About Least-Squares methods to solve direct and control problems

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ongoing works with Jérome Lemoine and Pablo Pedregal

Main motivation

Approximate (numerically) solution of direct problem and controllability problem for (nonlinear) PDEs using least-squares type method

• Example 1- The sine gordon equation : Find a control $v \in L^2(\Omega \times (0,T))$ such that the solution of

$$\begin{cases} y_{tt} - \Delta y + sin(y) = v 1_\omega & \Omega \times (0,T), \\ y = 0, & \partial\Omega \times (0,T), \\ (y(0), y_t(0)) = (y_0, y_1) & \Omega \times \{0\} \end{cases} \quad (1)$$

satisfies $(y(T), y_t(T)) = (0,0)$ in Ω.

• Example 2- The Navier-Stokes system: Find a control $v \in L^2(\partial\Omega \times (0,T))$ such that the solution of

$$\begin{cases} u_t - \nu\Delta u + (u \cdot \nabla)u + \nabla p = 0, \nabla \cdot u = 0 & \Omega \times (0,T), \\ u = v, & \partial\Omega \times (0,T), \\ u(0) = u_0, & \Omega \times \{0\} \end{cases} \quad (2)$$

satisfies $u(T) = u_d$, a trajectory (control of flows)

$\Longrightarrow$ Non trivial problem because fixed point or linearization technics may not converge and because duality arguments (when one wants control of minimal norm) make appear completed spaces.

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$\Longrightarrow$ Non trivial problem because fixed point or linearization technics may not converge and because duality arguments (when one wants control of minimal norm) make appear completed spaces.

Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$ be a bounded connected open set whose boundary $\partial\Omega$ is Lipschitz. We denote by $\boldsymbol{V} = \{v \in \mathcal{D}(\Omega)^d, \nabla \cdot v = 0\}$, $\boldsymbol{H}$ the closure of $\boldsymbol{V}$ in $L^2(\Omega)^d$ and $\boldsymbol{V}$ the closure of $\mathcal{V}$ in $H^1(\Omega)^d$.
The Navier-Stokes system describes a viscous incompressible fluid flow in the bounded domain Ω submitted to the external force F and reads as follows :

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = F, \quad \nabla\cdot u = 0 & \text{in } \Omega\times(0,T), \\ u = 0 & \text{on } \partial\Omega\times(0,T), \\ u(\cdot,0) = u_0, & \text{in } \Omega. \end{cases} \tag{3}$$

We refer to [Temam] [1]

Objective: Approximation of (u,p) using least-squares method ?

[1] Roger Temam, Navier-Stokes equations, AMS Chelsea Publishing, Providence, RI, 2001, Theory and numerical analysis

For any $\theta \in (0,1]$ and uniform discretization $\{t_n\}_{n=0\ldots N}$ of the time interval $(0,T)$, we consider time marching implicit schemes of the form

$$\begin{cases} y^0(\cdot,0) = u_0 \quad \text{in} \ \ \Omega, \\ \dfrac{y_\theta^{n+1} - y^n}{\theta\delta t} - \nu\Delta y_\theta^{n+1} + (y_\theta^{n+1}\cdot\nabla)y_\theta^{n+1} + \nabla\pi_\theta^{n+1} = \dfrac{1}{\delta t}\displaystyle\int_{t_n}^{t_{n+1}} F(\cdot,s)ds, \quad n \geq 0, \\ \nabla\cdot y_\theta^{n+1} = 0 \quad \text{in} \ \ \Omega, \quad n\geq 0, \\ y_\theta^{n+1} = 0 \quad \text{on} \ \ \partial\Omega, \quad n \geq 0, \\ y^{n+1} = \theta^{-1}(y_\theta^{n+1} - (1-\theta)y^n), \quad n \geq 0 \end{cases} \tag{4}$$

- $\delta t = T/N$ the time discretization step.
- $\theta = 1$ corresponds to the backward Euler scheme. Piecewise linear interpolation (in time) of $\{y^n\}_{n\in[0,N]}$ weakly converges in $L^2(0,T,\boldsymbol{V})$ toward a solution u of (3) as $\delta t \to 0$. It achieves a first order convergence with respect to δt.
- $\theta = 1/2$ corresponds to a Crank-Nicolson scheme and allows to achieve a second order convergence.

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The determination of y^{n+1} from y^n requires the resolution of a nonlinear PDE. Precisely, y_θ^{n+1} together with the pressure π_θ^{n+1}, solve the following problem: find $y \in \boldsymbol{V}$ and $\pi \in L^2_0(\Omega)$, solution of

$$\begin{cases} \alpha\, y - \nu\Delta y + (y\cdot\nabla)y + \nabla\pi = f + \alpha\, g, \quad \nabla\cdot y = 0 \quad \text{in } \Omega, \\ y = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5}$$

with

$$\alpha = \frac{1}{\theta\delta t} > 0, \quad f = \frac{1}{\delta t}\int_{t_n}^{t_{n+1}} F(\cdot,s)ds, \quad g = y^n. \tag{6}$$

• For any $f \in H^{-1}(\Omega)^d$ and $g \in L^2(\Omega)^d$, there exists at least one $(y,\pi) \in \boldsymbol{V} \times L^2_0(\Omega)$ solution of (5). $L^2_0(\Omega)$ stands for the space of functions in $L^2(\Omega)$ with zero means.
• If $\|g\|_2^2 + \alpha^{-1}\nu^{-1}\|f\|^2_{H^{-1}(\Omega)^d}$ is small enough, then the couple (y,π) is unique.
Here and in the sequel, $\|\cdot\|_2$ stands for $\|\cdot\|_{L^2(\Omega)^d}$.

One may use Newton-type method to solve the weak formulation of (5), i.e. find $y \in \boldsymbol{V}$ solution of

$$F(y,z) := \int_\Omega \alpha y\cdot z + \nu\nabla y\cdot\nabla z + (y\nabla)y\cdot z - < f, z >_{H^{-1}(\Omega)^d\times H_0^1(\Omega)^d} - \alpha\int_\Omega g\cdot z = 0, \forall z \in \boldsymbol{V}. \tag{7}$$

for all $z \in \boldsymbol{V}$. Equivalently, find $y \in \boldsymbol{V}$ such that

$$\sup_{z\in\boldsymbol{V}, z\neq 0} \frac{F(y,z)}{\|z\|_V} = 0.$$

The Newton algorithm reads as follows : construct $\{y_k\}_{k\in\mathbb{N}}$ such that

$$\begin{cases} y_0 \in \boldsymbol{V}, \\ \partial_y F(y_k, z)\cdot(y_{k+1} - y_k) = -F(y_k, z), \qquad \forall z \in \boldsymbol{V}, \quad \forall k \geq 0, \end{cases} \tag{8}$$

and converges to a solution $\overline{y}$ if $\partial_y F(y,z)$ is an isomorphism and Lipschitz-continuous with respect to y in the closed ball containing $\overline{y}$ [2].

