Approximation of controllability and inverse problems for PDE

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PART 3

Parabolic case

Introduction

$\Omega \subset \mathbb{R}^N$; $Q_T = \Omega \times (0, T)$; $q_T = \omega \times (0, T)$

$$\begin{cases} y_t - \nabla \cdot (c(x)\nabla y) + d(x,t)y = v\,1_\omega, & \text{in} \quad Q_T, \\ y = 0, & \text{in} \quad \Sigma_T, \\ y(x,0) = y_0(x), & \text{in} \quad \Omega. \end{cases} \tag{1}$$

$c := (c_{i,j}) \in C^1(\overline{\Omega}; \mathcal{M}_N(\mathbb{R}))$; $(c(x)\xi, \xi) \geq c_0|\xi|^2$ in $\overline{\Omega}$ $(c_0 > 0)$,

$d \in L^\infty(Q_T)$, $y_0 \in L^2(\Omega)$;

$v = v(x,t)$ is the *control* $\quad y = y(x,t)$ is the associated state.

We introduce the linear manifold

$$\mathcal{C}(y_0, T) = \{\, (y, v) : v \in L^2(q_T), \;\; y \text{ solves (1) and satisfies } y(T, \cdot) = 0 \,\}.$$

non empty (see FURSIKOV-IMANUVILOV'96, ROBBIANO-LEBEAU'95)).

NOTATIONS -
$Ly := y_t - \nabla \cdot (c(x)\nabla y) + d(x,t)y$; $\quad L^\star\varphi := -\varphi_t - \nabla \cdot (c(x)\nabla\varphi) + d(x,t)\varphi$

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PART I

Control of minimal $L^2(q_T)$-norm

$$(P) \qquad \inf_{(y,v)\in\mathcal{C}(y_0,T)} J(v,y) = \frac{1}{2}\|v\|^2_{L^2(q_T)}$$

$N = 1$ - $L^2(q_T)$-norm of the HUM control with respect to time

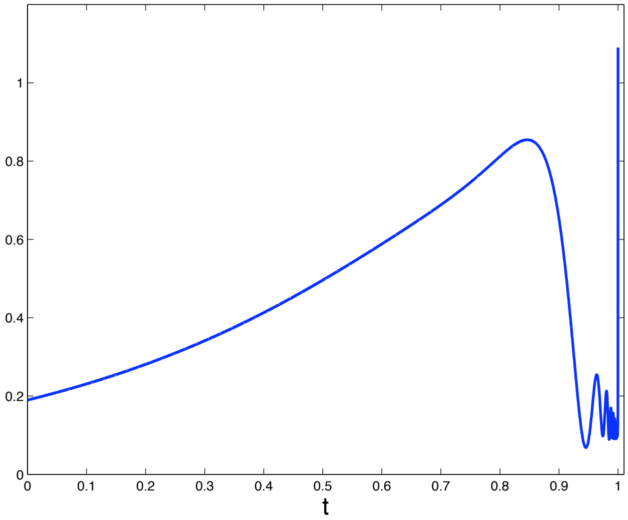

Figure: $y_0(x) = \sin(\pi x)$ - $T = 1$ - $\omega = (0.2, 0.8)$ - $t \to \|v(\cdot, t)\|_{L^2(0,1)}$ in $[0, T]$

$N = 1$ - L^2-norm of the HUM control with respect to time: Zoom near T

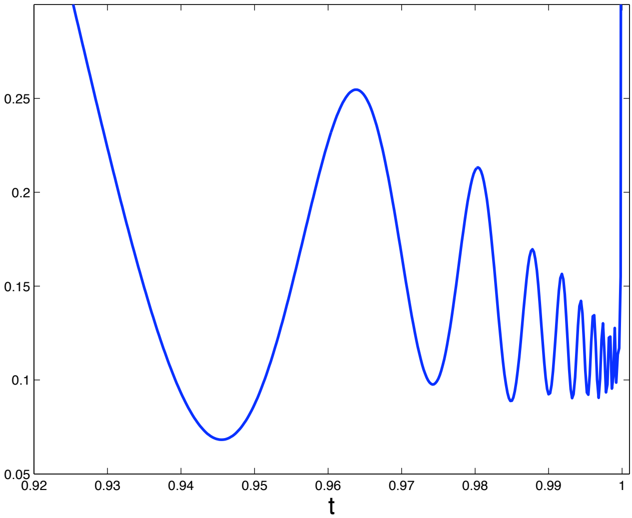

Figure: $y_0(x) = \sin(\pi x)$ - $T = 1$ - $\omega = (0.2, 0.8)$ - $t \to \|v(\cdot, t)\|_{L^2(0,1)}$ in $[0.92T, T]$

Minimal L^2 norm control

Since it is difficult to construct pairs $(v, y) \in \mathcal{C}(y_0, T)$ (*a fortiori* minimizing sequences for J !), it is standard to consider the corresponding dual :

$$\inf_{(y,v)\in\mathcal{C}(y_0,T)} J(y,v) = -\inf_{\phi_T\in H} J^\star(\phi_T), \; J^\star(\phi_T) = \frac{1}{2}\int_{q_T} \phi^2 dxdt + \int_\Omega \phi(0,\cdot)y_0 dx$$

where ϕ solves the backward system

$$\begin{cases} L^\star\phi = 0 & Q_T = (0,T)\times\Omega, \\ \phi = 0 \quad \Sigma_T = (0,T)\times\partial\Omega, & \phi(T,\cdot) = \phi_T \quad \Omega. \end{cases}$$

The Hilbert space H is defined as the completion of $\mathcal{D}(\Omega)$ with respect to the norm

$$\|\phi_T\|_H = \left(\int_{q_T} \phi^2(t,x)dxdt\right)^{1/2}.$$

From the observability inequality

$$C(T,\omega)\|\phi(0,\cdot)\|^2_{L^2(\Omega)} \leq \|\phi_T\|^2_H \quad \forall\phi_T \in L^2(\Omega),$$

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Ill-posedness

- The completed space H is huge:

$$H^{-s} \subset H \quad \forall s > 0!$$

(H may also contain elements which are not distribution !!):
Micu [1] proved in 1D that

the set of initial data y_0, for which the corresponding ϕ_T, minimizer of $J^\star$, does not belong to any negative Sobolev spaces, is dense in $L^2(0,1)$!!!

-The dual variable ϕ_T is the Lagrange multiplier for the constraint $y(\cdot,T)=0$ may belong to a "large" dual space, much larger than $L^2(\Omega)$:

$$< y(\cdot,T), \phi_T >= 0$$

-Ill-posedness here is therefore related to the hugeness of H, poorly approximated numerically.

-This phenomenon is unavoidable (unless $\omega = \Omega$!) and is independent of the choice of the norm !

[1] S. Micu, *Regularity issues for the null-controllability of the linear 1-d heat equation, 2011*

Optimal backward solution ϕ on $\partial\omega \times [0, T]$

$$T = 1, \quad y_0(x) = \sin(\pi x), \quad a(x) = a_0 = 1/10, \quad \omega = (0.2, 0.8)$$

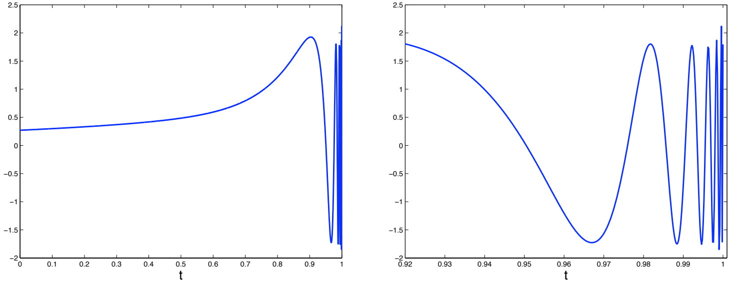

Figure: $T = 1$ - $\omega = (0.2, 0.8)$ - $\phi^N(\cdot, 0.8)$ for $N = 80$ on $[0, T]$ (**Left**) and on $[0.92T, T]$ (**Right**).

[Carthel-Glowinski-Lions'94, JOTA], [AM-Zuazua'11, Inverse Problems]

Regularization

For any $\epsilon > 0$, consider $J_\epsilon(y,v) = J(y,v) + \frac{\epsilon^{-1}}{2}\|y(T)\|^2_{H^{-s}(0,1)}$ and

$$\inf_{\phi_{T,\epsilon}\in L^2(0,1)} J^\star_\epsilon(\phi_{T,\epsilon}), \quad J^\star_\epsilon(\phi_{T,\epsilon}) = \frac{1}{2}\int_{q_T}\phi^2 dxdt + \int_\Omega \phi(0,\cdot)y_0 dx + \frac{\epsilon}{2}\|\phi_{T,\epsilon}\|^2_{H^s(0,1)}$$

and minimize in L^2 the quadratic and strictly convex function $J^\star_\epsilon$ by a conjugate gradient algorithm as initially proposed in Carthel-Glowinski-Lions'94 [2].

$$\phi_T(x) = \sum_{k\geq 1} a_k \sin(k\pi x) \Longleftrightarrow y_T(x) = \sum_{p\geq 1} b_p \sin(p\pi x), \quad x\in\Omega$$

and taking $y_0 = 0$ (for simplicity), we obtain the relation

$$b_p = \sum_{k\geq 1}\Big(c_{p,k}(\omega)g_{p,k}(T) + \epsilon(k\pi)^{2s}\delta_{p,k}\Big)a_{k,\epsilon}, \quad s = 0,1.$$

$$c_{p,k}(\omega) = 2\int_\omega \sin(k\pi x)\sin(p\pi x)dx, \quad g_{p,k}(T) = \frac{1-e^{-c(\lambda_p+\lambda_k)T}}{\lambda_k+\lambda_p}, \quad \lambda_k = (k\pi)^2$$

[3, 4]

[2] Carthel-Glowinski-Lions, *On exact and approximate boundary controllabilities for the heat equation: a numerical approach*, JOTA (1994)

[3] S. Kindermann, Convergence rates of the Hilbert uniqueness method via Tikhonov regularization, J. Optim. Theory Appl., (1999)

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Lack of uniform observability vs. ill-posedness

Discrete observability inequalities : Behavior of $C_{obs,h}$ w.r.t. h ?

$$\|\phi_h(0)\|^2_{L^2(\Omega)} \leq C_{obs,h} \int_0^T \int_\omega \phi_h^2(t,x)dxdt, \quad \forall \phi_{Th} \in L^2(\Omega)$$

Labbe-Trelat [5], Boyer [6] analyzed the case $\epsilon = h^\alpha$ and proposes some schemes leading to

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Remark- The condition number of the discrete HUM operator : $\Lambda_h : \phi_{Th} \to y_{Th}$ is estimated by

$$cond(\Lambda_h) \leq C_{obs,h} C_{2h} h^{-2}$$

where

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We have that

$$C_{2h} \to \infty \quad h \to 0$$

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Regular and singular perturbation of the controllability problem

Other regularization / perturbation are considered in [AM-Zuazua'10] [7]
1- Replace the heat equation by the hyperbolic equation

$$y_{\epsilon,t} - c\Delta y_\epsilon + \boldsymbol{\epsilon y_{\epsilon,tt}} = v_\epsilon 1_\omega, \quad \text{in} \quad Q_T,$$

2- Singular (non uniformly controllable w.r.t. ε) perturbation

$$y_{\epsilon,t} - c\Delta y_\epsilon - \boldsymbol{\epsilon \partial_t \Delta y_\epsilon} = v_\epsilon 1_\omega \quad \text{in} \quad Q_T.$$

Question:

$$\frac{\partial^\alpha y_\alpha}{\partial t^\alpha} - \Delta y_\alpha = v_\alpha 1_\omega \quad \text{in} \quad Q_T, \qquad \alpha < 1???$$

[7] AM, E. Zuazua, Numerical approximation of null controls for the heat equation : ill-posedness and remedies. Inverse Problems, (2010)

