(x,t) + (1 - \mathcal{X}_\varepsilon(x)) \sum_{k=0}^{m} \varepsilon^k Z^k\left(\frac{1-x}{\varepsilon}, t\right), \\ y_m^\varepsilon(x,t) = e^{\frac{M\alpha}{2\varepsilon}\left(x - \frac{(2-\alpha)Mt}{2}\right)} z_m^\varepsilon(x,t) \end{cases} \tag{32}$$

The main issue is now to find control functions $\overline{v}^k$ satisfying the matching conditions such that $\|L_\varepsilon y_m^\varepsilon\|_{C([0,T],L^2(0,1))}$ goes to zero with ε.

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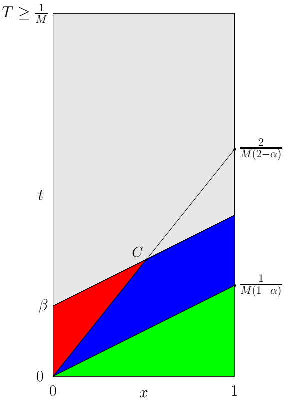
$\|L_\varepsilon y_m^\varepsilon\|_{C([0,T],L^2(0,1))}$



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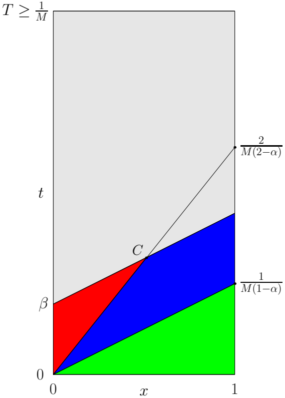
$$\|L_\varepsilon(y_0^\varepsilon)\|_{L^1(L^2(D_\beta^-\cap C_\alpha^+))} \approx (\overline{v}^0)(0)\mathcal{O}(\varepsilon^{1/2}) + (\overline{v}^0)^{(1)}(0)\mathcal{O}(\varepsilon^{3/2}).$$



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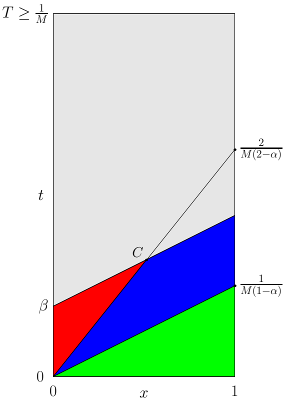

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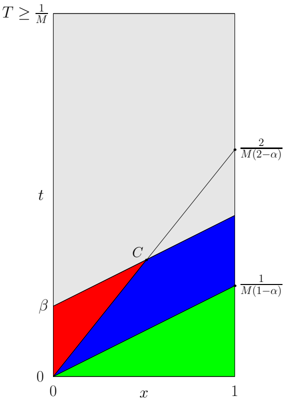

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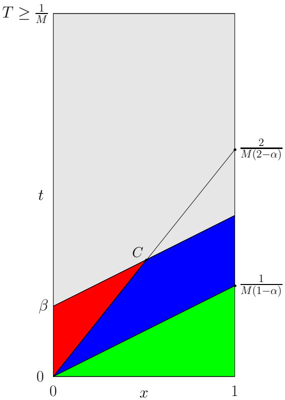

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$$\|L_\varepsilon(y_0^\varepsilon)\|_{L^1(L^2(D_\beta^-\cap C_\alpha^+))} \approx (\overline{v}^0)(0)\mathcal{O}(\varepsilon^{1/2}) + (\overline{v}^0)^{(1)}(0)\mathcal{O}(\varepsilon^{3/2}).$$

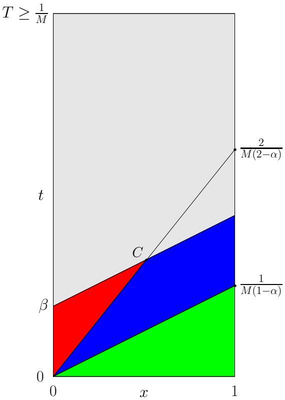

Figure: $(0,1)\times(0,T) = D_\beta^+ \cup (D_\beta^- \cap C_\alpha^-) \cup (D_\beta^- \cap C_\alpha^+)$.

Thus, the corresponding control is given

$$\begin{cases} v^0(t) = e^{-\frac{\gamma M^2 t}{4\varepsilon}} \overline{v}^0(t)\, 1_{[0,\beta]}(t), \quad \gamma = \alpha(2-\alpha), \\ \overline{v}^0(t) = \frac{kC_1 - 2C_2 + kC_2 t}{k^3} e^{kt} + C_3 t + C_4, \quad k := \frac{-\eta + \alpha(2-\alpha)}{4\varepsilon} \end{cases} \tag{34}$$

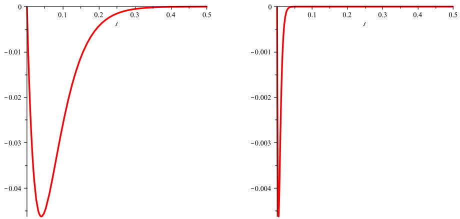

Figure: Control $v^0(t)$ for $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$ associated to $y_0^\varepsilon(x) = e^{\frac{-Mx}{2\varepsilon}} \sin(\pi x)$.