$\Longrightarrow$ Determine (for Navier-Stokes) necessary and sufficient conditions for $\partial_y F(y,z)$ to be isomorphism is an open problem.

[2] V. Girault, P.A. Raviart, Finite element methods for Navier-Stokes equations, Berlin, 1986, Theory and algorithms.

- L^2-LS method based on the functional

$$(y,\pi) \rightarrow \|\alpha\, y - \nu\Delta y + (y\cdot\nabla)y + \nabla\pi - f + \alpha\, g\|_2^2 + \|\nabla\cdot y\|_2^2$$

Resolution of the optimality conditions requires H^2 functional spaces. See Gunzburger's Book [3] for LSFEM.

- H^{-1}-LS method based on the minimization of

$$(y,\pi) \rightarrow \|\alpha\, y - \nu\Delta y + (y\cdot\nabla)y + \nabla\pi - f + \alpha\, g\|_{H^{-1}(\Omega)}^2 + \|\nabla\cdot y\|_2^2$$

considered in [Bristeau etal,1979][4] with experiments but without mathematical justification !

[3] Pavel B. Bochev and Max D. Gunzburger, *Least-squares finite element methods*, Springer, 2009.

[4] M. O. Bristeau, O. Pironneau, R. Glowinski, J. Periaux, and P. Perrier, On the numerical solution of nonlinear problems in fluid dynamics by least squares and finite element methods. I. Least square formulations and conjugate gradient, Comput. Methods Appl. Mech. Engrg. (1979)

Let $f \in H^{-1}(\Omega)^d$, $g \in L^2(\Omega)^d$ and $\alpha \in \mathbb{R}_+^\star$. The weak formulation of (5) reads as follows: find $y \in \boldsymbol{V}$ solution of

$$\alpha \int_\Omega y\cdot w + \nu \int_\Omega \nabla y \cdot \nabla w + \int_\Omega y \cdot \nabla y \cdot w = < f, w >_{H^{-1}(\Omega)^d \times H_0^1(\Omega)^d} + \alpha \int_\Omega g \cdot w, \quad \forall w \in \boldsymbol{V}. \tag{9}$$

Proposition

a) Assume $\Omega \subset \mathbb{R}^d$ is bounded and Lipschitz. There exists a least one solution y of (9) satisfying

$$\alpha \|y\|_2^2 + \nu \|\nabla y\|_2^2 \leq \frac{c}{\nu} \|f\|^2_{H^{-1}(\Omega)^d} + \alpha \|g\|_2^2 \tag{10}$$

for some constant $c > 0$ dependent on Ω. If moreover, Ω is C^2 and $f \in L^2(\Omega)^d$, then $y \in H^2(\Omega)^d \cap \boldsymbol{V}$.
b) Let us define $Q(g, f, \alpha, \nu)$ as follows :

$$Q(g,f,\alpha,\nu) = \begin{cases} \dfrac{1}{\nu^2}\left(\|g\|_2^2 + \dfrac{1}{\alpha\nu}\|f\|^2_{H^{-1}(\Omega)^d} \right), & if \quad d = 2, \\ \dfrac{\alpha^{1/2}}{\nu^{5/2}}\left(\|g\|_2^2 + \dfrac{1}{\alpha\nu}\|f\|^2_{H^{-1}(\Omega)^d} \right), & if \quad d = 3. \end{cases} \tag{11}$$

If $Q(g, f, \alpha, \nu)$ is small enough, then the solution of (9) is unique.

Mathematical justification of the H^{-1} -Least-squares method (interior case)

Let $f \in H^{-1}(\Omega)^d$, $g \in L^2(\Omega)^d$ and $\alpha \in \mathbb{R}_+^\star$. The weak formulation of (5) reads as follows: find $y \in \boldsymbol{V}$ solution of

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If $Q(g, f, \alpha, \nu)$ is small enough, then the solution of (9) is unique.

We now introduce our least-squares functional $E : \boldsymbol{V} \to \mathbb{R}^+$ as follows

$$E(y) := \frac{1}{2}\int_\Omega (\alpha|v|^2 + |\nabla v|^2) \tag{12}$$

where the corrector $v \in \boldsymbol{V}$ is the unique solution of

$$\begin{aligned}\alpha\int_\Omega v\cdot w + \int_\Omega \nabla v\cdot\nabla w = -\alpha\int_\Omega y\cdot w - \nu\int_\Omega \nabla y\cdot\nabla w - \int_\Omega y\cdot\nabla y\cdot w \\ + < f, w >_{H^{-1}(\Omega)^d\times H_0^1(\Omega)^d} + \alpha\int_\Omega g\cdot w, \quad \forall w\in \boldsymbol{V}.\end{aligned} \tag{13}$$

The infimum of E is equal to zero and is reached by a solution of (9). In this sense, the functional E is a so-called error functional which measures, through the corrector variable v, the deviation of the pair y from being a solution of the underlying equation (9).

Beyond this statement, we would like to argue why we believe it is a good idea to use a (minimization) least-squares approach to approximate the solution of (9) by minimizing the functional E. Our main result is a follows:

Theorem (Lemoine,Pedregal, M' 18)

Assume that $Q(g, f, \alpha, \nu)$ is small enough. There is a positive constant C, such that if $\{y_k\}_{k>0}$ is a sequence in

$$\mathbb{B} := \{y \in \boldsymbol{V} : \|y\|_{\mathbf{H}_0^1(\Omega)} \leq C\}$$

with $E'(y_k) \to 0$ as $k \to \infty$, then the whole sequence $\{y_k\}_{k\in\mathbb{N}}$ converges strongly as $k \to \infty$ in $\boldsymbol{V}$ to a solution $\overline{y}$ of (9).

We divide the proof in two main steps.

1. First, we use a typical a priori bound to show that leading the error functional E down to zero implies strong convergence to the (unique) solution of (9).
2. Next, we show that taking the derivative E' to zero actually suffices to take E to zero.

