Approximation of controllability and inverse problems for PDE

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CIMPA School - Marrakech - December 12-16, 2016

PART 1

Very general context and purposes

We discuss in this course the (numerical) approximation of control and inverse problem for mainly linear PDEs.

Problem 1: Controllability problem - Given a controllable PDE system

$$\begin{cases} \frac{d}{dt}y = \mathcal{L}(y) + C\boldsymbol{u}, t \in [0, T], \\ y(0) = y_0 \end{cases} \tag{1}$$

find a convergent approximation $\{\boldsymbol{u_h}\}_h$ of $\boldsymbol{u}$?

Problem 2: Inverse problem - Given an observable PDE system

$$\begin{cases} \frac{d}{dt}y = \mathcal{L}(y), t \in [0, T], \\ y(0) = y_0 \end{cases} \tag{2}$$

find a convergence approximation $\{y_h\}_h$ of y from a partial observation By of y ?

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We discuss in this course the (numerical) approximation of control and inverse problem for mainly linear PDEs.

Problem 1: Controllability problem - Given a controllable PDE system

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Problem 2: Inverse problem - Given an observable PDE system

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find a convergence approximation $\{y_h\}_h$ of y from a partial observation By of y ?

We discuss only hyperbolic equations and parabolic equations and emphasize space-time variational methods.

PART I - CONTROLLABILITY OF THE WAVE EQUATION : BOUNDARY CASE

Continuous case

$\Omega \subset \mathbb{R}^N$ bounded domain with boundary Γ of class C^2

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0, & (x,t) \in Q_T := \Omega \times (0,T) \\ \varphi = 0, & (x,t) \in \Sigma_T \\ (\varphi(\cdot,0), \varphi_t(\cdot,0)) = (\varphi_0, \varphi_1), & x \in \Omega \end{cases} \tag{3}$$

$(\varphi_0, \varphi_1) \in H_0^1(\Omega) \times L^2(\Omega)$

Conservative system with "energy" :

$$E(t) := \frac{1}{2} \|\varphi(\cdot,t), \varphi_t(\cdot,t)\|^2_{H_0^1(\Omega)\times L^2(\Omega)}$$

The wave equation with initial data in $L^2 \times H^{-1}$

$$\begin{cases} y_{tt} - \Delta y = 0, & Q_T \\ y = v\, 1_{\Gamma_0}(x), & \Sigma_T \\ (y(\cdot,0), y_t(\cdot,0)) = (y_0, y_1), & \Omega. \end{cases} \tag{4}$$

$v = v(t)$ - *control* function in $L^2(\Sigma_T)$. $\Gamma_0 \subset \partial\Omega$. $\Gamma_T := \Gamma_0 \times (0, T)$.

EXISTENCE - UNIQUENESS

$\forall (y_0, y_1) \in \boldsymbol{H} := L^2(\Omega) \times H^{-1}(\Omega)$ and $\forall v \in L^2(\Sigma_T)$, $\exists!$ solution y to (68), and (see Lions'88):

$$y = y(v) \in C^0([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega)). \tag{5}$$

and

$$\|y\|_{L^\infty(0,T;L^2(\Omega))} \leq C\Big(\|y_0, y_1\|_{L^2(\Omega)\times H^{-1}(\Omega)} + \|v\|_{L^2(\Sigma_T)}\Big)$$

The controllability of the wave equation

EXACT CONTROLLABILITY-
The system (68) is exactly controllable at time T if and only if for each $(y_0, y_1) \in \boldsymbol{Y}$ and $(z_0, z_1) \in \boldsymbol{Y}$, there exists $v \in L^2(0, T)$ such that

$$(y_v(\cdot\,, T), (y_v)_t(\cdot\,, T)) = (z_0, z_1), \quad \text{in } \Omega. \tag{6}$$

NULL CONTROLLABILITY-
The system (68) is null controllable at time T if and only if for each $(y_0, y_1) \in \boldsymbol{Y}$, there exists $v \in L^2(0, T)$ such that

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APPROXIMATE CONTROLLABILITY-
The system (68) is approximatively controllable at time T if and only if, for any $\epsilon > 0$ for each $(y_0, y_1) \in \boldsymbol{Y}$, there exists $v \in L^2(0, T)$ such that

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Link with the observability of the adjoint system

The controllability property of the wave equation (68) is related to the observability for the corresponding adjoint problem :

$$\begin{cases} L^\star\varphi := \varphi_{tt} - \Delta\varphi = 0 & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ (\varphi(\cdot\,,T), \varphi_t(\cdot\,,T)) = (\varphi_0, \varphi_1) \in \boldsymbol{V} & \text{in } \Omega \end{cases} \tag{9}$$

$\boldsymbol{V} := H_0^1(\Omega) \times L^2(\Omega)$.