PART II
Transmutation method [8]

[8] L. Miller, The control transmutation method and the cost of fast controls 2006

The control transmutation method

Let $L > 0$ and $y_0 \in H_0^1(\Omega)$. IF $f \in L^2([0, L] \times \omega)$ is a null-control for w, solution of the wave equation

$$\begin{cases} w_{ss} - w_{xx} = f\, 1_\omega & (s, x) \in (0, L) \times \Omega, \\ w = 0 & (0, L) \times \partial\Omega, \\ (w(0), w_s(0)) = (y_0, 0) \Longrightarrow (w(L), w_s(L)) = (0, 0) & \end{cases}$$

AND if $H \in C^0([0, T], \mathcal{M}(]-L, L[)$ is a fundamental controlled solution for the heat equation

$$\begin{cases} \partial_t H - \partial_s^2 H = 0 \quad \text{in} \quad \mathcal{D}'(]0, T[\times]-L, L[), \\ H(t = 0) = \delta, \quad H(t = T) = 0 \end{cases}$$

THEN the fonction

$$v(t, x) = 2 \int_0^L H(t, s) f(s, x) ds\, 1_\omega(x), \quad (0, T) \times \Omega$$

is a null control in $L^2(q_T)$ for $\quad y(t, x) = 2 \int_0^L H(t, s) w(s, x) ds \quad$ solution of the heat equation

$$\begin{cases} y_t - y_{xx} = v\, 1_\omega & (0, T) \times \Omega, \\ y = 0 & (0, T) \times \partial\Omega, \\ y(0) = y_0 & \end{cases}$$

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Computation of the fundamental solution for the heat equation

Jones [9], Rouchon [10]. Let $\delta \in (0, T)$. For $t \in (0, \delta)$, H is taken as the Gaussian :

$$H(t, s) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{s^2}{4t}}, \quad (t, s) \in (0, \delta) \times \mathbb{R}.$$

so that it remains to join $H(\delta, s)$ to 0 at time T. For any $a > 0$ and any $\alpha \geq 1$, we consider the *bump* function

$$h(n) = \exp\left(-\frac{a}{((n-\delta)(T-n))^\alpha}\right), \quad n \in (\delta, T)$$

and then the function

$$p(t) = \frac{1}{\sqrt{4\pi t}} \begin{cases} 1 & t \in (0, \delta) \\ \dfrac{\int_t^T h(n)dn}{\int_\delta^T h(n)dn} & t \in (\delta, T) \end{cases}$$

so that $p(T) = 0$. $h \in C_c^\infty([\delta, T])$ and $p \in C^\infty([0, T])$. h and p are both Gevrey functions of order $1 + 1/\alpha \in (1, 2]$ so that the serie

$$H(t, s) = \sum_{k \geq 0} p^{(k)}(t) \frac{s^{2k}}{(2k)!} \tag{2}$$

is convergent. (2) defines a solution of the heat equation and satisfies $H(T, s) = 0$ for all $s \in \mathbb{R}$ and

$\lim_{t \to 0^+} H(t, s) = \delta_{s=0}$.

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[9] B. Jones, A fundamental solution for the heat equation which is supported in a strip, J. Math. Anal. Appl. 1977

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$$H(t, s) = \sum_{k \geq 0} p^{(k)}(t) \frac{s^{2k}}{(2k)!} \tag{2}$$

is convergent. (2) defines a solution of the heat equation and satisfies $H(T, s) = 0$ for all $s \in \mathbb{R}$ and

$\lim_{t \to 0^+} H(t, s) = \delta_{s=0}$.

---

[9] B. Jones, A fundamental solution for the heat equation which is supported in a strip, J. Math. Anal. Appl. 1977

[10] B. Laroche, P. Martin and P. Rouchon, Motion planning for the heat equation, Int. Journal of Robust and Nonlinear Control, (2000)

Fundamental solution for the heat equation: example

$a_0 = 1$ by the change of variable $(\tilde{x}, \tilde{t}) = (a_0 t, x)$

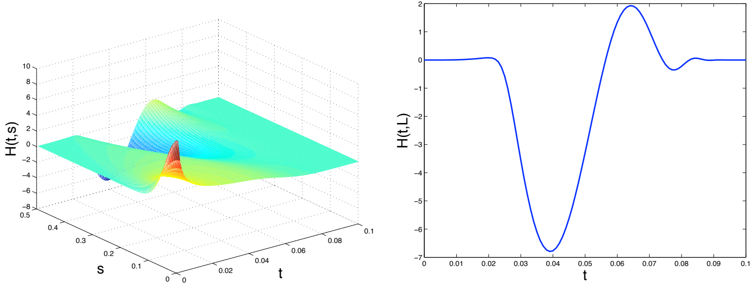

Figure: $L = 0.5$ - $T = 0.1$ - $(a, \alpha, \delta) = (10^{-2}, 1, T/5)$ - **Left**: fundamental solution H on $(0, T) \times (0, L)$ - **Right**: $H(t, L)$ vs. $t \in (0, T)$.

Fundamental solution for the heat equation: example

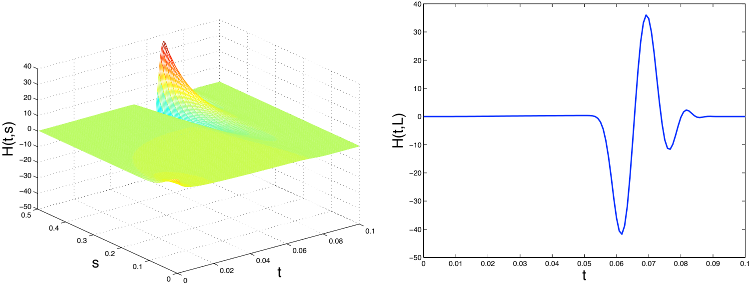

Figure: $L = 0.5$ - $T = 0.1$ - $(a, \alpha, \delta) = (10^{-2}, 1, T/2)$ - **Left**: fundamental solution H on $(0, T) \times (0, L)$ - **Right**: $H(t, L)$ vs. $t \in (0, T)$.

Control by the transmutation method

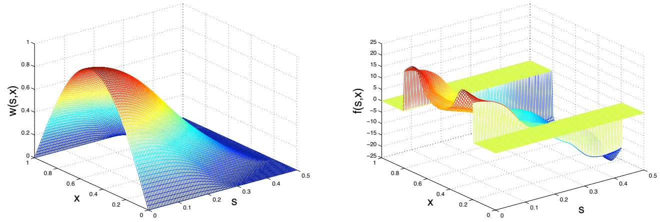

Figure: $y_0(x) = \sin(\pi x), L = 0.5$ - Controlled wave solution w (**Left**) and corresponding HUM control f (**Right**) on $(0, L) \times \Omega$.

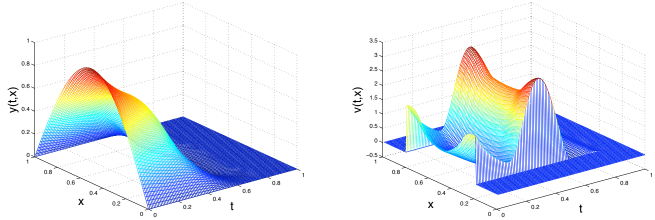

Figure: $y_0(x) = \sin(\pi x), T = 1, a_0 = 1/10, (\delta, \alpha) = (T/5, 1)$ - Controlled heat solution y (**Left**) and corresponding transmutted control v (**Right**) on $(0, T) \times \Omega$.

Control by the transmutation method

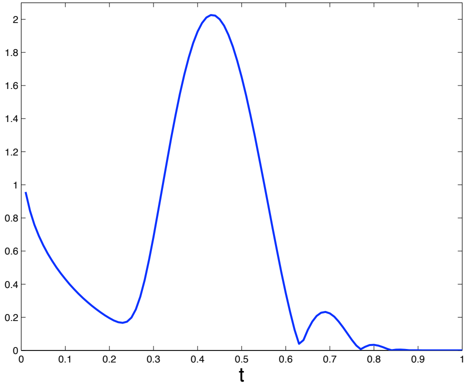

Figure: $L^2(\omega)$ norm of the control v vs $t \in [0, T]$ for $(y_0(x), T, a_0) = (\sin(\pi x), 1, 1/10)$

Transmutation to HUM ?

$\|v\|_{L^2(q_T)} \leq 2\|f\|_{L^2((0,L)\times\omega)}\|H\|_{L^2((0,T)\times(0,L))}$

$\|H\|_{L^2((0,T)\times(0,L))}$ is reduced if δ is small (reduce the time period where the dissipation is governed by the gaussian), and $\alpha_1 > 1$ (allows to take δ small) and $\alpha_2 < 1$ (increase the magnitude of the control near T).

$$h(s) = \exp\left(-\frac{a}{(s-\delta)^{\alpha_1}(T-s)^{\alpha_2}}\right)$$

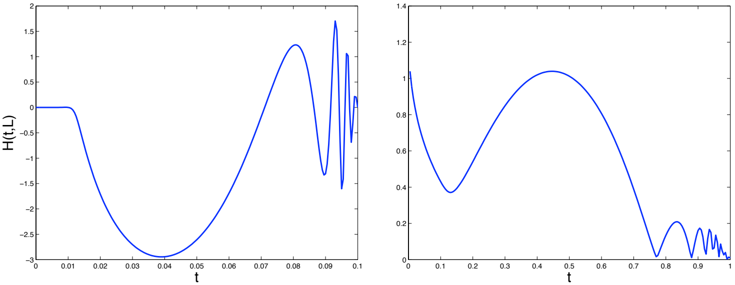

Figure: $(y_0(x), a_0) = (\sin(\pi x), 1/10)$ - Heat fundamental solution $H(t, L)$ vs. $t \in [0, \tilde{T}]$ (**Left**) and $L^2(\Omega)$-norm of corresponding control v (**Right**).
$\alpha_1 = 1.1, \alpha_2 = 0.7$ $\|g\|_{L^2(Q_T)} \approx 5.67 \times 10^{-1}$

-The transmuted control $v_h = (v)_{h>0}$ ensures that $\|y_h(T,\cdot)\|_{L^2(\Omega)} \approx 10^{-5}$

-Once a solution H in the one dimensional is constructed, we can take

$$H_n(t, x_1, x_2, \cdots, x_n) = H(t, x_1) \times H(t, x_2) \times \cdots \times H(t, x_n)$$

as a fundamental control solution for $(t, x) \in (0, T) \times [-L, L]^n$. Consequently, the transmutation provides also a control in any dimension, provided some geometric condition on the support ω.

-The transmutation method provides uniformly bounded discrete control $\{v_h\}$ discretization of

$$v(t, x) = 2 \sum_{k \geq 0} p^{(k)}(t) \int_0^L \frac{s^{2k}}{(2k)!} f(s, x) ds \, 1_\omega(x)$$

- The main difficulty is the robust evaluation of $p^{(k)}$. [11]

[11] P. Martin, P. Rouchon, L. Rosier, Null controllability of one-dimensional parabolic equations by the flatness approach (2016)

PART III

Change of the norm : framework of Fursikov-Imanuvilov'96 [12]

$$\begin{cases} \text{Minimize } J(y,v) = \dfrac{1}{2}\displaystyle\iint_{Q_T} \rho^2 |y|^2 \, dx\, dt + \dfrac{1}{2}\iint_{q_T} \rho_0^2 |v|^2 \, dx\, dt \\ \text{Subject to } (y,v) \in \mathcal{C}(y_0, T). \end{cases} \quad (3)$$

where ρ, ρ_0 are non-negative continuous weights functions such that $\rho, \rho_0 \in L^\infty(Q_{T-\delta}) \quad \forall \delta > 0$.

[12] A.V. Fursikov and O. Yu. Imanuvilov, *Controllability of Evolution Equations,* Lecture Notes Series, number 34. Seoul National University, Korea, (1996) 1–163.