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Proposition (first step)

Assuming that $Q(g, f, \alpha, \nu)$ is small enough and let $\overline{y}$ be a solution of (9). For every $y \in \boldsymbol{V}$, we have

$$\|y - \overline{y}\|_{H^1(\Omega)^d} \leq 2\nu^{-1}\sqrt{E(y)}. \tag{14}$$

This proposition very clearly establishes that as we take down the error E to zero, we get closer, in the strong norm, to the solution of the problem, and so, it justifies why a promising strategy to find good approximations of the solution of problem (9) is to look for global minimizers of the extremal problem:

$$\inf_{y \in \boldsymbol{V}} E(y). \tag{15}$$

Proposition (second step)

There exists a positive constant C such that if $\{y_k\}_{k\in\mathbb{N}}$ is a sequence in $\mathbb{B}$ defined by $\mathbb{B} = \{y \in \boldsymbol{V} : \frac{1}{\nu\alpha}\|\nabla y\|_2^2 < C\}$ if $d = 2$ and $\mathbb{B} = \{y \in \boldsymbol{V} : \frac{1}{\nu\alpha}\|\nabla y\|_2^4 < C\}$ if $d = 3$ with $E'(y_k) \to 0$ as $k \to \infty$, then $E(y_k) \to 0$ as $k \to \infty$.

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proof of the second step.

The error functional E is differentiable as functional defined on the Hilbert space $\boldsymbol{V}$, because the operator $y \to v$ taking each $y \in \boldsymbol{V}$ into its associated corrector v, as stated above is a differentiable operation. Indeed, $E'(y)$ can always be identified with an element of $\boldsymbol{V}$ itself. For any $Y \in \boldsymbol{V}$, we have

$$E'(y) \cdot Y = \int_\Omega \alpha\, v \cdot V + \nabla v \cdot \nabla V \tag{16}$$

where $V \in \boldsymbol{V}$ is the unique solution of

$$\alpha \int_\Omega V \cdot w + \int_\Omega \nabla V \cdot \nabla w = -\alpha \int_\Omega Y \cdot w - \nu \int_\Omega \nabla Y \cdot \nabla w - \int_\Omega (y \cdot \nabla Y + Y \cdot \nabla y) \cdot w, \forall w \in \boldsymbol{V}. \tag{17}$$

There exists an element $Y_1 = Y_1(y)$, uniformly bounded in $\boldsymbol{V}$ such that

$$E'(y) \cdot Y_1 = 2E(y), \quad \forall y \in \boldsymbol{V}. \tag{18}$$

Let now, for any $k \in \mathbb{N}$, $Y_{1,k}$ be associated to y_k. The previous equality writes $E'(y_k) \cdot Y_{1,k} = 2E(y_k)$ and implies our statement, since $Y_{1,k}$ is uniformly bounded in $\boldsymbol{V}$. □

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For any $y \in \boldsymbol{V}$, we now look for an element $Y_1 \in \boldsymbol{V}$ solution of the following formulation

$$\alpha \int_\Omega Y_1 \cdot w + \nu \int_\Omega \nabla Y_1 \cdot \nabla w + \int_\Omega (y \cdot \nabla Y_1 + Y_1 \cdot \nabla y) \cdot w = -\alpha \int_\Omega v \cdot w - \int_\Omega \nabla v \cdot \nabla w, \forall w \in \boldsymbol{V} \tag{19}$$

where $v \in \boldsymbol{V}$ is the corrector associated to y. Y_1 enjoys the following property

Proposition

There exists $c > 0$ such that, for all $y \in \boldsymbol{V}$ satisfying $\frac{1}{\nu\alpha}\|\nabla y\|_2^2 < c$ if $d = 2$ and $\frac{1}{\nu\alpha}\|\nabla y\|_2^4 < c$ if $d = 3$, there exists a unique solution Y_1 of (19) associated to y. This solution satisfies

$$\|Y_1\|_{\boldsymbol{V}} \leq M$$

for some constant $M > 0$, independent of y.

Proposition

Assume that $Q(g, f, \alpha, \nu)$ is small enough. There is a known, specific positive constant C such that if $\|y_0 - \overline{y}\|_{\mathbf{H}_0^1(\Omega)} < C$, then a gradient method for E in (41) starting from y_0 will always converge to $\overline{y}$.

Proof.

Let $(y_k)_{k\in\mathbb{N}}$ a minimizing sequence for E based on the gradient E', i.e. $(y_{k+1} - y_k, w) = -\lambda E'(y_k) \cdot w$, for all $w \in \boldsymbol{V}$ and $\lambda > 0$. We check that, taking $w = y_k - \overline{y} - \lambda g_k$, we obtain

$$\|y_{k+1} - \overline{y}\|^2_{\boldsymbol{V}} - \|y_k - \overline{y}\|^2_{\boldsymbol{V}} = 2\lambda E'(y_k) \cdot (\overline{y} - y_k) + \lambda^2 \|g_k\|^2, \quad \forall k \in \mathbb{N} \tag{20}$$

where g_k is given by $(g_k, w)_{\boldsymbol{V}} = -E'(y_k) \cdot w$ for all $w \in \boldsymbol{V}$, i.e.

$$(g_k, w)_{\boldsymbol{V}} = \int_\Omega \alpha v_k w + \nu \nabla v_k \nabla w + [(w \cdot \nabla) y_k + (y_k \cdot \nabla) w] v_k. \tag{21}$$

v_k is the corrector associated to y_k. We deduce that $\|g_k\|_{\boldsymbol{V}}$ is uniformly bounded as soon as y_k is uniformly bounded.
The strategy is then to show that the quantity $E'(y_0) \cdot (\overline{y} - y_0)$ becomes non-positive, if the initial guess y_0 is sufficiently close to the exact solution $\overline{y}$.

Proposition

Assume that $Q(g, f, \alpha, \nu)$ is small enough. There is a known, specific positive constant C such that if $\|y_0 - \overline{y}\|_{\mathbf{H}_0^1(\Omega)} < C$, then a gradient method for E in (41) starting from y_0 will always converge to $\overline{y}$.

Proof.