Definition (Observability inequality)

System (9) is observable in time T if there exists a positive constant $C_{obs} > 0$ such that

$$\|(\varphi_0, \varphi_1)\|_{\boldsymbol{V}}^2 \leq C_{obs} \int_0^T \int_{\Gamma_0} \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt \quad \forall (\varphi_0, \varphi_1) \in \boldsymbol{V}. \tag{10}$$

Hidden regularity : If $(\varphi_0, \varphi_1) \in \boldsymbol{V}$, then $\frac{\partial\varphi}{\partial\nu} \in L^2(\partial\Omega \times (0, T))$.

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Characterization of the controls

Let φ the solution of the adjoint problem

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Lemma (Caracterization of the controls for (68))

The initial data $(y_0, y_1) \in \boldsymbol{H}$ *is controllable to zero IFF there exists* $v \in L^2(\Gamma_0 \times (0,T))$ *such that*

$$\int_0^T \int_{\Gamma_0} \frac{\partial \varphi}{\partial \nu} v \, d\sigma dt + < y_0, \varphi_1 >_{L^2, L^2} - < y_1, \varphi_0 >_{H^{-1}, H_0^1} = 0 \qquad (12)$$

for any $(\varphi_0, \varphi_1) \in \boldsymbol{V} := H_0^1(\Omega) \times L^2(\Omega)$ *and the corresponding solution* φ.

Notation - $< y_1, \varphi_0 >_{H^{-1}, H_0^1} = \int_\Omega \nabla(-\Delta^{-1} y_1) \nabla \varphi_0 dx$

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Variational approach - Introduction of $J^\star$

The characterization of the controls leads to the introduction of the following functional $J^\star : \boldsymbol{V} \to \mathbb{R}$ defined by

$$J^\star(\varphi_0, \varphi_1) := \frac{1}{2} \int_0^T \int_{\Gamma_0} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\sigma dt + < (\varphi_0, \varphi_1), (y_0, y_1) >$$

Theorem

Let $(y_0, y_1) \in \boldsymbol{H}$. and suppose that $(\hat{\varphi}_0, \hat{\varphi}_1) \in \boldsymbol{V}$ is a minimizer of $J^\star$. If $\hat{\varphi}$ is the corresponding adjoint solution, then $v = \frac{\partial \hat{\varphi}}{\partial \nu} 1_{\Gamma_0}$ is a null control for (y_0, y_1)

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Existence of minimizers for $J^\star$

Theorem

Suppose that the adjoint system is observable at time T and let $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$. The functional $J^\star$ has a unique minimizer $(\hat{\varphi}_0, \hat{\varphi}_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Observability at time T implies the coercivity of $J^\star$

$$J^\star(\varphi_0, \varphi_1) \geq \frac{C_{obs}^{-1}}{2} \|(\varphi_0, \varphi_1)\|_{\boldsymbol{V}}^2 - \|(\varphi_0, \varphi_1)\|_{\boldsymbol{V}} \|(y_0, y_1)\|_{\boldsymbol{H}}$$

Observability of $\varphi \Longrightarrow$ Controllability of y

Observability inequality

True if Γ_0 and $T > 0$ are "large" enough !

MULTIPLIER TECHNIQUES - [Ho 1986] [1] proved that if one considers subsets of Γ of the form

$$\Gamma_0 = \Gamma(x_0) = \{x \in \Gamma; (x - x^0) \cdot \nu(x) > 0\}$$

for some $x^0 \in \mathbb{R}^N$ and $T > T(x^0) = 2\|x - x^0\|_{L^\infty(\Omega)}$, the observability holds.
Multiplier $(x - x^0) \cdot \nabla\varphi$.