Primal (direct) approach

Following Fursikov-Imanuvilov'96, we assume Carleman type weights :

$$\left\{\begin{array}{l}\rho(x,t)=\exp\left(\dfrac{\beta(x)}{T-t}\right),\ \ \rho_0(x,t)=(T-t)^{3/2}\rho(x,t),\ \ \beta(x)=K_1\left(e^{K_2}-e^{\beta_0(x)}\right)\\ \text{where the } K_i \text{ are sufficiently large positive constants (depending on } T,\, c_0 \text{ and } \|c\|_{C^1})\\ \text{and } \beta_0\in C^\infty(\overline{\Omega}),\ \beta_0>0 \text{ in } \Omega,\ (\beta_0)_{|\partial\Omega}=0,\ |\nabla\beta_0|>0 \text{ outside } \omega.\end{array}\right. \tag{4}$$

We introduce

$$P_0=\{\, q\in C^2(\overline{Q}_T) : q=0 \ \text{ on } \ \Sigma_T\,\}.$$

In this linear space, the bilinear form

$$(p,q)_P := \iint_{Q_T}\rho^{-2}L^*p\,L^*q\,dx\,dt+\iint_{q_T}\rho_0^{-2}p\,q\,dx\,dt$$

is a scalar product (unique continuation property).

Let P be the completion of P_0 for this scalar product.

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Carleman estimates

Lemma (Fursikov-Imanuvilov'96)

Let ρ and ρ_0 be given by (4) and let $(\rho_1, \rho_2) = ((T-t)^{1/2}, (T-t)^{-1/2})\rho$. Then there exists $C > 0$, only depending on ω, T, c_0 and $\|c\|_{C^1}$, such that

$$\left\{ \begin{aligned} &\iint_{Q_T} \left[\rho_2^{-2}\left(|q_t|^2 + |\Delta q|^2\right) + \rho_1^{-2}|\nabla q|^2 + \rho_0^{-2}|q|^2\right] dx\,dt \\ &\qquad \le C\left(\iint_{Q_T} \rho^{-2}|L^* q|^2\,dx\,dt + \iint_{q_T} \rho_0^{-2}|q|^2\,dx\,dt\right), \forall q \in P. \end{aligned} \right. \tag{5}$$

Lemma (Fursikov-Imanuvilov'96, Fernández-Cara-Guerrero'06)

Under the same assumptions, for any $\delta > 0$, one has

$$P \hookrightarrow C^0([0, T-\delta]; H_0^1(\Omega)),$$

where the embedding is continuous. In particular, there exists $C > 0$, only depending on ω, T, a_0 and $\|a\|_{C^1}$, such that, for all $q \in P$,

$$\|q(\cdot\,, 0)\|^2_{H_0^1(\Omega)} \le C\left(\iint_{Q_T} \rho^{-2}|L^* q|^2\,dx\,dt + \iint_{q_T} \rho_0^{-2}|q|^2\,dx\,dt\right). \tag{6}$$

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Primal (direct) approach

Proposition

Let ρ and ρ_0 be given by (4). Let (y, v) be the corresponding optimal pair for J. Then there exists $p \in P$ such that

$$y = \rho^{-2} L^* p, \quad v = -\rho_0^{-2} p|_{q_T}\,. \tag{7}$$

The function p is the unique solution in P of

$$\iint_{Q_T} \rho^{-2} L^* p \, L^* q \, dx\, dt + \iint_{q_T} \rho_0^{-2} p\, q\, dx\, dt = \int_0^1 y_0(x)\, q(x,0)\, dx, \quad \forall q \in P \tag{8}$$

Remark

p solves, at least in $\mathcal{D}'$, the following differential problem, that is second order in time and fourth order in space:

$$\begin{cases} L(\rho^{-2} L^* p) + \rho_0^{-2} p \, 1_\omega = 0, & (x,t) \in (0,1) \times (0,T) \\ p(x,t) = 0, \ \ (-\rho^{-2} L^* p)(x,t) = 0 & (x,t) \in \{0,1\} \times (0,T) \\ (-\rho^{-2} L^* p)(x,0) = y_0(x), \ \ (-\rho^{-2} L^* p)(x,T) = 0, & x \in (0,1). \end{cases} \tag{9}$$

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Conformal approximation

For large integers N_x and N_t, we set $\Delta x = 1/N_x$, $\Delta t = T/N_t$ and $h = (\Delta x, \Delta t)$. Let us introduce the associated uniform triangulation $\mathcal{T}_h$, with

$$\overline{Q_T} = \bigcup_{K \in \mathcal{T}_h} K.$$

The following (conformal) finite element approximations of the space P are introduced:

$$P_h = \{\, q_h \in C^{1,0}_{x,t}(\overline{Q}_T) : q_h|_K \in (\mathbb{P}_{3,x} \otimes \mathbb{P}_{1,t})(K) \ \ \forall K \in \mathcal{T}_h, \ \ q_h|_{\Sigma_T} \equiv 0 \,\},$$

where $C^{1,0}_{x,t}(\overline{Q}_T)$ is the space of the functions in $C^0(\overline{Q}_T)$ that are continuously differentiable with respect to x in $\overline{Q}_T$.

The variational equality (28) is approximated as follows:

$$\left\{ \begin{array}{l} \displaystyle\iint_{Q_T} \rho^{-2} L^* p_h \, L^* q_h \, dx\, dt + \iint_{q_T} \rho_0^{-2} p_h \, q_h \, dx \, dt = \int_0^1 y_0(x) \, q_h(x,0) \, dx \\ \forall q_h \in P_h; \quad p_h \in P_h . \end{array} \right. \tag{10}$$

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Conformal approximation

Theorem (Fernandez-Cara, AM)

Let $p_h \in P_h$ be the unique solution to (11). Let us set

$$y_h := \rho^{-2} L^\star p_h, \quad v_h := -\rho_0^{-2} p_h \, 1_{q_T}.$$

Then one has

$$\|y - y_h\|_{L^2(Q_T)} \to 0 \text{ and } \|v - v_h\|_{L^2(q_T)} \to 0, \qquad as \quad h \to 0$$

where (y, v) is the minimizer of J.

In practice, we introduce the variable $m_h := \rho_0^{-1} p_h \in \rho_0^{-1} P_h \subset \rho_0^{-1} P \subset C([0,T], H_0^1(\Omega))$ and we solve

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Experiment with $\omega = (0.2, 0.8)$

$\Delta x = \Delta t$	$1/20$	$1/40$	$1/80$	$1/160$	$1/320$
conditioning	1.33×10^{14}	1.76×10^{22}	7.86×10^{32}	2.17×10^{44}	2.30×10^{54}
$\|p_h(\cdot, T)\|_{L^2(0,1)}$	2.85×10^{1}	2.04×10^{2}	1.59×10^{3}	4.70×10^{4}	6.12×10^{6}
$\|y_h(\cdot, T)\|_{L^2(0,1)}$	4.37×10^{-2}	2.18×10^{-2}	1.09×10^{-2}	5.44×10^{-3}	2.71×10^{-3}
$\|v_h\|_{L^2(q_T)}$	1.228	1.251	1.269	1.281	1.288

Table: $T = 1/2$, $y_0(x) \equiv \sin(\pi x)$, $a(x) \equiv 10^{-1}$. $\|y_h(\cdot, T)\|_{L^2(0,1)} = \mathcal{O}(h)$.

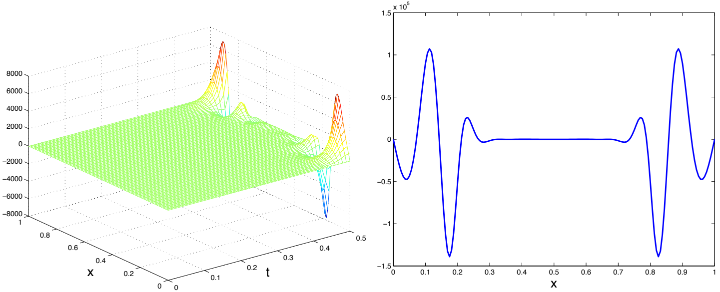
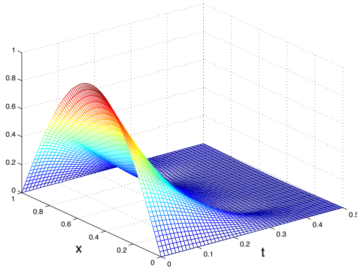


Figure: $\omega = (0.2, 0.8)$. The adjoint state p_h and its restriction to $(0, 1) \times \{T\}$.

Experiments with $\omega = (0.2, 0.8)$

$$T = 1/2,\ y_0(x) \equiv \sin(\pi x),\ a(x) \equiv 10^{-1}.$$



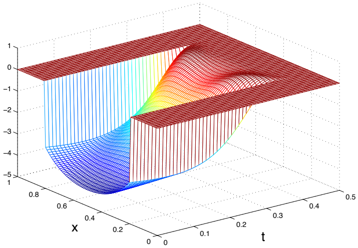


Figure: $\omega = (0.2, 0.8)$. The state y_h (**Left**) and the control v_h (**Right**).

Experiments with $\omega = (0.3, 0.4)$

$\Delta x = \Delta t$	$1/20$	$1/40$	$1/80$	$1/160$	$1/320$
conditioning	3.06×10^{14}	5.24×10^{22}	2.13×10^{33}	5.11×10^{44}	4.03×10^{54}
$\Vert p_h(\cdot, T)\Vert_{L^2(0,1)}$	1.37×10^{3}	5.51×10^{3}	5.12×10^{4}	2.16×10^{6}	3.90×10^{6}
$\Vert y_h(\cdot, T)\Vert_{L^2(0,1)}$	1.55×10^{-1}	9.46×10^{-2}	6.12×10^{-2}	3.91×10^{-2}	2.41×10^{-2}
$\Vert v_h\Vert_{L^2(q_T)}$	5.813	8.203	10.68	13.20	15.81

Table: $T = 1/2$, $y_0(x) \equiv \sin(\pi x)$, $a(x) \equiv 10^{-1}$. $\Vert y_h(\cdot, T)\Vert_{L^2(0,1)} = \mathcal{O}(h^{0.66})$.

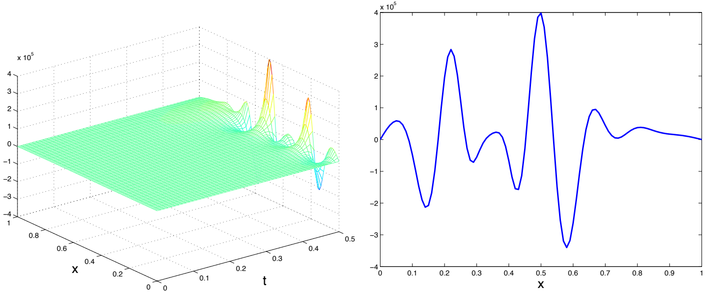


Figure: $\omega = (0.3, 0.4)$. The adjoint state p_h in Q_T (**Left**) and its restriction to $(0, 1) \times \{T\}$ (**Right**).

Experiments with $\omega = (0.3, 0.4)$

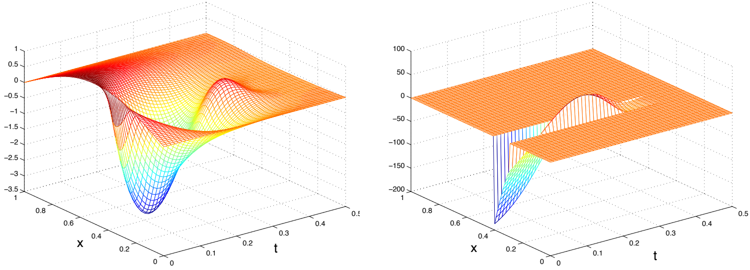


Figure: $\omega = (0.3, 0.4)$. The state y_h (**Left**) and the control v_h (**Right**).