Let $(y_k)_{k\in\mathbb{N}}$ a minimizing sequence for E based on the gradient E', i.e. $(y_{k+1} - y_k, w) = -\lambda E'(y_k) \cdot w$, for all $w \in \boldsymbol{V}$ and $\lambda > 0$. We check that, taking $w = y_k - \overline{y} - \lambda g_k$, we obtain

$$\|y_{k+1} - \overline{y}\|^2_{\boldsymbol{V}} - \|y_k - \overline{y}\|^2_{\boldsymbol{V}} = 2\lambda E'(y_k) \cdot (\overline{y} - y_k) + \lambda^2 \|g_k\|^2, \quad \forall k \in \mathbb{N} \tag{20}$$

where g_k is given by $(g_k, w)_{\boldsymbol{V}} = -E'(y_k) \cdot w$ for all $w \in \boldsymbol{V}$, i.e.

$$(g_k, w)_{\boldsymbol{V}} = \int_\Omega \alpha v_k w + \nu \nabla v_k \nabla w + [(w \cdot \nabla) y_k + (y_k \cdot \nabla) w] v_k. \tag{21}$$

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The strategy is then to show that the quantity $E'(y_0) \cdot (\overline{y} - y_0)$ becomes non-positive, if the initial guess y_0 is sufficiently close to the exact solution $\overline{y}$.

□

We now introduce our least-squares functional $E : \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega) \to \mathbb{R}^+$ as follows

$$E(y,p) := \frac{1}{2}\int_\Omega (\alpha|v|^2 + |\nabla v|^2) + \frac{1}{2}\int_\Omega |\nabla \cdot y|^2 \tag{22}$$

where the corrector $v \in \mathbf{H}^1_0(\Omega)$ is the unique solution of

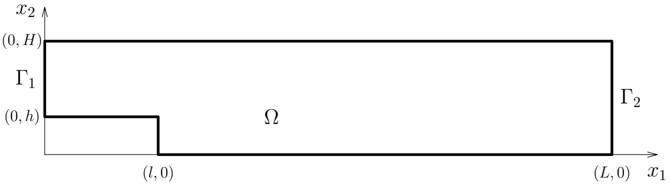
$$\alpha\int_\Omega v\cdot w + \int_\Omega \nabla v\cdot\nabla w = -\alpha\int_\Omega y\cdot w - \nu\int_\Omega \nabla y\cdot\nabla w - \int_\Omega y\cdot\nabla y\cdot w$$
$$+\int_\Omega p\nabla\cdot w + < f, w >_{H^{-1}(\Omega)^d\times H^1_0(\Omega)^d} + \alpha\int_\Omega g\cdot w, \quad \forall w\in \mathbf{H}^1_0(\Omega). \tag{23}$$

Theorem (Lemoine, Pedregal,M' 18)

Assume that $Q(g,f,\alpha,\nu)$ is small enough. There is a positive constant C, such that if $\{y_k,p_k\}_{k>0}$ is a sequence in $\mathbf{H}^1_0(\Omega)\times L^2_0(\Omega)$ with y_k belonging to the ball $\mathbb{B} := \{y\in \boldsymbol{V} : \|y\|_{\mathbf{H}^1_0(\Omega)} \le C\}$ with $E'(y_k,p_k)\to 0$ as $k\to\infty$, then the whole sequence $\{y_k,p_k\}_{k\in\mathbb{N}}$ converges strongly as $k\to\infty$ in $\mathbf{H}^1_0(\Omega)\times L^2_0(\Omega)$ to a solution $(\overline{y},p)$ of (9).

We consider the celebrated test problem of a two-dimensional channel with a backward facing step described for instance in Jian-Povinelli [5].

Dirichlet conditions of the Poiseuille type are imposed on the entrant and sortant sides Γ_1 and Γ_2 of the channel: we impose $y = (4(H-y)(y-h)/(H-h)^2, 0)$ on Γ_1 and $y = (4(H-h)y(H-y)/H^2, 0)$ on Γ_2, with $h = 1, H = 3, l = 3$ and $L = 30$. On the remaining part $\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$, the fluid flow is imposed to zero. The external force $\mathbf{f}$ is zero.



A two-dimensional channel with a step.

[5] Bo-Nan Jiang and Louis A. Povinelli, Least-squares finite element method for fluid dynamics, Comput. Methods Appl. Mech. Engrg. (1990)

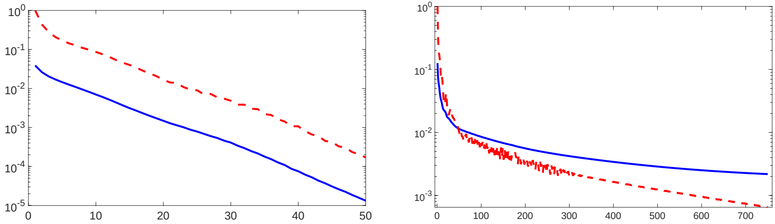
Minimization of E though the CG algorithm (Polak-Ribiere). The gradient g_k of $E(y_k)$ is defined as follows :

$$(g_k, w)_{\boldsymbol{V}} = -E'(y_k) \cdot w, \quad \forall w \in \boldsymbol{V},$$

i.e.

$$(g_k, w)_{\boldsymbol{V}} = \int_\Omega \alpha v_k w + \nu \nabla v_k \nabla w + [(w \cdot \nabla) y_k + (y_k \cdot \nabla) w] v_k. \tag{24}$$

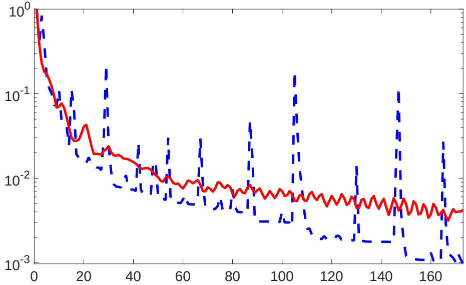
where v_k is the corrector associated to y_k.
The discretization in space is performed with $\mathbb{P}_2/\mathbb{P}_1$ Taylor-Hood finite element.



$\nu = 1/50$; $\sqrt{E(y_k)}$ (blue line) and $\|\mathbf{g}_k\|_{\mathbf{H}^1} / \|\mathbf{g}_0\|_{\mathbf{H}^1}$ (red line) w.r.t. iterate k; Left: Interior case; Right: Exterior case

Barzilai-Borwein[6] two steps algorithm

$$\begin{cases}(y_{k+1} - y_k, Y)_{\boldsymbol{V}} = -\alpha_k E'_\varepsilon(y_k) \cdot Y, \quad \forall Y \in \boldsymbol{V}, \quad k \geq 0, \\ \alpha_k = < y_k - y_{k-1}, g_k - g_{k-1} >_{\boldsymbol{V}} / \|g_k - g_{k-1}\|^2_{\boldsymbol{V}}.\end{cases}$$



$\nu = 1/50$; Evolution of $\|g_k\|_{\mathcal{A}}/\|g_0\|_{\mathcal{A}}$ for the CG algorithm (red line) and for the BB algorithm (blue dashed line) w.r.t. iterate k.