[1] L.F. Ho, Observabilité frontière de l'équation des ondes, 1986

Multiplier method for the $1D$ case , $\Omega = (0,1) \subset \mathbb{R}$, $\Gamma_0 = \{1\}$, $x_0 = \{0\}$

We multiply the φ equation by $x\varphi_x$ and integrate over Q_T:

$$
\begin{aligned}
0 &= \iint_{Q_T} (\varphi_{tt} - \varphi_{xx}) x\varphi_x \, dx\, dt = \iint_{Q_T} x\varphi_{tt}\varphi_x \, dxdt - \int_0^T \int_0^1 x\varphi_{xx}\varphi_x \, dxdt \\
&= \iint_{Q_T} -x\varphi_t\varphi_{xt} \, dxdt + \int_0^1 [x\varphi_t\varphi_x]_0^T dx - \iint_{Q_T} x\frac{1}{2}(|\varphi_x|^2)_x dx \\
&= \iint_{Q_T} -x\frac{1}{2}(|\varphi_t|^2)_x \, dxdt + \int_0^1 [x\varphi_t\varphi_x]_0^T dx + \frac{1}{2}\iint_{Q_T} |\varphi_x|^2 \, dxdt - \frac{1}{2}\int_0^T |\varphi_x(1,t)|^2 dt \\
&= \iint_{Q_T} \frac{1}{2}(|\varphi_t|^2) \, dxdt - \frac{1}{2}\int_0^T |\varphi_t(1,t)|^2 dt + \int_0^1 [x\varphi_t\varphi_x]_0^T dx \\
&\qquad + \frac{1}{2}\iint_{Q_T} |\varphi_x|^2 \, dxdt - \frac{1}{2}\int_0^T |\varphi_x(1,t)|^2 dt
\end{aligned}
$$

leading to

$$
\frac{1}{2}\iint_{Q_T} (|\varphi_t|^2 + |\varphi_x|^2) \, dxdt + \int_0^1 [x\varphi_t\varphi_x]_0^T dx = \frac{1}{2}\int_0^T |\varphi_x(1,t)|^2 dt
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But

$$\frac{1}{2}\iint_{Q_T} (|\varphi_t|^2 + |\varphi_x|^2)\, dxdt = TE(t) = TE(0) \tag{13}$$

and

$$\begin{aligned}\int_0^1 [x\varphi_t\varphi_x]_0^T dx &\leq \int_0^1 [\varphi_t\varphi_x]_0^T dt = \int_0^1 \Big((\varphi_t\varphi_x)(T) - (\varphi_t\varphi_x)(0)\Big) dt \\ &\leq \frac{1}{2}\int_0^1 (|\varphi_t(\cdot,T)|^2 + |\varphi_x(\cdot,T)|^2) dx + \frac{1}{2}\int_0^1 (|\varphi_t(\cdot,0)|^2 + |\varphi_x(\cdot,0)|^2) dx \\ &\leq 2\,E(0)\end{aligned} \tag{14}$$

leading to

$$(T-2)E(0) \leq \frac{1}{2}\int_0^T |\varphi_x(1,t)|^2 dt$$

Spectral method for the $1D$ case , $\Omega = (0,1) \subset \mathbb{R}$, $\Gamma_0 = \{1\}$

We expand the solution of the adjoint system as follows:

$$\begin{cases} \varphi_0(x) = \sum_{k>0} a_k \sin(k\pi x), \quad \varphi_1(x) = \sum_{k>0} b_k \cos(k\pi x), \\ \varphi(x,t) = \sum_{k>0} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sin(k\pi x) \end{cases} \tag{15}$$

leading to

$$\begin{cases} \varphi(x,t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} x_k e^{i\Lambda_k t} \sin(k\pi x), \\ x_k = \frac{a_k - ib_k/(k\pi)}{2}, k > 0, \quad x_k = \frac{-a_{-k} + ib_{-k}/(k\pi)}{2}, k < 0, \qquad \Lambda_k = k\pi \end{cases} \tag{16}$$

leading to

$$\int_0^T |\varphi_x(1,t)|^2 dt = \int_0^T \left| \sum_{k\in\mathbb{Z}\setminus\{0\}} (-1)^k k\pi x_k e^{i\Lambda_k t} \right|^2 dx,$$