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Numerical null controllability of the 1-d heat equation: primal algorithms,
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Numerical null controllability of the 1-d heat equation: Carleman weights and duality,
JOTA, (2013) - Available at
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Part III

$$\begin{cases} \text{Minimize } J(y,v) = \frac{1}{2}\iint_{q_T} \rho_0^2 |v|^2 \, dx\, dt \\ \text{Subject to } (y,v) \in \mathcal{C}(y_0, T). \end{cases} \quad (12)$$

where ρ_0 are non-negative continuous weights functions such that $\rho, \rho_0 \in L^\infty(Q_{T-\delta}) \quad \forall \delta > 0$.

$$\min_{\varphi \in \widetilde{W}_{\rho_0,\rho}} \hat{J}^\star(\varphi) = \frac{1}{2}\iint_{q_T} \rho_0^{-2} |\varphi(x,t)|^2 dx\, dt + (y_0, \varphi(\cdot,0))_{L^2(\Omega)}. \quad (13)$$

$$\widetilde{W}_{\rho_0,\rho} = \{\varphi \in \widetilde{\Phi}_{\rho_0,\rho} : \rho^{-1} L^\star \varphi = 0 \text{ in } L^2(Q_T)\}$$

13

[13] D. Araujo de Souza, AM, A mixed formulation for the direct approximation of the control of minimal L^2-weighted norm for the linear heat equation. Advances in Computational Mathematics, (2016)

PART III

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PART IV
Semi-linear case

E. Fernández-Cara and A. Münch,
Numerical null controllability of a semi-linear 1D heat equation via a least squares reformulation,
C.R. Acad. Sci. Paris, Série. I, 349, 867-871 (2011)

E. Fernández-Cara and A. Münch,
Numerical null controllability of semi-linear 1D heat equations : fixed point, least squares and Newton methods,
Mathematical Control and Related Fields, 3(2), 217-246 (2012)

Framework

$\omega \subset \Omega$, $a \in C^1(\overline{\Omega}, \mathbb{R}_*^+)$, $y_0 \in L^2(\Omega)$, $Q_T = \Omega \times (0,T)$, $q_T = \omega \times (0,T)$, $v \in L^\infty(q_T)$

$$\begin{cases} y_t - (c(x)y_x)_x + f(y) = v 1_\omega, & (x,t) \in Q_T \\ y(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T) \\ y(x,0) = y_0(x), & x \in (0,1). \end{cases} \tag{14}$$

$f : \mathbb{R} \to \mathbb{R}$ is, at least, locally Lipschitz-continuous.

$$|f'(s)| \leq C(1+|s|^p) \quad \text{a.e., with } \; p \leq 5. \tag{15}$$

Under this condition, (14) possesses exactly one local in time solution.
Under the growth condition [Cazenave-Haraux'89]

$$|f(s)| \leq C(1+|s|\log(1+|s|)) \quad \forall s \in \mathbb{R}, \tag{16}$$

the solutions to (14) are globally defined in $[0,T]$ and one has

$$y \in C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H_0^1(0,1)). \tag{17}$$

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Statement

The goal is to analyze numerically the null controllability properties of (14), in particular when blow-up occurs.
Without a growth condition of the kind (16), the solutions to (14) can blow up before $t = T$; in general, the blow-up time depends on the sizes of $\|y_0\|_{L^2(\Omega)}$ and $\|c\|_{L^\infty}$.

Assume $f(0) = 0$. The system (14) is said to be *"null-controllable"* at time T if, for any $y_0 \in L^2(\Omega)$, there exist controls $v \in L^\infty(q_T)$ and associated states y that are again globally defined in $[0, T]$ and satisfy (17) and

$$y(x, T) = 0, \quad x \in (0, 1). \tag{18}$$

Controllability results

The first one states that, if f is "too super-linear" at infinity, then the control cannot compensate the blow-up phenomena occurring in $(0,1)\backslash\overline{\omega}$:

Theorem (Fernandez-Cara and Zuazua'00)

There exist locally Lipschitz-continuous functions f with $f(0)=0$ and

$$|f(s)| \sim |s| \log^{p}(1+|s|) \quad as \quad |s| \to \infty, \quad p>2, \tag{19}$$

such that (14) fails to be null-controllable for all $T>0$.

The second result provides conditions under which (14) is null-controllable:

Theorem (Fernandez-Cara and Zuazua'00, Barbu'00)

Let $T>0$ be given. Assume that $f:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz-continuous and satisfies (15) and

$$\frac{f(s)}{|s| \log^{3/2}(1+|s|)} \to 0 \quad as \quad |s| \to \infty. \tag{20}$$

Then (14) is null-controllable at time T.

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Then (14) is null-controllable at time T.

The proof in [Fernandez-Cara & Zuazua, 2000] is based on

- a linearization of the eq.

$$y_t - (c(x)y_x)_x + g(z)y = v\,1_\omega, \quad Q_T \tag{21}$$

  with

$$g(z) = \frac{f(z)}{z} \tag{22}$$

- a fixed point argument : it is shown that the operator $\Lambda_0 : z \longrightarrow y$ is continuous compact from $L^2(Q_T)$ to $L^2(Q_T)$ and maps the closed ball $B(0, M) \subset L^2(Q_T)$ into itself. Then, Schauder Theorem provides the existence of at least one fixed point for Λ_0.

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Step 1: A linear control problem

First, we deal with the controllability properties of the following linear system

$$\begin{cases} L_A y := y_t - (c(x)y_x)_x + A(x,t)y = v\,1_\omega + B(x,t), & (x,t) \in Q_T \\ y(x,t) = 0, & (x,t) \in \Sigma_T \\ y(x,0) = y_0(x), & x \in (0,1) \end{cases} \tag{23}$$

that arises naturally after linearization of (14). From Lebeau-Robbiano'95 and Fursikov-Imanuvilov'96, (23) is null-controllable.

We give some numerical methods to address the extremal problem

$$\begin{cases} \text{Minimize } \; J(y,v) = \dfrac{1}{2}\displaystyle\iint_{Q_T} \rho^2 |y|^2\,dx\,dt + \dfrac{1}{2}\iint_{q_T} \rho_0^2 |v|^2\,dx\,dt \\ \text{Subject to } \; (y,v) \in \mathcal{C}_{lin}(y_0,T) \end{cases} \tag{24}$$

where $\mathcal{C}_{lin}(y_0,T)$ is the linear manifold

$$\mathcal{C}_{lin}(y_0,T) = \{\, (y,v) : v \in L^2(q_T),\; y \text{ solves (23) and satisfies } y(T) = 0 \,\}.$$

We assume that $A \in L^\infty(Q_T)$ and $B \in L^2(Q_T)$ and, also, that B vanishes at $t = T$ in an appropriate sense (i.e. $\displaystyle\iint_{Q_T} \rho_0^2 |B|^2\,dx\,dt < +\infty$).

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Numerical solution via a dual method [Fernandez-Cara, AM-2010]

$$\begin{cases} \text{Minimize } \; J^\star(\mu, \varphi_T) = \frac{1}{2}\left(\iint_{Q_T} \rho^{-2}|\mu|^2\,dx\,dt + \iint_{q_T} \rho_0^{-2}|\varphi|^2\,dx\,dt\right) \\ \qquad\qquad + \iint_{Q_T} B(x,t)\varphi\,dx\,dt + \int_0^1 y_0(x)\,\varphi(x,0)\,dx \\ \text{Subject to } \; (\mu, \varphi_T) \in \mathcal{V} \end{cases} \tag{25}$$

where $\mathcal{V}$ is defined as the completion of $\mathcal{D}(Q_T) \times \mathcal{D}(0,1)$ with respect to the norm

$$(\mu, \varphi_T) \to \left(\iint_{Q_T} \rho^{-2}|\mu|^2\,dx\,dt + \iint_{q_T} \rho_0^{-2}|\varphi|^2\,dx\,dt\right)^{1/2}.$$

where ψ solves

$$L_A^\star \varphi = \mu \quad Q_T, \qquad \varphi = 0 \quad \Sigma_T, \qquad \varphi(\cdot, T) = \varphi_T \quad (0,1). \tag{26}$$

Primal (direct) approach

Proposition

Let ρ and ρ_0 be given by (4). Let (y, v) be the corresponding optimal pair for J. Then there exists $p \in P$ such that

$$y = \rho^{-2} L_A^* p, \quad v = -\rho_0^{-2} p|_{q_T} \,. \tag{27}$$

The function p is the unique solution in P of

$$(p, q)_P = \int_0^1 y_0(x)\, q(x, 0)\, dx + \iint_{Q_T} Bq\, dx\, dt, \quad \forall q \in P \tag{28}$$

Remark

p solves, at least in $\mathcal{D}'$, the following differential problem, that is second order in time and fourth order in space:

$$\begin{cases} L_A(\rho^{-2} L_A^* p) + \rho_0^{-2} p 1_\omega = B, & (x, t) \in (0, 1) \times (0, T) \\ p(x, t) = 0, \ \ (\rho^{-2} L^* p)(x, t) = 0 & (x, t) \in \{0, 1\} \times (0, T) \\ (\rho^{-2} L_A^* p)(x, 0) = y_0(x), \ \ (\rho^{-2} L_A^* p)(x, T) = 0, & x \in (0, 1). \end{cases} \tag{29}$$

The "boundary" conditions at $t = 0$ and $t = T$ appear as Neumann conditions.

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Step 2: Fixed points

Step 2 : Back to the nonlinear problem

For simplicity, we will assume that $y_0 \in L^\infty(0,1)$ and $f \in C^1(\mathbf{R})$ and is globally Lipschitz-continuous. Let us introduce the function g, with

$$g(s) = \frac{f(s)}{s} \ \text{ if } \ s \neq 0, \quad g(0) = f'(0) \ \text{ otherwise.}$$

Then $g \in C^0_b(\mathbf{R})$ and $f(s) = g(s)\,s$ for all s (recall that $f(0) = 0$). We will set $G_0 = \|g\|_{L^\infty(\mathbf{R})}$.

For any $z \in L^1(Q_T)$, let us introduce the bilinear form

$$m(z;p,q) = \iint_{Q_T} \rho^{-2} L^*_{g(z)} p \, L^*_{g(z)} q \, dx\, dt + \iint_{q_T} \rho_0^{-2} p\, q\, dx\, dt \quad \forall p,q \in P_0. \tag{30}$$

Then $m(z;\cdot,\cdot)$ is a scalar product in P_0 and can be used to construct a Hilbert space P that, in principle, may depend on z. We will use the following result, which is a direct consequence of the Carleman estimates :

Lemma

Under the previous conditions, if the constants K_i in (4) are large enough (depending on ω, T, c_0, $\|c\|_{C^1}$ and G_0), then there exist $C_1, C_2 > 0$ such that

$$C_1\, m(0;p,p) \leq m(z;p,p) \leq C_2\, m(0;p,p) \quad \forall p \in P_0 \tag{31}$$

for all $z \in L^1(Q_T)$.