[6] J. Barzilai and J. M. Borwein, Two-point step size gradient methods, IMA J. Numer. Anal. 1988

$$\begin{cases} y^0 \in \boldsymbol{V}, \\ y_{k+1} = y_k - \lambda_k Y_{1,k}, \quad k > 0. \end{cases} \tag{25}$$

where $Y_{1,k}$ solves the formulation, for all $w \in \boldsymbol{V}$

$$\alpha \int_\Omega Y_{1,k} \cdot w + \nu \int_\Omega \nabla Y_{1,k} \cdot \nabla w + \int_\Omega (y_k \cdot \nabla Y_{1,k} + Y_{1,k} \cdot \nabla y_k) \cdot w = -\alpha \int_\Omega v_k \cdot w - \int_\Omega \nabla v_k \cdot \nabla w, \tag{26}$$

leading to $E'(y_k) \cdot Y_{1,k} = 2E(y_k)$.

Lemma

$$E(y_k - \lambda Y_{1,k}) = (1-\lambda)^2 E(y_k) + \lambda^2 (1-\lambda) A_k + \lambda^4 B_k \tag{27}$$

with $A_k = \int_\Omega \alpha v_k \overline{\overline{v}}_k + \nabla v_k \nabla \overline{\overline{v}}_k$ *and* $B_k = \frac{1}{2} \int_\Omega \alpha |\overline{\overline{v}}_k|^2 + |\nabla \overline{\overline{v}}_k|^2$
where $v_k \in \boldsymbol{V}$ *is the corrector associated to* y_k *and* $\overline{\overline{v}}^k \in \boldsymbol{V}$ *solves*

$$\alpha \int_\Omega \overline{\overline{v}}_k \cdot w + \int_\Omega \nabla \overline{\overline{v}}_k \cdot \nabla w + \int_\Omega Y_{1,k} \cdot \nabla Y_{1,k} \cdot w = 0, \quad \forall w \in \boldsymbol{V}. \tag{28}$$

Proposition

In a neighborhood of a zero of E, the algorithm (25) is of order at least two.

Proof.

$$\begin{aligned} E(y_{k+1}) =&(1-\lambda_k)^2 E(y_k) + \lambda_k^2(1-\lambda_k)A_k + \lambda_k^4 B_k \\ \leq&(1-\lambda_k)^2 E(y_k) + \lambda_k^2|1-\lambda_k| c_\nu E(y_k)^{3/2} + \lambda_k^4 c_\nu^2 E(y_k)^2, \quad c_\nu := 4\nu^{-2} \end{aligned} \tag{29}$$

Taking $\lambda_k = 1$ leads to $(c_\nu^2 E(y_{k+1})) \leq (c_\nu^2 E(y_k))^2$ and the order 2 of convergence as soon as $E(y_k) \leq c\nu^{-4}$ for some $c > 0$, i.e. $\|y_k - \overline{y}\|_V \leq c\nu^2$. □

Remark

Actually, for $\lambda_k = 1$, the algorithm for the functional E coincides exactly with the Newton algorithm for the variational formulation F !!
The optimization of λ_k allows a faster convergence together with a larger radius of convergence.

Use of the element Y_1 to minimize E

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Remark

Actually, for $\lambda_k = 1$, the algorithm for the functional E coincides exactly with the Newton algorithm for the variational formulation F !!
The optimization of λ_k allows a faster convergence together with a larger radius of convergence.

Use of the element Y_1

$\sharp$ iterate k	(25) with $\lambda_k = 1$	(25) (λ_k)	quasi (25)	CG
1	4.442×10^{-1}	3.798×10^{-1} (0.8545)	3.796×10^{-1}	5.214×10^{-2}
2	1.959×10^{-1}	1.810×10^{-1} (0.9573)	1.592×10^{-1}	4.195×10^{-2}
3	5.609×10^{-2}	4.045×10^{-2} (0.9949)	4.375×10^{-2}	3.276×10^{-2}
4	3.986×10^{-3}	2.223×10^{-3} (1.0006)	6.055×10^{-3}	2.946×10^{-2}
5	2.082×10^{-5}	5.719×10^{-6} (0.9999)	6.808×10^{-3}	2.568×10^{-2}
6	5.912×10^{-10}	4.959×10^{-11} (1)	9.899×10^{-4}	2.290×10^{-2}
7	4.881×10^{-15}	3.299×10^{-15} (1)	9.009×10^{-4}	2.219×10^{-2}
8	–	–	1.486×10^{-4}	2.024×10^{-2}
9	–	–	9.553×10^{-5}	1.952×10^{-2}
10	–	–	2.092×10^{-5}	1.819×10^{-2}
11	–	–	1.396×10^{-5}	1.764×10^{-2}
12	–	–	3.170×10^{-6}	1.723×10^{-2}
13	–	–	1.839×10^{-6}	1.674×10^{-2}
14	–	–	3.809×10^{-7}	1.657×10^{-2}
15	–	–	1.987×10^{-7}	1.606×10^{-2}
26	–	–	4.321×10^{-13}	1.120×10^{-2}
50	–	–	–	3.325×10^{-3}
100	–	–	–	1.756×10^{-3}
200	–	–	–	2.091×10^{-5}

Table: $\nu = 1/150$; Evolution of $\|y_{k+1} - y_k\|_V / \|y_k\|_V$ with respect to k.

Use of the element Y_1: result for the 2D channel with a backward facing step.