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Theorem (Ingham, 1936 [2])

Let $K \in \mathbb{Z}$ and $(w_k)_{k\in K}$ be a family of real numbers satisfying the uniform gap condition $\gamma := \inf_{k\neq n} |w_k - w_n| > 0$. If I is a bounded interval of length $|I| > 2\pi/\gamma$, then

$$\sum_{k\in K} |y_k|^2 \asymp \int_I |\sum_{k\in K} y_k e^{iw_k t}|^2 dt$$

for all square-summable complex coefficients y_k

Application - $K = \mathbb{Z}\setminus\{0\}$ - $w_k = \Lambda_k$ leads to $\gamma = \pi$. $I = (0,T)$ leads to : if $T > 2$, then

$$\int_0^T |\varphi_x(1,t)|^2 dt \asymp \sum_{k\in\mathbb{Z}} |y_k|^2 = \frac{1}{2}\sum_{k>0}\Big((k\pi)^2|a_k|^2 + |b_k|^2\Big) = \|\varphi_0, \varphi_1\|_V^2 \qquad (17)$$

[2] V. Komornik, P. Loreti, Fourier series in control theory, 2005

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[2] V. Komornik, P. Loreti, Fourier series in control theory, 2005

Spectral method for the 1D case , $\Omega = (0,1) \subset \mathbb{R}$, $\Gamma_0 = \{1\}$

$$\int_0^T |\varphi_x(1,t)|^2 dt = \int_0^T \Big| \sum_{k\in\mathbb{Z}\setminus\{0\}} \underbrace{(-1)^k k\pi x_k}_{y_k} e^{i\Lambda_k t} \Big|^2 dt,$$

Theorem (Ingham, 1936 [2])

Let $K \in \mathbb{Z}$ and $(w_k)_{k\in K}$ be a family of real numbers satisfying the uniform gap condition $\gamma := \inf_{k\neq n} |w_k - w_n| > 0$. If I is a bounded interval of length $|I| > 2\pi/\gamma$, then

$$\sum_{k\in K} |y_k|^2 \asymp \int_I |\sum_{k\in K} y_k e^{iw_k t}|^2 dt$$

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The control $v = \frac{\partial \varphi}{\partial \nu}$

Theorem

The control $v = \frac{\partial \hat{\varphi}}{\partial \nu}$ associated to the minimization of $J^\star$ is the control of minimal L^2-norm.

PROOF - Let $v = \frac{\partial \hat{\varphi}}{\partial \nu}$ and v_1 another control. They both satisfy the characterization:

$$\begin{aligned} \int_0^T \int_{\Gamma_0} \frac{\partial \varphi}{\partial \nu} v d\sigma dt + < (\varphi_0, \varphi_1), (y_0, y_1) >= 0, \quad \forall (\varphi_0, \varphi_1) \in \boldsymbol{V}, \\ \int_0^T \int_{\Gamma_0} \frac{\partial \varphi}{\partial \nu} v_1 d\sigma dt + < (\varphi_0, \varphi_1), (y_0, y_1) >= 0, \quad \forall (\varphi_0, \varphi_1) \in \boldsymbol{V}, \end{aligned} \tag{18}$$

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Equivalence with the minimization of $J^\star$

$$\begin{cases} \text{Minimize } \; J(y,v) = \dfrac{1}{2}\displaystyle\int_0^T |v|^2\, dt \\ \text{Subject to } \; (y,v) \in \mathcal{C}(y_0, y_1; T) \end{cases} \tag{20}$$

where $\mathcal{C}(y_0, y_1; T)$ denotes the non-empty linear manifold

$$\mathcal{C}(y_0, y_1; T) = \{\, (y,v) : v \in L^2(0,T), \; y \text{ solves (68) and satisfies (8)} \,\}.$$

Using the Fenchel-Rockafellar theorem [Ekeland-Temam 74], [Brezis 84] we get that

$$\inf_{(y,v)\in\mathcal{C}(y_0,y_1;T)} J(y,v) = - \min_{(\varphi_0,\varphi_1)\in \boldsymbol{V}} J^\star(\varphi_0,\varphi_1)$$