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Back to the nonlinear problem

Accordingly, all the spaces P provided by the bilinear forms $m(z;\cdot,\cdot)$ are the same and, in fact, (31) holds for all $p \in P$:

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We will fix the following norm in P:

$$\|p\|_P = m(0;p,p)^{1/2} \quad \forall p \in P. \tag{33}$$

The operator Λ_0

Let us introduce the mapping $\Lambda_0 : L^2(Q_T) \mapsto L^2(Q_T)$ where, for any $z \in L^2(Q_T)$, $y_z = \Lambda_0(z)$ is, together with v_z, the unique solution to the linear extremal problem

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such that $y(\cdot, T) = 0$. $\Lambda_0 : L^2(Q_T) \mapsto L^2(Q_T)$ is well defined. Furthermore, applying proposition 2 with $A = g(z)$ and $B = 0$, we obtain that y_z and v_z are characterized as follows :

$$y_z = \Lambda_0(z) = \rho^{-2} L^*_{g(z)} p_z, \quad v_z = -\rho_0^{-2} p_z|_{q_T}, \tag{36}$$

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A fixed point method

In order to solve the null controllability problem for (14), it suffices to find a solution to the fixed point equation

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ALG 1 (fixed point):

$$y^0 \in L^2(Q_T), \quad y^{n+1} = \Lambda_0(y^n), \, n \geq 0 \tag{39}$$

If $(y^n, v^n) \rightharpoonup (y, v)$ in $L^2(Q_T) \times L^2(q_T)$, then (y, v) solves the nonlinear null controllability problem. Indeed, since the $g(y^n)$ are uniformly bounded in $L^\infty(Q_T)$, after extraction of a subsequence it can be assumed that y^n (resp. y^n_t) converges weakly in $L^2(0, T; H^1_0(0, 1))$ (resp. $L^2(0, T; H^{-1}(0, 1))$). Therefore, y^n converges strongly in $L^2(Q_T)$ and a.e., $g(y^n)$ converges to $g(y)$ weakly-$*$ in $L^\infty(Q_T)$ and we can take limits and deduce that y solves, together with v, the nonlinear system.

This fixed point formulation has been used in [Fernandez-Cara, Zuazua, 2000] to prove Theorem 3. Precisely, it is shown there that $\Lambda_0 : L^2(Q_T) \mapsto L^2(Q_T)$ is continuous and compact and, also that there exists $M > 0$ such that Λ_0 maps the whole space $L^2(Q_T)$ inside the ball $B(0; M)$. Then, Schauder's Theorem provides the existence of at least one fixed point for Λ_0.

It is however important to note that this does not imply the convergence of the sequence $\{y^n\}$ defined by $y^{n+1} = \Lambda_0(y^n)$.

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A fixed point : a numerical application

$$f(s) = C_f s \ \log^p(1 + |s|) \quad \forall s \in \mathbf{R}. \tag{40}$$

We consider the following data:

$$a(x) = 1/10, \ \ p = 1.4, \ \ C_f = -5, \ \ T = 1/2, \ \ y_0(x) = \alpha \sin(\pi x)$$

In the uncontrolled situation, these data lead to the blow-up of the solution of (14) at time $t_c \approx 0.406, 0.367, 0.339, 0.318$ for $\alpha = 10, 20, 40$ and 80, respectively.

We first take $\omega = (0.2, 0.8)$ and initialize **ALG 1** with

$$y^0(x,t) = y_0(x)(1 - t/T)^2.$$

A fixed point : a numerical application - Lack of convergence

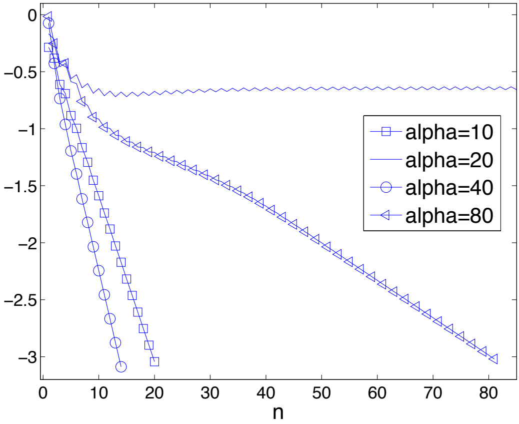

Figure: Fixed point method - $h = (1/60, 1/60)$ - $y_0(x) = \alpha \sin(\pi x)$ - Evolution of $log_{10}(\|\Lambda_0(y_h^n) - y_h^n\|_{L^2(Q_T)} / \|y_h^n\|_{L^2(Q_T)})$ vs. n for $\alpha = 10, 20, 40$ and 80.

A fixed point : a numerical application

	$\|v_h\|_{L^2(q_T)}$	$\|v_h\|_{L^\infty(q_T)}$	$\|y_h\|_{L^2(Q_T)}$	$\sharp$ iterates
$\alpha = 10$	3.531×10^1	2.542×10^2	1.742	20
$\alpha = 40$	2.142×10^2	2.053×10^3	6.654	14
$\alpha = 80$	5.109×10^2	7.021×10^3	14.410	81

Table: Fixed point method - $h = (1/60, 1/60)$ - $y_0(x) = \alpha \sin(\pi x)$ - Norms for $\alpha = 10, 40$ and $\alpha = 80$.

A fixed point : a numerical application

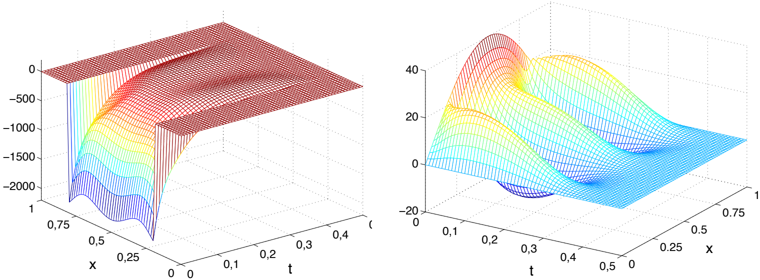

Figure: Fixed point method - $h = (1/60, 1/60)$ - $y_0(x) = 40\sin(\pi x)$ - Control v_h (**Left**) and corresponding controlled solution y_h (**Right**) in Q_T.

A fixed point : a numerical application : $p \geq 3/2$

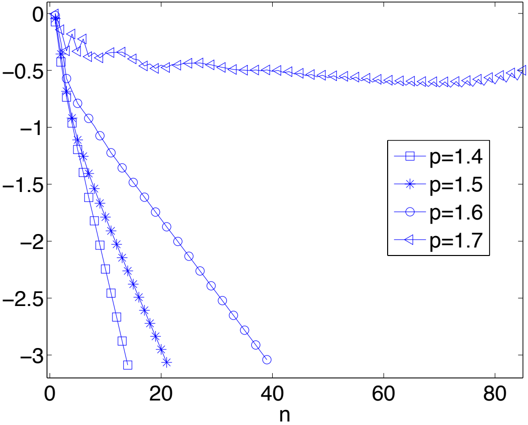

Figure: Fixed point method - $h = (1/60, 1/60)$ - $y_0(x) \equiv 40\sin(\pi x)$ - $p = 1.4$ - Evolution of $\log_{10}(\|\Lambda_0(y_h^n) - y_h^n\|_{L^2(Q_T)}/\|y_h^n\|_{L^2(Q_T)})$ for $p = 1.4, 1.5, 1.6$ and $p = 1.7$.

Least squares reformulation

We now introduce the function $\zeta(t) = (T-t)^{-1/2}$ for all t in $(0, T)$ and the space $Z := L^2(\zeta^2, Q_T)$. We will denote by Λ the restriction to Z of the mapping Λ_0. Obviously, $\Lambda(z) \in Z$ for all $z \in Z$.

Let us consider the following least squares reformulation of (38):

$$\begin{cases} \text{Minimize } \ R(z) := \dfrac{1}{2}\|z - \Lambda(z)\|_Z^2 \\ \text{Subject to } \ z \in Z. \end{cases} \tag{41}$$

Any solution to (38) solves (41). Conversely, if y solves (41), we necessarily have $R(y) = 0$ (because (14) is null controllable with control-states (y, v) such that $J(z; y, v) < +\infty$); hence, y also solves (38). This shows that (38) and (41) are, in the present context, equivalent.

Least squares reformulation

Proposition

Let us assume that $g \in C_b^1(\mathbf{R})$. *Then* $R \in C^1(Z)$. *Moreover, for any* $z \in Z$, *the gradient of* R *with respect to the inner product of* Z *is given by*

$$DR(z) = \big(1 - \rho^{-2} g'(z) p_z\big)(z - y_z) + \zeta^{-2} g'(z)\big(y_z \lambda_z + p_z \mu_z\big), \qquad (42)$$

where p_z *is the unique solution to* (37), $y_z = \rho^{-2} L^*_{g(z)} p_z$, λ_z *is the unique solution to the linear (adjoint) problem*

$$m(z; q, \lambda_z) = (z - y_z, \rho^{-2} L^*_{g(z)} q)_Z \quad \forall q \in P; \quad \lambda_z \in P \qquad (43)$$

and, finally, $\mu_z = \rho^{-2} L^*_{g(z)} \lambda_z$.

Least squares reformulation

Proposition

Let the assumptions in proposition 3 be satisfied and let us introduce $G_1 := \|g'\|_{L^\infty(\mathbf{R})}$. There exists a constant K that depends on ω, T, c_0, $\|c\|_{C^1}$ and G_0 but is independent of z and y_0, such that the following holds for all $z \in Z$:

$$\|DR(z)\|_Z \geq \left(1 - K\, G_1\, \|y_0\|_{L^2}\right) \|z - \Lambda(z)\|_Z\,. \tag{44}$$

Least squares : Gradient method for R

ALG 2 (Least-squares):

$$z^0 \in L^2(Q_T), \quad (z^{n+1}, h)_Z = (z^n, h)_Z - \eta\,(DR(z^n), h)_Z, \quad n \geq 0$$

Least squares : a numerical application

$$f_\eta(s) = C_f s \log^p(1 + |s|_n) \quad \forall s \in \mathbf{R}, \quad |s|_\eta := \sqrt{s^2 + \eta^2} - \eta \tag{45}$$

so that, for all $\eta > 0$, $g_\eta := C_f \log^p(1 + |s|_\eta)$ belongs to $C_b^1(\mathbf{R})$. We have $f_\eta(0) = 0$ and Theorem applies for f_η, since f_η and f are equivalent at infinity. We take $\eta = 10^{-1}$.

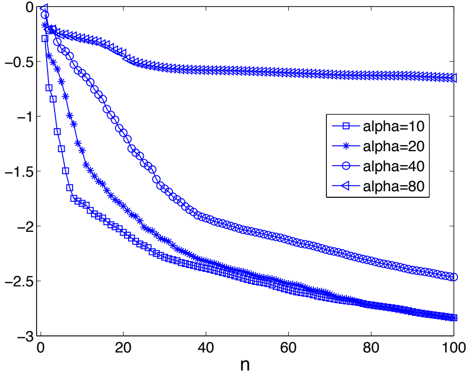

Figure: Least squares method - $h = (1/60, 1/60)$ - Evolution of $\log_{10}(\|\Lambda(z_h^n) - z_h^n\|_{L^2(Q_T)}/\|z_h^n\|_{L^2(Q_T)})$ - $\alpha = 10, 20, 40, 80$ and algorithm **ALG 2'**.

Least squares reformulation : numerical application

$$f_\eta(s) = C_f s \ \log^p(1+|s|_n) \quad \forall s \in \mathbf{R}, \quad |s|_\eta := \sqrt{s^2+\eta^2} - \eta \qquad (46)$$

so that, for all $\eta > 0$, $g_\eta := C_f \log^p(1+|s|_\eta)$ belongs to $C_b^1(\mathbf{R})$. Moreover, we have $f_\eta(0) = 0$ and Theorem applies for f_η, since f_η and f are equivalent at infinity. We shall take $\eta = 10^{-1}$.

	$\|v_h\|_{L^2(q_T)}$	$\|v_h\|_{L^\infty(q_T)}$	$\|z_h\|_{L^2(Q_T)}$	$\|R'(z_h)\|_{L^2(Q_T)}$	e^n
$\alpha = 10$	3.507×10^1	2.532×10^2	1.753	1.27×10^{-3}	1.43×10^{-3}
$\alpha = 20$	8.781×10^1	7.323×10^2	3.180	1.44×10^{-3}	1.54×10^{-3}
$\alpha = 40$	2.137×10^2	2.048×10^3	6.651	5.42×10^{-3}	3.39×10^{-3}
$\alpha = 80$	2.526×10^2	3.299×10^3	14.73	2.23×10^{-1}	7.89×10^{-1}

Table: Least squares method approach after 100 iterates - $h = (1/60, 1/60)$ - $y_0(x) \equiv \alpha \sin(\pi x)$ - $p = 1.4$ - Norms for $\alpha = 10, 20, 40, 80$. Here, $e^n = \|\Lambda(z_h^n) - z_h^n\|_{L^2(Q_T)} / \|z_h^n\|_{L^2(Q_T)}$.