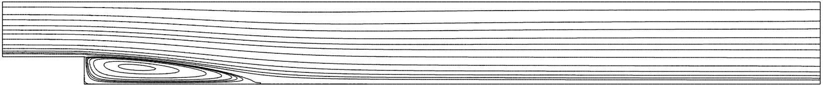
$\sharp$ iterate k	(25) with $\lambda_k = 1$	(25)	quasi (25)	CG
1	5.467×10^{-2}	5.467×10^{-2}	5.476×10^{-2}	5.467×10^{-2}
2	2.398×10^{-2}	2.224×10^{-2}	2.222×10^{-2}	3.701×10^{-2}
3	4.953×10^{-3}	4.601×10^{-3}	5.457×10^{-3}	2.917×10^{-2}
4	3.201×10^{-4}	1.565×10^{-4}	9.322×10^{-4}	2.492×10^{-2}
5	1.530×10^{-6}	5.437×10^{-7}	5.191×10^{-4}	2.201×10^{-2}
6	3.650×10^{-11}	4.227×10^{-12}	1.712×10^{-4}	1.995×10^{-2}
7	6.541×10^{-16}	2.541×10^{-16}	1.712×10^{-4}	1.840×10^{-2}
8	–	–	7.852×10^{-5}	1.709×10^{-2}
9	–	–	2.472×10^{-5}	1.603×10^{-2}
10	–	–	8.953×10^{-6}	1.511×10^{-2}
11	–	–	3.424×10^{-6}	1.433×10^{-2}
12	–	–	1.205×10^{-6}	1.363×10^{-2}
13	–	–	4.251×10^{-7}	1.301×10^{-2}
14	–	–	1.366×10^{-7}	1.242×10^{-2}
15	–	–	4.478×10^{-8}	1.187×10^{-2}
26	–	–	1.599×10^{-14}	6.259×10^{-3}
50	–	–	–	2.673×10^{-3}
100	–	–	–	7.583×10^{-4}
200	–	–	–	1.551×10^{-5}

Table: $\nu = 1/150$; Evolution of $\|v_k\|_{\mathbf{V}} = \sqrt{2E(y_k)}$ with respect to k.

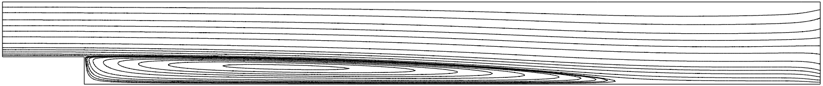
$\sharp$ iterate k	$\Vert\mathbf{y}^{k+1}-\mathbf{y}^k\Vert_V/\Vert\mathbf{y}^k\Vert_V$	$\sqrt{2E(\mathbf{y}^k)}$	λ_k	$\sqrt{2E(\mathbf{y}^k)}$ with $\lambda_k=1$
1	7.153×10^{-1}	5.467×10^{-2}	0.727	5.467×10^{-2}
2	1.424×10^{-4}	2.791×10^{-2}	4.77×10^{-5}	3.452×10^{-2}
3	2.073×10^{-1}	2.791×10^{-2}	2.01×10^{-2}	8.089×10^{-2}
4	3.538×10^{-1}	2.737×10^{-2}	0.958	5.344×10^{-2}
5	9.138×10^{-2}	7.270×10^{-3}	4.81×10^{-6}	2.409
6	6.244×10^{-2}	2.622×10^{-3}	1.73×10^{-3}	6.115×10^{-1}
7	2.028×10^{-2}	1.078×10^{-3}	0.358	3.944
8	3.695×10^{-3}	2.610×10^{-4}	0.521	9.851×10^{1}
9	7.522×10^{-4}	4.184×10^{-5}	1.098	8.186×10^{1}
10	9.886×10^{-6}	6.014×10^{-7}	0.963	4.385×10^{4}
11	3.872×10^{-6}	1.692×10^{-7}	1.032	1.093×10^{4}
12	6.820×10^{-11}	4.404×10^{-12}	0.9983	3.169×10^{4}
13	1.288×10^{-10}	2.880×10^{-12}	0.9999	1.576×10^{5}
14	6.879×10^{-15}	3.263×10^{-16}	$1.$	4.068×10^{4}

Table: $\nu=1/700$; Results for the algorithm (25).

Newton algorithm fails to converge for $\nu^{-1}>250$.
Algorithm (25) fails to converge for $\nu^{-1}>725$.

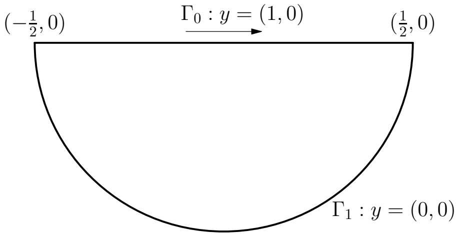


Streamlines of the steady state solution at $\nu = 1/150$; $L = 30$. length of recirculating zone ≈ 6.26



Streamlines of the steady state solution at $\nu = 1/700$; $L = 30$. length of recirculating zone ≈ 17.31

Case considered by Glowinski etal[2006] [7] for which a Hopf bifurcation phenomenon occurs : for $\nu^{-1} \geq 6650$, the unsteady solution does not converge toward the steady solution.



Semi-disk geometry: $\Omega = \{(x_1, x_2) \in \mathbb{R}^2, x_1^2 + x_2^2 \leq 1/4, x_2 \leq 0\}$

Initialized with the solution of the corresponding Stokes problem,

- Newton ($\lambda_k = 1$) fails to converge for $\nu^{-1} > 1/500$.
- the algorithm (25) fails to converge for $\nu^{-1} > 1/910$.

[7] Glowinski, R. and Guidoboni, G. and Pan, T.-W., Wall-driven incompressible viscous flow in a two-dimensional semi-circular cavity, J. Comput. Phys., 2006

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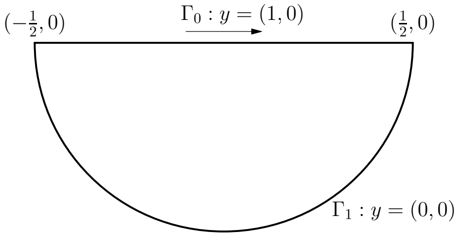

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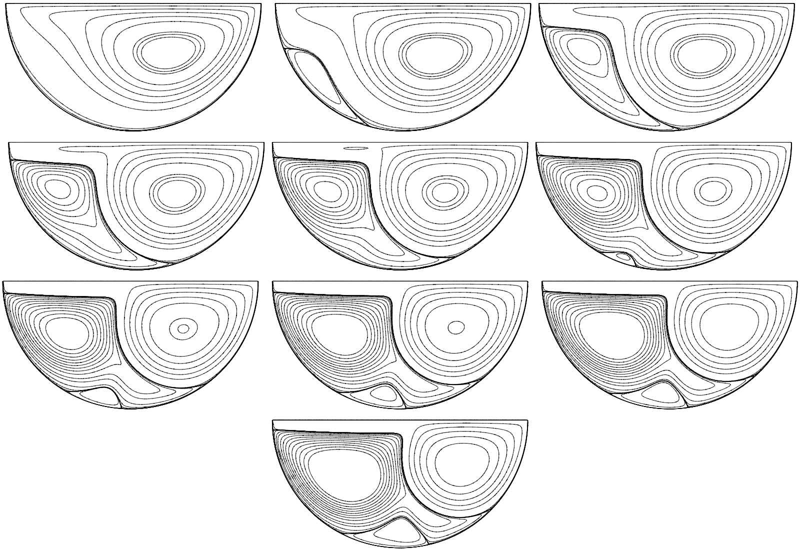

Streamlines of the steady state solution for
$Re = 500, 1000, 2000, 3000, 4000, 5000, 6000, 7000, 8000$ and $Re = 9000$.