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Extension 1 : more general cost

$$\begin{cases} \text{Minimize } \; J(y,v) = \dfrac{1}{2}\displaystyle\iint_{Q_T} \rho^2 |y|^2 \, dx\, dt + \dfrac{1}{2}\int_0^T\!\!\int_{\Gamma_0} \rho_0^2 |v|^2 \, d\sigma \, dt \\ \text{Subject to } \; (y,v) \in \mathcal{C}(y_0, y_1; T) \end{cases} \tag{22}$$

$$v = -\rho_0^{-2}\frac{\partial \varphi}{\partial \nu} \text{ in } (0,T)\times\Gamma_0 \text{ and } y = -\rho^{-2}\mu \text{ in } Q_T.$$

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where φ solves the nonhomogeneous backward problem

$$L^\star \varphi := \mu \quad \text{in} \quad Q_T, \qquad \varphi = 0 \quad \text{on} \quad \Sigma_T, \qquad (\varphi(\cdot,0),\varphi'(\cdot,0)) = (\varphi_0,\varphi_1) \tag{24}$$

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Extension 2 : wave equation with non constant coefficient [3]

$T>0$ $a\in C^3([0,1])$ with $a(x)\geq a_0>0$ in $[0,1]$, $b\in L^\infty((0,1)\times(0,T))$, $(y_0,y_1)\in \boldsymbol{H}$ and $v\in L^2(0,T)$

$$\begin{cases} y_{tt}-(a(x)y_x)_x+b(x,t)y=0, & (x,t)\in(0,1)\times(0,T) \\ y(0,t)=0, \qquad y(1,t)=v(t), & t\in(0,T) \\ y(x,0)=y_0(x), \qquad y_t(x,0)=y_1(x), & x\in(0,1). \end{cases} \tag{25}$$

Theorem

Let $x_0<0$ and

$$\mathcal{A}(x_0,a_0)=\Big\{\, a\in C^3([0,1]) : a(x)\geq a_0>0, \; -\min_{[0,1]}\Big(a(x)+(x-x_0)a_x(x)\Big)<\min_{[0,1]}\Big(a(x)+\frac{1}{2}(x-x_0)a_x(x)\Big)\,\Big\} \tag{26}$$

If

$$a(x)\in\mathcal{A}(x_0,a_0) \quad \text{and} \quad T>\frac{1}{\beta}\max_{[0,1]} a(x)^{1/2}(x-x_0).$$

with $\beta\in\Big]-\min_{[0,1]}\Big(a(x)+(x-x_0)a_x(x)\Big),\min_{[0,1]}\Big(a(x)+\frac{1}{2}(x-x_0)a_x(x)\Big)\Big[$

then the system is null-controllable.

[3] N. Cindea, E. Fernandez-Cara, AM, Numerical controllability of the wave equation through primal methods and Carleman estimates, 2012

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Extension 3 : wave equation with inner control

$q_T = \omega \times (0,T)$, $Q_T := \Omega \times (0,T)$, $\boldsymbol{V} := H_0^1(0,1) \times L^2(0,1)$, $a, b \in C([0,T],]0,1[)$

$$\begin{cases} y_{tt} - \Delta y = v 1_{q_T}, & (x,t) \in Q_T \\ y = 0, & (x,t) \in \partial\Omega \times (0,T) \\ (y(\cdot,0), y_t(\cdot,0)) = (y_0, y_1) \in \boldsymbol{V}, & x \in (0,1). \end{cases} \tag{27}$$

Theorem (Bardos-Lebeau-Rauch'92, Burq'97)

If the triplet (ω, T) satisfies the following geometric optic condition in Ω:

Every ray of geometric optics that propagates in Ω and is reflected on its boundary Γ enters ω in time less than T

then (27) is null controllable.