Newton-Raphson Algorithm (a different way to linearize $f(y^{n+1})$)

ALG 3' (Newton):

1. Choose $(y^0, z^0) \in Y$.
2. Then, given $n \geq 0$ and $(y^n, v^n) \in Y$, solve in $(y^{n+1}, v^{n+1}) \in Y$ the linear problem
$$F'(y^n, v^n) \cdot (y^{n+1} - y^n, v^{n+1} - v^n) = -F(y^n, v^n),$$
i.e. find y^{n+1} and v^{n+1} such that $(y^{n+1}, v^{n+1}) \in Y$ and

$$\begin{cases} y_t^{n+1} - (c(x)y_x^{n+1})_x + f'(y^n)\,y^{n+1} = v^{n+1}\,1_\omega + f'(y^n)\,y^n - f(y^n), & Q_T \\ y^{n+1} = 0, & \Sigma_T \\ y^{n+1}(\cdot, 0) = y_0, & (0,1). \end{cases} \quad (47)$$

Newton-Raphson Algorithm (a different way to linearize $f(y^{n+1})$)

ALG 3' (Newton):

1. Choose $(y^0, z^0) \in Y$.
2. Then, given $n \geq 0$ and $(y^n, v^n) \in Y$, solve in $(y^{n+1}, v^{n+1}) \in Y$ the linear problem

$$F'(y^n, v^n) \cdot (y^{n+1} - y^n, v^{n+1} - v^n) = -F(y^n, v^n),$$

i.e. find y^{n+1} and v^{n+1} such that $(y^{n+1}, v^{n+1}) \in Y$ and

$$\begin{cases} y_t^{n+1} - (c(x) y_x^{n+1})_x + f'(y^n)\, y^{n+1} = v^{n+1}\, 1_\omega + f'(y^n)\, y^n - f(y^n), & Q_T \\ y^{n+1} = 0, & \Sigma_T \\ y^{n+1}(\cdot, 0) = y_0, & (0,1). \end{cases} \tag{47}$$

smaller support ω

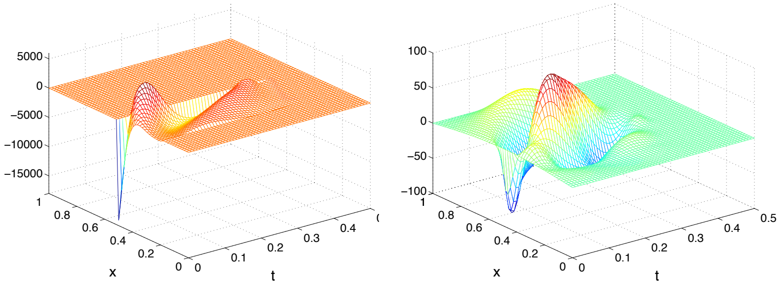

Figure: Fixed point method - $h = (1/60, 1/60)$ - $y_0(x) \equiv 10\sin(\pi x)$ - $p = 1.4$ - $\omega = (0.2, 0.5)$ - Control v_h (Left) and corresponding controlled solution y_h (Right) in Q_T.

PART IV

A variational least-squares approach [14]

[14] AM, P. Pedregal, Numerical null controllability of the heat equation through a least squares and variational approach. European Journal of Applied Mathematics, (2014).

Least-squares approach

We define the non-empty set

$$\mathcal{A} = \Big\{(u,f);\ u \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega));\, u' \in L^2(0,T,H^{-1}(\Omega)),$$
$$u(\cdot,0) = u_0, u(\cdot,T) = 0, f \in L^2(q_T)\Big\}$$

and find $(u,f) \in \mathcal{A}$ solution of the heat eq. !

For any $(u,f) \in \mathcal{A}$, we define the "corrector" $v = v(u,f) \in H^1(Q_T)$ solution of the Q_T-elliptic problem

$$\begin{cases} -v_{tt} - \nabla\cdot(c(x)\nabla v) + (Lu - f 1_\omega) = 0, & (x,t) \in Q_T, \\ v_t = 0, & x \in \Omega, t \in \{0,T\} \\ v = 0, & x \in \Sigma_T. \end{cases} \tag{48}$$

Least-squares approach

We define the non-empty set

$$\mathcal{A} = \Big\{(u,f);\ u \in C([0,T];L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega));\ u' \in L^2(0,T,H^{-1}(\Omega)),$$
$$u(\cdot,0)=u_0, u(\cdot,T)=0, f \in L^2(q_T)\Big\}$$

and find $(u,f) \in \mathcal{A}$ solution of the heat eq. !

For any $(u,f) \in \mathcal{A}$, we define the "corrector" $v = v(u,f) \in H^1(Q_T)$ solution of the Q_T-elliptic problem

$$\begin{cases} -v_{tt} - \nabla\cdot(c(x)\nabla v) + (Lu - f\,1_\omega) = 0, & (x,t) \in Q_T, \\ v_t = 0, & x \in \Omega, t \in \{0,T\} \\ v = 0, & x \in \Sigma_T. \end{cases} \tag{48}$$

Least-squares approach (2)

Theorem
u is a controlled solution of the heat eq. by the control function $f\,1_\omega \in L^2(q_T)$ if and only if (u,f) is a solution of the extremal problem

$$\inf_{(u,f)\in\mathcal{A}} E(u,f) := \frac{1}{2}\iint_{Q_T} (|v_t|^2 + c(x)|\nabla v|^2)dx\,dt. \tag{49}$$

Proof.
$\Longleftarrow$ From the null controllability of the heat eq., the extremal problem is well-posed in the sense that the infimum, equal to zero, is reached by any controlled solution of the heat eq. (the minimizer is not unique).

$\Longrightarrow$ Conversely, we check that any minimizer of E is a solution of the (controlled) heat eq.:
We define the vector space

$$\mathcal{A}_0 = \Big\{(u,f);\ u \in C([0,T];L^2(\Omega))\cap L^2(0,T;H^1_0(\Omega));\ u' \in L^2(0,T,H^{-1}(\Omega)),$$
$$u(\cdot,0) = u(\cdot,T) = 0, x \in \Omega, f \in L^2(q_T)\Big\}$$

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$$u(\cdot,0) = u(\cdot,T) = 0, x \in \Omega, f \in L^2(q_T)\Big\}$$

Least-squares approach (2)

The first variation of E at (u,f) in the admissible direction $(U,F)\in\mathcal{A}_0$ defined by

$$< E'(u,f),(U,F) >= \lim_{\eta\to 0}\frac{E((u,f)+\eta(U,F))-E(u,f)}{\eta} \tag{50}$$

exists and is given by

$$< E'(u,f),(U,F) >= \iint_{Q_T}(v_t V_t + a(x)\nabla v\cdot\nabla V)dx\,dt, \tag{51}$$

where the corrector $V\in H^1(Q_T)$ associated to (U,F) is the solution of

$$\begin{cases} U_t - V_{tt} - \nabla\cdot(a(x)(\nabla U+\nabla V)) - F\,1_\omega = 0, & (x,t)\in Q_T,\\ V_t(x,0)=V_t(x,T)=0, & x\in\Omega,\\ V(0,t)=V(1,t)=0, & t\in(0,T). \end{cases} \tag{52}$$

Using that

$$-\int_0^T < U_t, v >_{H^{-1}(\Omega),H^1(\Omega)}\ dt = \iint_{Q_T} Uv_t\,dx\,dt - \int_0^1 [Uv]_0^T\,dx = \iint_{Q_T} Uv_t\,dx\,dt,$$

we get that

$$< E'(u,f),(U,F) >= \iint_{Q_T}(Uv_t - a(x)\nabla U\cdot\nabla v + Fv\,1_\omega)\,dx\,dt, \quad \forall (U,F)\in\mathcal{A}_0$$

Least-squares approach (2)

Therefore, if (u, f) minimizes E, the equality $< E'(u, f), (U, F) >= 0$ for all $(U, F) \in A_0$ implies that the corrector $v = v(u, f)$ satisfies

$$\begin{cases} -v_t - \nabla \cdot (c(x)\nabla v) + dv = 0, & (x, t) \in Q_T, \\ v = 0, & (x, t) \in q_T \end{cases}$$

in addition to the boundary conditions: $v = 0$ on Σ_T and $v_t = 0$ on $\Omega \times \{0, T\}$.
Unique continuation property implies that $v = 0$ in Q_T and so $E(u, f) = 0$ and so $(u, f) \in \mathcal{A}$ solves the heat eq.

Remark The proposition educes the search of ONE control f distributed in ω to the minimization of the functional E over $\mathcal{A}$.

Remark Least squares terminology :

$$E(u, f) := \frac{1}{2}\|u_t - \nabla \cdot (a(x)\nabla u) + d\,u - f1_\omega\|^2_{H^{-1}(Q_T)}$$

Least-squares approach (2)

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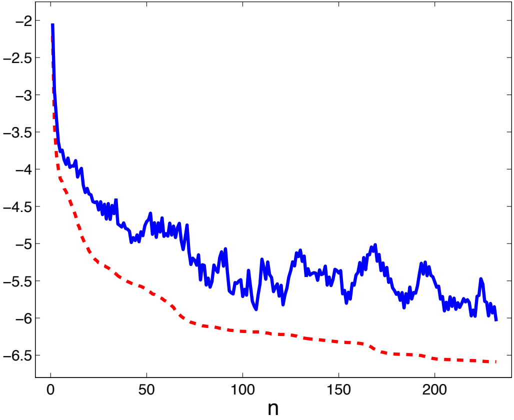
$$E(u, f) := \frac{1}{2}\|u_t - \nabla \cdot (a(x)\nabla u) + d\,u - f 1_\omega\|^2_{H^{-1}(Q_T)}$$

A numerical application in 1D (inner controllability)

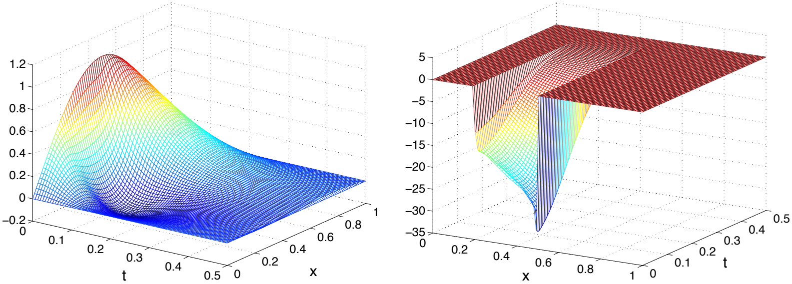
$N = 1$, $\Omega = (0, 1)$, $\omega = (0.2, 0.5)$, $u_0(x) = sin(\pi x)$, $c(x) = c_0 = 0.25$, $T = 1/2$, $d := 0$

Starting point of the algorithm: $(u, f) = (u_0(x)(1 - t/T)^2, 0) \in \mathcal{A}$



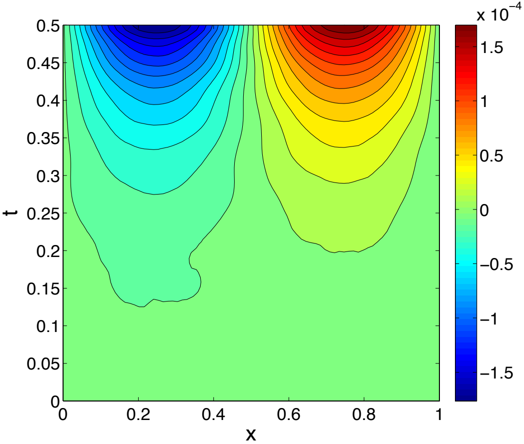
$u_0(x) = \sin(\pi x)$ - Control acting on $\omega = (0.2, 0.5)$ - $\varepsilon = 10^{-6}$ - $\log_{10}(E_h(u_h^n)$ (**dashed line**) and $\log_{10}(\|g_h^n\|_{\mathcal{A}})$ (**full line**) vs. the iteration n of the CG algorithm.