In order to compute y^{n+1} from y^n, one consider the following extremal problem

$$\inf_{y\in \boldsymbol{V}} E_n(y), \quad E_n(y) = \frac{1}{2}\int_\Omega \frac{1}{\theta\delta t}|v|^2 + |\nabla v|^2 \tag{30}$$

where the corrector $v \in \boldsymbol{V}$ solves

$$\begin{aligned} \frac{1}{\theta\delta t}\int_\Omega v\cdot w + \int_\Omega \nabla v\cdot\nabla w = -\alpha\int_\Omega y\cdot w - \nu\int_\Omega \nabla y\cdot\nabla w - \int_\Omega y\cdot\nabla y\cdot w \\ + < f^n, w >_{H^{-1}(\Omega)^d\times H_0^1(\Omega)^d} + \frac{1}{\theta\delta t}\int_\Omega y^n\cdot w, \quad \forall w\in \boldsymbol{V} \end{aligned} \tag{31}$$

The natural choice is to initialize the minimizing sequence, says $(y_k^{n+1})_{k\in\mathbb{N}}$ for E_n with $y_0^{n+1} = y^n$:

Theorem (Lemoine, M' 2019)

Assume $\theta = 1$. Assume Ω is C^2 and $\nabla y^0 \in L^2(\Omega)^d$. Then, the sequence $\{y^n\}_n$ satisfies

$$\|y^{n+1} - y^n\|_2 = \mathcal{O}\left(\frac{\delta t}{\nu^{3/2}}\right)^{1/2} \tag{32}$$

$Q_T = \Omega \times (0, T)$. We define the functional $E : C([0, T], H) \cap L^2(0, T, V) \to \mathbb{R}^+$ by putting

$$E(y) = \frac{1}{2}\iint_{Q_T} |\nabla v|^2 \, dx \, dt \tag{33}$$

where the corrector $v \in C([0, T]; H) \cap L^2(0, T, V)$ solves the boundary value problem

$$\begin{cases} v_t - \nu\Delta v + \nabla\pi + y_t - \nu\Delta y + (y\cdot\nabla)y = f, \quad \nabla\cdot v = 0 \quad \text{in } \; Q_T, \\ v = 0 \quad \text{on } \; \Sigma_T := \partial\Omega \times (0, T), \quad v(\cdot, 0) = 0 \quad \text{in } \; \Omega \end{cases} \tag{34}$$

Proposition (Lemoine, M' 19)

Let y_0 be a solution of (NS). Let $M \in \mathbb{R}$ such that $\|y_0\|_{L^\infty(0,T,H)} \leq M$ and $\|\nabla y_0\|_{L^2(Q_T)^4} \leq M$. If $\|y\|_{L^\infty(0,T,H)} \leq M$ and $\|\nabla y\|_{L^2(Q_T)^4} \leq M$, then there exists a constant $c(M)$ such that

$$\nu \iint_{Q_T} |\nabla(y - y_0)|^2 \leq c(M)\Big(E(y)^{1/2} + E(y)\Big) \tag{35}$$

and for all $t \in [0, T]$,

$$\int_\Omega |y(t) - y_0(t) + v(t)|^2 \leq c(M)\Big(E(y)^{1/2} + E(y)\Big). \tag{36}$$

A space-time least-squares approach for NS

$Q_T = \Omega \times (0,T)$. We define the functional $E : C([0,T],H) \cap L^2(0,T,V) \to \mathbb{R}^+$ by putting

$$E(y) = \frac{1}{2} \iint_{Q_T} |\nabla v|^2 \, dx \, dt \tag{33}$$

where the corrector $v \in C([0,T];H) \cap L^2(0,T,V)$ solves the boundary value problem

$$\begin{cases} v_t - \nu \Delta v + \nabla \pi + y_t - \nu \Delta y + (y \cdot \nabla) y = f, \quad \nabla \cdot v = 0 \quad \text{in } \; Q_T, \\ v = 0 \quad \text{on } \; \Sigma_T := \partial\Omega \times (0,T), \quad v(\cdot,0) = 0 \quad \text{in } \; \Omega \end{cases} \tag{34}$$

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and for all $t \in [0,T]$,

$$\int_\Omega |y(t) - y_0(t) + v(t)|^2 \leq c(M) \Big(E(y)^{1/2} + E(y) \Big). \tag{36}$$

Proposition (Lemoine, M' 19)

For all $y \in C([0,T],H) \cap L^2(0,T,V)$ such that $\|y\|_{L^\infty(0,T;H)} \leq M$ and $\|\nabla y\|_{L^2(Q_T)^4} \leq M$, there exists a unique solution of

$$\begin{cases} \partial_t \int_\Omega Y_1 \cdot w + \nu \int_\Omega \nabla Y_1 \cdot \nabla w + \int_\Omega y \cdot \nabla Y_1 \cdot w \\ \qquad + \int_\Omega Y_1 \cdot \nabla y \cdot w = -\partial_t \int_\Omega v \cdot w - \nu \int_\Omega \nabla v \cdot \nabla w, \quad \forall w \in V, \\ Y(0) = 0. \end{cases}$$

Moreover,

$$\|Y_1\|_{L^\infty(0,T;H)} \leq c(M), \quad \|\nabla Y_1\|_{L^2(Q_T)^4} \leq c(M)$$

and

$$E'(y) \cdot Y_1 = 2\,E(y) \tag{37}$$

Corollary

If $(y_n)_{n\in\mathbb{N}^*}$ is a bounded sequence in $L^\infty(0,T;H) \cap L^2(0,T;V)$ and If $(E'(y_n))_{n\in\mathbb{N}^*} \to 0$, then $(E(y_n))_{n\in\mathbb{N}^*} \to 0$.