For any $\varepsilon > 0$ and $\beta > 0$, the optimality system associated to the extremal problem

$$\min_{v^\varepsilon \in L^2(0,T)} \|v^\varepsilon\|^2_{L^2(0,T)} + \beta^{-1}\|y^\varepsilon(\cdot,T)\|^2_{L^2(0,1)} \tag{35}$$

where the pair $(v^\varepsilon, y^\varepsilon)$ solves (5) is given by

$$\begin{cases} L_\varepsilon y^\varepsilon = 0, \quad L^\star_\varepsilon \varphi^\varepsilon = 0, & (x,t) \in Q_T, \\ y^\varepsilon(\cdot,0) = y^\varepsilon_0, & x \in (0,1), \\ v^\varepsilon(t) = y^\varepsilon(0,t) = \varepsilon\varphi^\varepsilon_x(0,t), & t \in (0,T), \\ y^\varepsilon(1,t) = 0, \ \varphi^\varepsilon(0,t) = \varphi^\varepsilon(1,t) = 0, & t \in (0,T), \\ -\beta\varphi^\varepsilon_{xx}(\cdot,T) + y^\varepsilon(\cdot,T) = 0, & x \in (0,1). \end{cases} \tag{36}$$

Optimality condition

$$v^0(t) + \varepsilon v^1(t) + \cdots = \Phi_z^0(0,t) + \varepsilon \Phi_z^1(0,t) + \cdots, \quad \forall t \in (0,T).$$

At the zero order, we get therefore $v^0(t) = \Phi_z^0(0,t)$ leading simultaneously, to

$$v^0(t) = M\varphi^0(0,t) = \begin{cases} M\varphi_T^0(M(T-t)), & t \in]T-1/M, T], \\ 0, & t \in [0, T-1/M] \end{cases}$$

The last equality contradicts the matching conditions (17).

$$\begin{aligned} J_\varepsilon^\star(\varphi_T^\varepsilon) &= \frac{1}{2}\int_0^T (\varepsilon \varphi_x^\varepsilon(0,\cdot))^2 dt + (y_0, \varphi^\varepsilon(\cdot,0))_{L^2(0,1)}, \\ &= \frac{1}{2}\int_0^T (\Phi_z^0(0,t))^2\, dt + (y_0, \varphi^0(\cdot,0)) + \varepsilon \ldots, \\ &= \frac{M^2}{2}\int_{T-1/M}^T \Big(\varphi_T^0(M(T-t))\Big)^2 dt + \varepsilon \ldots \end{aligned}$$

$$v^0(t) + \varepsilon v^1(t) + \cdots = \Phi_z^0(0,t) + \varepsilon \Phi_z^1(0,t) + \cdots, \quad \forall t \in (0,T).$$

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Final remarks

- Instead of imposing regularity assumptions and matching conditions, we may introduce an additional C^2 cut-off $\mathcal{X}$ function to take into account the discontinuity of the solutions y^k on the characteristic line. This allows to deal with the initial optimality system.

- The negative case is very similar except that the control v_ε lives in the boundary layer. (still in progress !)

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- The negative case is very similar except that the control v_ε lives in the boundary layer. (still in progress !)

$$\begin{cases} y^\varepsilon_{tt} + \varepsilon\Delta^2 y^\varepsilon - \Delta y^\varepsilon = 0, & \text{in } Q_T, \\ y^\varepsilon = 0, \quad \partial_\nu y^\varepsilon = v^\varepsilon 1_{\Gamma_T}, & \text{on } \Sigma_T, \\ (y^\varepsilon(\cdot,0), y^\varepsilon_t(\cdot,0)) = (y_0, y_1), & \text{on } \Omega. \end{cases}$$

Theorem (Lions)

Assume $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$. Assume that (Ω, Γ_T, T) satisfies a geometric control condition. For any $\varepsilon > 0$, let v^ε be the control of minimal $L^2(\Gamma_T)$ for y^ε. Then,

$$(\sqrt{\varepsilon} v^\varepsilon, y^\varepsilon) \to (v, y) \quad in \quad L^2(\Gamma_T) \times L^\infty(0, T; L^2(\Omega)), \quad as \quad \varepsilon \to 0$$

where v is the control of minimal $L^2(\Gamma_T)$-norm for y, solution in $C^0([0,T]; L^2(\Omega)) \times C^1([0,T]; H^{-1}(\Omega))$ of :

$$\begin{cases} y_{tt} - \Delta y = 0, & in \ Q_T, \\ y = v 1_{\Gamma_T}, & on \ \Sigma_T, \\ (y(\cdot,0), y_t(\cdot,0)) = (y_0, y_1), & in \ \Omega. \end{cases}$$

Fin

• A. Münch : *Numerical estimations of the cost of boundary controls for the equation* $y_t - \varepsilon y_{xx} + M y_x = 0$ *with respect to* ε.

• Y. Amirat, A. Münch : *Asymptotic analysis of the equation* $y_t - \varepsilon y_{xx} + M y_x = 0$ *and controllability results*.

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