A numerical application in 1D (inner controllability)



$(u, f) \in \mathcal{A}$ along Q_T at convergence

A numerical application in 1D (inner controllability)



Isovalues along Q_T of the corresponding corrector v: $\|v\|_{H^1(Q_T)} \approx 10^{-4}$

PART V

Inverse problem - Reconstruction of y from y_{q_T} [15]

[15] D. Araujo de Souza, AM, Inverse problems for linear parabolic equations using mixed formulations - Part 1 : Theoretical analysis. Journal of Inverse and Ill posed problems. (2016)

Inverse problem - Reconstruction of y from y_{q_T}

$\Omega \subset \mathbb{R}^N$ ($N \geq 1$) - $T > 0$, $c \in C^1(\overline{\Omega}, \mathbb{R})$), $d \in L^\infty(Q_T)$, $y_0 \in \boldsymbol{H}$

$$\begin{cases} Ly := y_t - \nabla \cdot (c\nabla y) + dy = f, & Q_T := \Omega \times (0,T) \\ y = 0, & \Sigma_T := \partial\Omega \times (0,T) \\ y(\cdot, 0) = y_0, & \Omega. \end{cases} \tag{53}$$

- Inverse Problem : Distributed observation on $q_T = \omega \times (0,T), \omega \subset \Omega$

$$\begin{cases} X = L^2(q_T), \\ \text{Given } (y_{obs}, f) \in (L^2(q_T), X), \text{find } y \; s.t. \{(53) \text{ and } y - y_{obs} = 0 \quad \text{on} \quad q_T\} \end{cases}$$

Well-known Difficulty:

$$\left(Ly \in L^2(Q_T), y \in L^2(q_T), y_{|\Sigma_T} = 0\right) \Longrightarrow y \in C([\delta, T], H_0^1(\Omega)), \quad \forall \delta > 0$$

Carleman ineq.

$$\begin{cases} \iint_{Q_T} \rho_0^{-2}|q|^2\,dx\,dt \\ \qquad\qquad \leq C\left(\iint_{Q_T} \rho^{-2}|L^*q|^2\,dx\,dt + \iint_{q_T} \rho_0^{-2}|q|^2\,dx\,dt\right), \forall q \in P. \end{cases} \tag{54}$$

$t \Longrightarrow T - t$

$$\tilde{\rho}(x,t) = \rho(x, T-t), \quad \tilde{\rho}_0(x,t) = \rho_0(x, T-t)$$

$$\begin{cases} \iint_{Q_T} \tilde{\rho}_0^{-2}|y|^2\,dx\,dt \\ \qquad\qquad \leq C\left(\iint_{Q_T} \tilde{\rho}^{-2}|Ly|^2\,dx\,dt + \iint_{q_T} \tilde{\rho}_0^{-2}|y|^2\,dx\,dt\right), \forall y \in Y. \end{cases} \tag{55}$$

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$$\left\{\begin{aligned} &\iint_{Q_T} \rho_0^{-2}|q|^2\,dx\,dt \\ &\qquad\qquad \leq C\left(\iint_{Q_T} \rho^{-2}|L^*q|^2\,dx\,dt + \iint_{q_T} \rho_0^{-2}|q|^2\,dx\,dt\right), \forall q \in P. \end{aligned}\right. \tag{54}$$

$t \Longrightarrow T - t$

$$\tilde{\rho}(x,t) = \rho(x, T-t), \quad \tilde{\rho}_0(x,t) = \rho_0(x, T-t)$$

$$\left\{\begin{aligned} &\iint_{Q_T} \tilde{\rho}_0^{-2}|y|^2\,dx\,dt \\ &\qquad\qquad \leq C\left(\iint_{Q_T} \tilde{\rho}^{-2}|Ly|^2\,dx\,dt + \iint_{q_T} \tilde{\rho}_0^{-2}|y|^2\,dx\,dt\right), \forall y \in Y. \end{aligned}\right. \tag{55}$$

Second order mixed formulation .... as in the previous part

We then define the following extremal problem :

$$\left\{ \begin{array}{l} \text{Minimize } \; J(y) := \frac{1}{2}\iint_{q_T} \rho_0^{-2}|y(x,t)-y_{obs}(x,t)|^2\,dx\,dt + r\iint_{Q_T}(\rho^{-1}L\,y)^2\,dx\,dt \\ \text{Subject to } \; y \in \mathcal{W} := \Big\{ y \in \mathcal{Y} : \rho^{-1}Ly = 0 \text{ in } L^2(Q_T) \Big\} \end{array} \right. \qquad (P)$$

with $\rho_0, \rho \in \mathcal{R}$ where ($\rho_\star \in \mathbb{R}_\star^+$)

$$\mathcal{R} := \{w : w \in C(Q_T);\, w \geq \rho_\star > 0 \text{ in } Q_T;\, w \in L^\infty(\Omega \times (\delta, T))\ \forall \delta > 0\}$$

Let $\mathcal{Y}_0 := \Big\{ y \in C^2(\overline{Q}_T) : y = 0 \text{ on } \Sigma_T \Big\}$ and for $\eta > 0$, $\rho \in \mathcal{R}$, the bilinear form by

$$(y, \overline{y})_{\mathcal{Y}_0} := \iint_{q_T} \rho_0^{-2} y\,\overline{y}\,dx\,dt + \eta \iint_{Q_T} \rho^{-2} L\,y L\,\overline{y}\,dx\,dt, \quad \forall y, \overline{y} \in \mathcal{Y}_0.$$

Let $\mathcal{Y}$ be the completion of $\mathcal{Y}_0$ for this scalar product endowed with the norm

$$\|y\|_{\mathcal{Y}}^2 := \|\rho_0^{-1} y\|_{L^2(q_T)}^2 + \eta \|\rho^{-1} L\,y\|_{L^2(Q_T)}^2, \quad \forall y \in \mathcal{Y}.$$

Second order mixed formulation .... as in the previous part

We then define the following extremal problem :

$$\begin{cases} \text{Minimize } J(y) := \frac{1}{2}\iint_{q_T} \rho_0^{-2}|y(x,t) - y_{obs}(x,t)|^2\,dx\,dt + r\iint_{Q_T}(\rho^{-1}L\,y)^2\,dx\,dt \\ \text{Subject to } y \in \mathcal{W} := \left\{ y \in \mathcal{Y} : \rho^{-1}Ly = 0 \text{ in } L^2(Q_T) \right\} \end{cases} \qquad (P)$$

with $\rho_0, \rho \in \mathcal{R}$ where ($\rho_\star \in \mathbb{R}_\star^+$)

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Let $\mathcal{Y}$ be the completion of $\mathcal{Y}_0$ for this scalar product endowed with the norm

$$\|y\|_{\mathcal{Y}}^2 := \|\rho_0^{-1} y\|_{L^2(q_T)}^2 + \eta\|\rho^{-1} L\,y\|_{L^2(Q_T)}^2, \quad \forall y \in \mathcal{Y}.$$

Mixed formulation

Find $(y, \lambda) \in \mathcal{Y} \times L^2(Q_T)$ solution of

$$\begin{cases} a_r(y, \overline{y}) + b(\overline{y}, \lambda) & = & l(\overline{y}) & \forall \overline{y} \in \mathcal{Y}, \\ b(y, \overline{\lambda}) & = & 0 & \forall \overline{\lambda} \in L^2(Q_T), \end{cases} \tag{56}$$

where

$$a_r : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}, \quad a(y, \overline{y}) := \iint_{q_T} \rho_0^{-2} y \, \overline{y} \, dx \, dt + r \iint_{Q_T} \rho^{-2} Ly \, L\overline{y} \, dx \, dt$$

$$b : \mathcal{Y} \times L^2(Q_T) \to \mathbb{R}, \quad b(y, \lambda) := \iint_{Q_T} \rho^{-1} Ly \, \lambda \, dx \, dt$$

$$l : \mathcal{Y} \to \mathbb{R}, \quad l(y) := \iint_{q_T} \rho_0^{-2} y \, y_{obs} \, dx \, dt.$$

Corollary

Let $\rho_0 \in \mathcal{R}, \rho \in \mathcal{R} \cap L^\infty(Q_T)$ *and assume* $\exists K$ *s.t.*

$$\rho_0 \leq K\rho_{c,0}, \quad \rho \leq K\rho_c \quad in \quad Q_T.$$

If (y, λ) *is the solution of the mixed formulation (56), then* $\exists C > 0$ *such that*

$$\|\rho_{c,0}^{-1} y\|_{L^2(Q_T)} \leq C\|y\|_{\mathcal{Y}}.$$

$H_0^1 - L^2$ first order formulation

First order formulation involving y and the flux $\mathbf{p} = c(x)\nabla y$.

$$\begin{cases} \mathcal{I}(y,\mathbf{p}) := y_t - \nabla\cdot\mathbf{p} + d\,y = f, \quad \mathcal{J}(y,\mathbf{p}) := c(x)\nabla y - \mathbf{p} = \mathbf{0} & \text{in} \quad Q_T, \\ y = 0 & \text{on} \quad \Sigma_T, \\ y(x,0) = y_0(x) & \text{in} \quad \Omega. \end{cases} \tag{57}$$

$$(y_0, f) \in L^2(\Omega) \times L^2(Q_T) \Longrightarrow p \in \mathbf{L}^2(Q_T), y \in L^2(0,T,H_0^1(\Omega)), y_t \in L^2(0,T,H^{-1}(\Omega))$$

- Inverse Problem : Distributed observation on $q_T = \omega \times (0,T), \omega \subset \Omega$

$$\begin{cases} X = L^2(q_T), \\ \text{Given } (y_{obs}, f) \in (L^2(q_T), X), \text{ find } (y,\mathbf{p}) \; s.t. \, \{(57) \text{ and } y - y_{obs} = 0 \quad \text{on} \quad q_T\} \end{cases}$$

Parabolic case: $H_0^1 - L^2$ first order formulation of the parabolic

The extremal problem is then :

$$\begin{cases} \text{Minimize } \; J(y,\mathbf{p}) := \dfrac{1}{2}\displaystyle\iint_{q_T} \rho_0^{-2}|y(x,t) - y_{obs}(x,t)|^2\,dx\,dt + \mathbf{r}\dots \\ (y,\mathbf{p}) \in \mathcal{V} := \Big\{ (y,\mathbf{p}) \in \mathcal{U} \;:\; \rho_1^{-1}\mathcal{J}(y,\mathbf{p}) = 0 \text{ in } \mathbf{L}^2(Q_T), \quad \rho^{-1}\mathcal{I}(y,\mathbf{p}) = 0 \text{ in } L^2(Q_T) \Big\} \end{cases}$$

$\mathcal{U}$ - completion of $\mathcal{U}_0 := \Big\{ (y,\mathbf{p}) \in C^1(\overline{Q}_T) \times \mathbf{C}^1(\overline{Q}_T) : y = 0 \text{ on } \Sigma_T \Big\}$ for

$$((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}}))_{\mathcal{U}_0} = \iint_{q_T} \rho_0^{-2} y\,\overline{y}\,dx\,dt + \eta_1 \iint_{Q_T} \rho_1^{-2}\mathcal{J}(y,\mathbf{p})\cdot\mathcal{J}(\overline{y},\overline{\mathbf{p}})\,dx\,dt$$
$$+ \eta_2 \iint_{Q_T} \rho^{-2}\mathcal{I}(y,\mathbf{p})\mathcal{I}(\overline{y},\overline{\mathbf{p}})\,dx\,dt \quad \forall (y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}}) \in \mathcal{U}_0.$$

for any $\eta_1, \eta_2 > 0$ and any $\rho,\ \rho_0,\ \rho_1 \in \mathcal{R}$

$$\|(y,\mathbf{p})\|_{\mathcal{U}}^2 := \|\rho_0^{-1}y\|_{L^2(q_T)}^2 + \eta_1\|\rho_1^{-1}\mathcal{J}(y,\mathbf{p})\|_{\mathbf{L}^2(Q_T)}^2 + \eta_2\|\rho^{-1}\mathcal{I}(y,\mathbf{p})\|_{L^2(Q_T)}^2.$$