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$Q_T = \Omega \times (0, T)$. We define the functional $E : C([0, T], H) \cap L^2(0, T, V) \to \mathbb{R}^+$ by putting

$$E(y) = \frac{1}{2} \iint_{Q_T} (|v_t|^2 + |\nabla v|^2)\, dx\, dt \tag{38}$$

where the corrector $v \in H^1(Q_T)$ solves the boundary value problem

$$\begin{cases} -v_{tt} - \nu \Delta v + y_t - \nu \Delta y + (y \cdot \nabla) y + \nabla \pi = f, \quad \nabla \cdot v = 0 \quad \text{in } \; Q_T, \\ v = 0 \quad \text{on } \; \Sigma_T := \partial\Omega \times (0, T), \quad v_t = 0 \quad \text{on } \; \Omega \times \{0, T\}. \end{cases} \tag{39}$$

Find a distributed control $u \in L^2(\Omega \times (0,T))$ such that the solution of

$$\begin{cases} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p = u1_\omega, \nabla\cdot y = 0 & \Omega\times(0,T), \\ y = 0, & \partial\Omega\times(0,T), \\ y(0) = y_0, & \Omega\times\{0\} \end{cases} \tag{40}$$

satisfies $y(T) = y_d$, a trajectory (control of flows).

The least-squares problem is

$$inf_{y,u} E(y,u) = \frac{1}{2}\iint_{Q_T} |\nabla v|^2 \, dx\, dt \tag{41}$$

over

$$\mathcal{A} = \left\{ (y,u) : y \in C([0,T],H) \cap L^2(0,T,V), y(\cdot,T) = y_d, u \in L^2(\Omega\times(0,T)) \right\}$$

where the corrector $v \in C([0,T];H) \cap L^2(0,T,V)$ solves the boundary value problem

$$\begin{cases} v_t - \nu\Delta v + \nabla\pi + y_t - \nu\Delta y + (y\cdot\nabla)y = u1_\omega, \quad \nabla\cdot v = 0 \quad \text{in} \ \ Q_T, \\ v = 0 \quad \text{on} \ \ \Sigma_T := \partial\Omega\times(0,T), \quad v(\cdot,0) = 0 \quad \text{in} \ \ \Omega \end{cases} \tag{42}$$

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$$\begin{cases} y_t - y_{xx} - 5y\, log^{1.4}(1+|y|) = 0, & (x,t) \in (0,1)\times(0,1/2), \\ y(\cdot,0) = 3sin(\pi x), & x \in \Omega, \\ y(0,t) = 0, y(1,t) = u(t), & t \in (0,1/2) \end{cases} \tag{43}$$

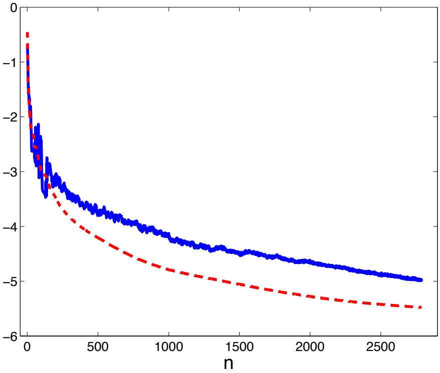
Find a null control u such that for $y(T=1/2)=0$?

Uniform null controllability is given in [Barbu 99], [Fernandez-Cara Zuazua 00]. [8]

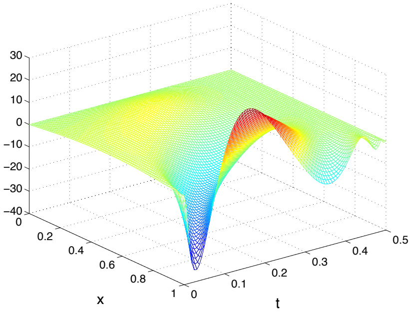
The controllability is obtained by linearization and fixed point argument, useless in practice if the fixed point operator is not a contraction. [Fernandez-Cara, Munch 2012][9]

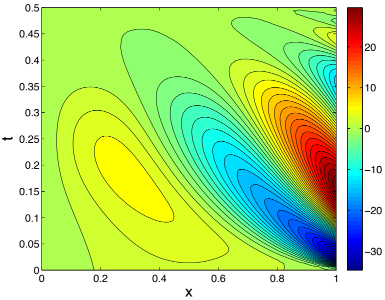
[8] E. Fernandez-Cara and E. Zuazua, Null and approximate controllability for weakly blowing up semilinear, Ann. Inst. Henri Poincaré, Analyse non linéaire. (2000).

[9] E. Fernandez-Cara, A. Munch, Numerical null controllability of semi-linear 1D heat equations : fixed point, least squares and Newton methods, Mathematical Control and Related Fields (2012).



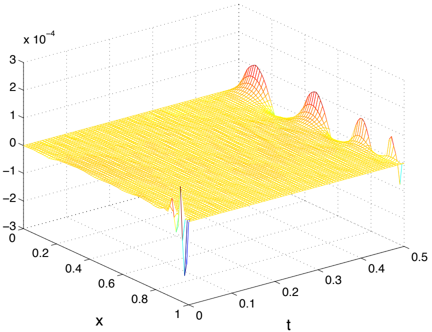
$\log_{10}(E(y)$ (**dashed line**) and $\log_{10}(\|g\|_{\mathcal{A}})$ (**full line**) vs. the iteration n of the CG algorithm.





Controlled solution $y \in \mathcal{A}$ along $Q_T = (0,1) \times (0,T)$ and its isovalues.

Corrector



Corrector function $v \in H^1(Q_T)$ along $Q_T = (0,1) \times (0,T)$

Some conclusions

- We have analyzed rigorously the H^{-1}-LS method introduced by Glowinski in 1979 and extended it to space-time domain.
- The analysis leads to an improvement of the Newton method and can be extended to any "reasonable" nonlinearities.
- The numerical analysis (w.r.t. approximation) seems doable since inequality like

$$\|y_h - \overline{y}\|_V \leq C\sqrt{E(y_h)}, \quad \forall y_h \in V_h \subset \boldsymbol{V}$$

  remains true.
- We may adapt the method to solve Inverse Problems.

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THANK YOU FOR YOUR ATTENTION !!!