PART II - CONTROLLABILITY OF THE WAVE EQUATION : BOUNDARY CASE

Numerical issues

Approximation and minimization of $J^\star$ over $\boldsymbol{H} := H_0^1(\Omega) \times L^2(\Omega)$

$$\begin{cases} \text{Min } \; J^\star(\varphi_0,\varphi_1) = \dfrac{1}{2}\displaystyle\int_0^T\int_{\Gamma_0}\left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt + < (\varphi_0,\varphi_1),(y_0,y_1) > \\ \text{Subject to } \; (\varphi_0,\varphi_1) \in \boldsymbol{V} = H_0^1(\Omega)\times L^2(\Omega) \quad \text{where} \quad L^\star\varphi = 0 \end{cases} \tag{28}$$

Gradient descent iterative method :

$$DJ^\star(\varphi_0,\varphi_1)\cdot(\overline{\varphi_0},\overline{\varphi_1}) = < z(\cdot,0) - y_0, \overline{\varphi_1} >_{L^2} + < z_t(\cdot,0) - y_1, \overline{\varphi_0} >_{H^{-1},H_0^1}$$

The difficulty is the constraint $L^\star\varphi = 0$!!

It is impossible in general to find an approximation of finite dimension φ_h such that $L^\star\varphi_h = 0$!!! (h is an approximation parameter)

In practice, one may find φ_h such that $\|L^\star\varphi_h\| = \mathcal{O}(h^\alpha), \alpha > 0$.

And the previous arguments are no more valid !?!!!!.

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And the previous arguments are no more valid !?!!!!.

First method to bypass the fact that $L^{\star}\varphi_h \neq 0$

The trick, initially used by Roland Glowinski [4] in the nineties and many others is :

Replace the operator L by a discrete operator L_h and control a finite dimensional differential system

[4] R. Glowinski, C.H. Li, J.-L. Lions, A numerical approach to the exact boundary controllability of the wave equation. I. Dirichlet controls: description of the numerical methods, (1990)

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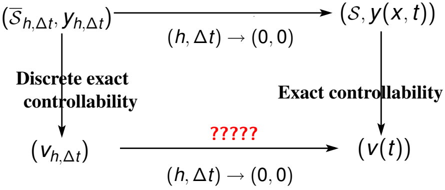
$h, \Delta t$: approximation parameters



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The case of the wave equation in $1d$ [6]

Let $J \in \mathbb{N}$, $h = 1/(J+1)$ and a uniform grid of $(0,1)$:

$$0 = x_0 < x_1 < ... < x_J < x_{J+1} = 1, \quad x_j = jh, j = 0, ..., J+1.$$

Let $N \in \mathbb{N}$, $\Delta t = T/N$ and a uniform grid of $(0, T)$:

$$0 = t_0 < t_1 < ... < t_N = 1, \quad t_n = n\Delta t, n = 0, ..., N.$$

h and Δt are the space and time step.

Defining for $j = 1, ..., J$ and $n = 0, ..., N$ the operators

$$\Delta_h y_j^n := \frac{y_{j+1}^n - 2y_j^n + y_{j-1}^n}{h^2}, \quad \Delta_{\Delta t} y_j^n := \frac{y_j^{n+1} - 2y_j^n + y_j^{n-1}}{\Delta t^2}. \tag{29}$$

$$(S_{h,\Delta t}^{\theta_\nu,0}) \begin{cases} (1 + \theta_\nu h^2 \Delta_h)\Delta_{\Delta t} y_j^n = \Delta_h y_j^n, & 1 \le j \le J, \quad 0 \le n \le N, \\ y_0^n = 0, y_{J+1}^n = v_h^n, & 0 \le n \le N, \\ \frac{y_j^0 + y_j^1}{2} = y_{0j}, \quad \frac{y_j^1 - y_j^0}{\Delta t} = y_{1j}, & 0 \le j \le J+1. \end{cases} \tag{30}$$

where

$$\theta \ge 0, \quad \alpha \ge 0, \quad \theta_\nu \equiv \theta - \alpha\nu^2; \qquad \nu \equiv \Delta t/h. \tag{31}$$

[6] A. Münch, A uniformly controllable and implicit scheme for the 1-D wave equation,(2005)