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$\mathcal{U}$ - completion of $\mathcal{U}_0 := \Big\{(y,\mathbf{p}) \in C^1(\overline{Q}_T) \times \mathbf{C}^1(\overline{Q}_T) : y = 0 \text{ on } \Sigma_T\Big\}$ for

$$\begin{aligned}((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}}))_{\mathcal{U}_0} = &\iint_{q_T} \rho_0^{-2} y\,\overline{y}\,dx\,dt + \eta_1 \iint_{Q_T} \rho_1^{-2}\mathcal{J}(y,\mathbf{p})\cdot\mathcal{J}(\overline{y},\overline{\mathbf{p}})\,dx\,dt \\ &+ \eta_2 \iint_{Q_T} \rho^{-2}\mathcal{I}(y,\mathbf{p})\mathcal{I}(\overline{y},\overline{\mathbf{p}})\,dx\,dt \quad \forall (y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}}) \in \mathcal{U}_0.\end{aligned}$$

for any $\eta_1, \eta_2 > 0$ and any $\rho,\ \rho_0,\ \rho_1 \in \mathcal{R}$

$$\|(y,\mathbf{p})\|_{\mathcal{U}}^2 := \|\rho_0^{-1} y\|_{L^2(q_T)}^2 + \eta_1\|\rho_1^{-1}\mathcal{J}(y,\mathbf{p})\|_{\mathbf{L}^2(Q_T)}^2 + \eta_2\|\rho^{-1}\mathcal{I}(y,\mathbf{p})\|_{L^2(Q_T)}^2.$$

Parabolic case: $H_0^1 - L^2$ first order formulation - Mixed formulation

Precisely, we set $\mathcal{X} := L^2(Q_T) \times \mathbf{L}^2(Q_T)$ and then we consider the following mixed formulation : find $((y, \mathbf{p}), (\lambda, \boldsymbol{\mu})) \in \mathcal{U} \times \mathcal{X}$ solution of

$$\begin{cases} a_{\boldsymbol{r}}((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}})) + b((\overline{y},\overline{\mathbf{p}}),(\lambda,\boldsymbol{\mu})) & = & l(\overline{y},\overline{\mathbf{p}}) & \forall(\overline{y},\overline{\mathbf{p}}) \in \mathcal{U}, \\ b((y,\mathbf{p}),(\overline{\lambda},\overline{\boldsymbol{\mu}})) & = & 0 & \forall(\overline{\lambda},\overline{\boldsymbol{\mu}}) \in \mathcal{X}, \end{cases} \tag{58}$$

where

$$a_{\boldsymbol{r}} : \mathcal{U} \times \mathcal{U} \to \mathbb{R}, \quad a_{\boldsymbol{r}}((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}})) := \iint_{q_T} \rho_0^{-2} y\, \overline{y}\, dx\, dt$$

$$+ r_1 \iint_{Q_T} \rho_1^{-2} \mathcal{J}(y,\mathbf{p}) \cdot \mathcal{J}(\overline{y},\overline{\mathbf{p}})\, dx\, dt + r_2 \iint_{Q_T} \rho^{-2} \mathcal{I}(y,\mathbf{p}) \mathcal{I}(\overline{y},\overline{\mathbf{p}})\, dx\, dt$$

$$b : \mathcal{U} \times \mathcal{X} \to \mathbb{R}, \quad b((y,\mathbf{p}),(\lambda,\boldsymbol{\mu})) := \iint_{Q_T} \rho_1^{-1} \mathcal{J}(y,\mathbf{p}) \cdot \boldsymbol{\mu}\, dx\, dt + \iint_{Q_T} \rho^{-1} \mathcal{I}(y,\mathbf{p}) \lambda\, dx\, dt$$

$$l : \mathcal{U} \to \mathbb{R}, \quad l(y,\mathbf{p}) := \iint_{q_T} \rho_0^{-2} y\, y_{obs}\, dx\, dt.$$

$\forall \boldsymbol{r} = (r_1, r_2) \in (\mathbb{R}^+)^2$

Parabolic case: $H_0^1 - L^2$ first order formulation - Global stability

Proposition (Imanuvilov-Puel-Yamamoto, 2010)

$$\rho_p(x,t) := \exp\left(\frac{\beta(x)}{t^2}\right), \quad \beta(x) := K_1\left(e^{K_2} - e^{\beta_0(x)}\right),$$
$$\rho_{p,0}(x,t) := t\rho_p(x,t), \quad \rho_{p,1}(x,t) := t^{-1}\rho_p(x,t), \quad \rho_{p,2}(x,t) := t^{-2}\rho_p(x,t)$$

$\exists C = C(\omega, T) > 0$ *s.t.*

$$\|\rho_{p,0}^{-1}y\|^2_{L^2(Q_T)} + \|\rho_{p,1}^{-1}\nabla y\|^2_{L^2(Q_T)} \le C\left(\|\rho_p^{-1}\mathbf{G}\|^2_{\mathbf{L}^2(Q_T)} + \|\rho_{p,2}^{-1}g\|^2_{L^2(Q_T)} + \|\rho_{p,0}^{-1}y\|^2_{L^2(q_T)}\right),$$

for any

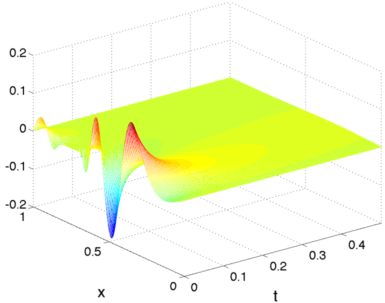
$$\begin{cases} y \in \mathcal{K} := \left\{y \in L^2(0,T;H_0^1(\Omega)) : y_t \in L^2(0,T;H^{-1}(\Omega))\right\}, \\ Ly = g + \nabla\cdot\mathbf{G} \ in\ Q_T, \quad (g,\mathbf{G}) \in L^2(Q_T)\times\mathbf{L}^2(Q_T). \end{cases}$$

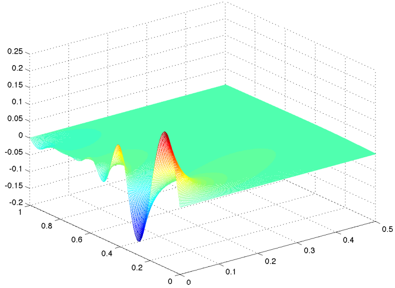
$$\begin{cases} Ly = \mathcal{I}(y,\mathbf{p}) - \nabla\cdot\mathcal{J}(y,\mathbf{p}), \\ \mathcal{J}(y,\mathbf{p}) := c(x)\nabla y - \mathbf{p}, \quad \mathcal{I}(y,\mathbf{p}) := y_t - \nabla\cdot\mathbf{p} + dy \end{cases}$$

$N = 1$ - Heat eq. Comparison with the standard method

$$y_0(x) = \sin(\pi x)^{20}, \quad Q_T = (0,1) \times (0,T), \quad q_T = (0.7, 0.8) \times (0,T), \quad T = 1/2$$

$$\min_{y_{0h}} \left(J_h(y_{0h}) + \frac{h^2}{2} \|y_{0h}\|^2_{L^2(\Omega)} \right) \quad vs. \quad \min_{\lambda_h} J^{\star\star}(\lambda_h) \quad \text{over} \quad \Lambda_h \tag{59}$$





$$\frac{\|y - y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} \approx 5.86 \times 10^{-2}, \qquad \frac{\|y - y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} \approx 7.70 \times 10^{-2}$$

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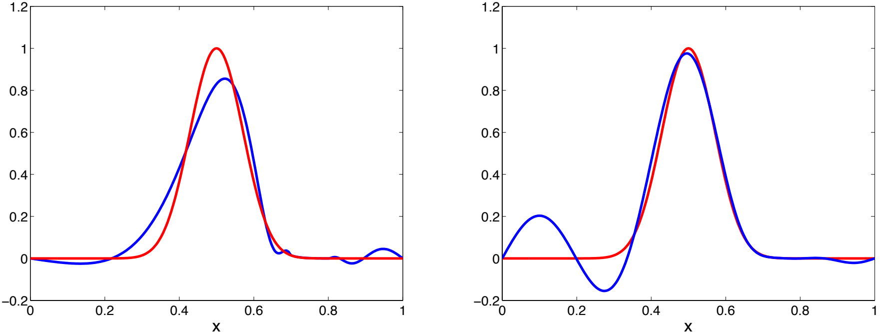
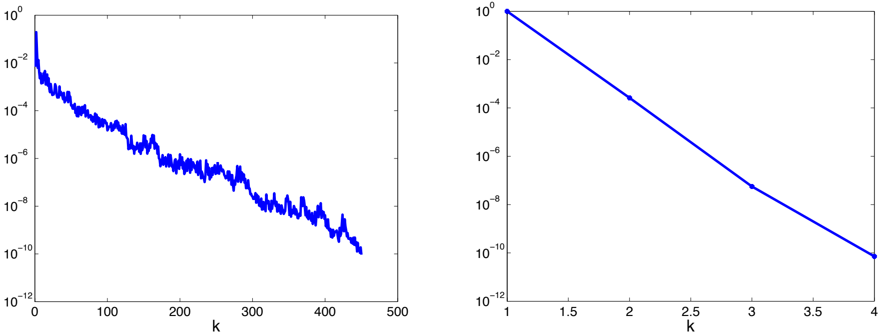


Restriction at $(0,1) \times \{0\}$

$N = 1$ - Comparison with the standard method



Evolution of the relative residu $\frac{\|g^k\|}{\|g^0\|}$ w.r.t. iterate k

Final comments

THE VARIATIONAL APPROACH CAN BE USED IN THE CONTEXT OF MANY OTHER CONTROLLABLE SYSTEMS FOR WHICH APPROPRIATE CARLEMAN ESTIMATES ARE AVAILABLE.

THE APPROXIMATION IS ROBUST (WE HAVE TO INVERSE SYMMETRIC DEFINITE POSITIVE AND VERY SPARSE MATRICE WITH DIRECT LU AND CHOLESKY SOLVERS)

WITH CONFORMAL TIME-SPACE FINITE ELEMENTS APPROXIMATIONS, WE OBTAIN STRONG CONVERGENCE RESULTS WITH RESPECT TO $h = (\Delta x, \Delta t)$.

THE PRICE TO PAY IS TO USED C^1 FINITE ELEMENTS (AT LEAST IN SPACE) ..... BUT WE MAY INTRODUCE LOWER ORDER SYSTEM.

IN THAT SPACE-TIME APPROACH, THE SUPPORT OF THE CONTROL MAY VARIES IN TIME (WITHOUT ADDITIONAL DIFFICULTIES).

THIS APPROACH MAY BE APPLIED FOR INVERSE PROBLEMS, OBSERVATION PROBLEMS, RECONSTRUCTION OF DATA, ...

Final comments - Application

- Approximation of observability constant

$$C_{obs}(T,\omega) = \sup_{\varphi_T \in H_0^1(\Omega)} \frac{\|\varphi(\cdot,0)\|^2_{L^2(\Omega)}}{\|\varphi\|^2_{L^2(\omega\times(0,T))}}. \tag{60}$$

Example : $y_t - \epsilon y_{xx} + y_x = 0$, $y(0,t) = v_\epsilon$, $\epsilon > 0$

- Optimization of the support of the control

$$\inf_{\omega\subset\Omega, |\omega|=L|\Omega|} \|v_\omega\|_{L^2(\omega\times(0,T))}, \qquad L \in (0,1) \tag{61}$$

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