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h and Δt are the space and time step.
Defining for $j = 1, ..., J$ and $n = 0, ..., N$ the operators

$$\Delta_h y_j^n := \frac{y_{j+1}^n - 2y_j^n + y_{j-1}^n}{h^2}, \quad \Delta_{\Delta t} y_j^n := \frac{y_j^{n+1} - 2y_j^n + y_j^{n-1}}{\Delta t^2}. \tag{29}$$

$$(S_{h,\Delta t}^{\theta_\nu,0}) \begin{cases} (1 + \theta_\nu h^2 \Delta_h)\Delta_{\Delta t} y_j^n = \Delta_h y_j^n, & 1 \le j \le J, \quad 0 \le n \le N, \\ y_0^n = 0, y_{J+1}^n = v_h^n, & 0 \le n \le N, \\ \frac{y_j^0 + y_j^1}{2} = y_{0j}, \quad \frac{y_j^1 - y_j^0}{\Delta t} = y_{1j}, & 0 \le j \le J+1. \end{cases} \tag{30}$$

where

$$\theta \ge 0, \quad \alpha \ge 0, \quad \theta_\nu \equiv \theta - \alpha\nu^2; \qquad \nu \equiv \Delta t / h. \tag{31}$$

[6] A. Münch, A uniformly controllable and implicit scheme for the 1-D wave equation,(2005)

The case of the wave equation in $1d$ [6]

Let $J \in \mathbb{N}$, $h = 1/(J+1)$ and a uniform grid of $(0,1)$:

$$0 = x_0 < x_1 < ... < x_J < x_{J+1} = 1, \quad x_j = jh, j = 0, ..., J+1.$$

Let $N \in \mathbb{N}$, $\Delta t = T/N$ and a uniform grid of $(0,T)$:

$$0 = t_0 < t_1 < ... < t_N = 1, \quad t_n = n\Delta t, n = 0, ..., N.$$

h and Δt are the space and time step.
Defining for $j = 1, ..., J$ and $n = 0, ..., N$ the operators

$$\Delta_h y_j^n := \frac{y_{j+1}^n - 2y_j^n + y_{j-1}^n}{h^2}, \quad \Delta_{\Delta t} y_j^n := \frac{y_j^{n+1} - 2y_j^n + y_j^{n-1}}{\Delta t^2}. \tag{29}$$

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Vectorial form of the scheme $(\mathcal{S}_{h,\Delta t}^{\theta_\nu,0})$

we introduce $K, M_0^{\theta_\nu} \in \mathcal{M}_{J\times J}(\mathbb{R})$ by

$$K = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & (0) & \\ & -1 & \ddots & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & (0) & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}_{J\times J}, \quad M_0^{\theta_\nu} = I - \theta_\nu K \qquad (32)$$

We denote the vector $\boldsymbol{Y_h^n} = (y_1^n, y_2^n, ..., y_J^n)^T$, $n = 0, ..., N$. $(\mathcal{S}_{h,\Delta t}^{\theta_\nu,0})$ takes the vectorial form :

$$\begin{cases} M_0^{\theta_\nu}(\boldsymbol{Y_h^{n+1}} - 2\boldsymbol{Y_h^n} + \boldsymbol{Y_h^{n-1}}) + \nu^2 K \boldsymbol{Y_h^n} = \boldsymbol{F_h^n}, \quad 0 \le n \le N, \\ \frac{\boldsymbol{Y_h^0} + \boldsymbol{Y_h^1}}{2} = \boldsymbol{y_{0h}}, \quad \frac{\boldsymbol{Y_h^1} - \boldsymbol{Y_h^0}}{\Delta t} = \boldsymbol{y_{1h}}, \end{cases} \qquad (33)$$

where $\boldsymbol{F_h^n} = (f_1^n, ..., f_{J-1}^n, f_J^n)$ with $f_j^n = 0$, $j = 1...J-1$ and

$$f_J^n = -\theta_\nu(v_h^{n+1} - 2v_h^n + v_h^{n-1}) + \nu^2 v_h^n, \quad 0 \le n \le N, \qquad (34)$$

taking into account that $y_{J+1}^n = v_h^n$ and $y_0^n = 0$, for $n = 0...N$.

The discrete controllability problem

Given T large enough independent of h and Δt and $(\boldsymbol{y_{0h}}, \boldsymbol{y_{1h}}) \in \mathbb{R}^{2J}$, does there exist a control function $(v_h^n)_n$, $n = 0...N$, such that the solution $\boldsymbol{Y_h^n}$ of (33) satisfies

$$\boldsymbol{Y_h^N} = \mathbf{0}, \quad \frac{\boldsymbol{Y_h^N} - \boldsymbol{Y_h^{N-1}}}{\Delta t} = \mathbf{0}, \tag{35}$$

and therefore $\boldsymbol{Y_h^N} = \boldsymbol{Y_h^{N-1}} = \mathbf{0}$? If this holds for any $(\boldsymbol{y_{0h}}, \boldsymbol{y_{1h}}) \in \mathbb{R}^{2J}$, we say that the discrete system (33) is null controllable.

Lemma

The discrete system is null controllable if $\forall(\boldsymbol{y_{0h}}, \boldsymbol{y_{1h}}) \in \mathbb{R}^{2J}$, $\exists(v_h^n)_n$ *such that*

$$\Delta t \sum_{n=0}^{N-1} (v_h^n)^2 < \infty,\ \Delta t \sum_{n=0}^{N-1} \Big(\frac{v_h^{n+1}-v_h^n}{\Delta t}\Big)^2 < \infty,\ \Delta t \sum_{n=0}^{N-1} \Big(\frac{v_h^{n+1}-2v_h^n+v_h^{n-1}}{\Delta t^2}\Big)^2 < \infty$$

and

$$\Delta t \sum_{n=0}^{N-1} \left(v_h^n \frac{w_J^n}{h} + \theta_\nu h \frac{v_h^{n+1}-v_h^n}{\Delta t} \frac{w_J^{n+1}-w_J^n}{\Delta t} \right) - \left(\left(-K_h^{-1} M_0^{\theta_\nu} \boldsymbol{y_{1h}}, (M_1^{\theta_\nu})^{-1} M_0^{\theta_\nu} \boldsymbol{y_{0h}} \right), \left(\frac{\boldsymbol{W_h^0}+\boldsymbol{W_h^1}}{2}, \frac{\boldsymbol{W_h^1}-\boldsymbol{W_h^0}}{\Delta t} \right) \right)_1 = 0, \tag{36}$$

for any $(\boldsymbol{w_{0h}}, \boldsymbol{w_{1h}}) \in \mathbb{R}^{2J}$, *where* $\boldsymbol{W_h^n}$ *is the solution of the adjoint homogeneous system :*

$$\begin{cases} M_0^{\theta_\nu}(\boldsymbol{W_h^{n+1}} - 2\boldsymbol{W_h^n} + \boldsymbol{W_h^{n-1}}) + \nu^2 K \boldsymbol{W_h^n} = 0, \quad 0 \le n \le N, \\ \frac{\boldsymbol{W_h^{N-1}}+\boldsymbol{W_h^N}}{2} = \boldsymbol{w_{0h}}, \ \frac{\boldsymbol{W_h^N}-\boldsymbol{W_h^{N-1}}}{\Delta t} = \boldsymbol{w_{1h}}. \end{cases} \tag{37}$$

Homogeneous system

We then define the discrete version $\mathcal{J}_h : \mathbb{R}^{2J} \to \mathbb{R}$ of the functional $\mathcal{J}$

$$\mathcal{J}_h(\boldsymbol{w_{0h}}, \boldsymbol{w_{1h}}) = \frac{\Delta t}{2} \sum_{n=0}^{N-1} \left[\left(\frac{w_J^n}{h} \right)^2 + \rho^n \theta_\nu \left(\frac{w_J^{n+1} - w_J^n}{\Delta t} \right)^2 \right]$$
$$- \left(\left(-K_h^{-1} M_0^{\theta_\nu} \boldsymbol{y_{1h}}, (M_1^{\theta_\nu})^{-1} M_0^{\theta_\nu} \boldsymbol{y_{0h}} \right), \left(\frac{\boldsymbol{W_h^0} + \boldsymbol{W_h^1}}{2}, \frac{\boldsymbol{W_h^1} - \boldsymbol{W_h^0}}{\Delta t} \right) \right)_1, \tag{38}$$

where $\boldsymbol{W_h^n}$ is the solution of the following adjoint homogeneous system :

$$\begin{cases} M_0^{\theta_\nu} (\boldsymbol{W_h^{n+1}} - 2\boldsymbol{W_h^n} + \boldsymbol{W_h^{n-1}}) + \nu^2 K \boldsymbol{W_h^n} = 0, \quad 0 \le n \le N, \\ \frac{\boldsymbol{W_h^{N-1}} + \boldsymbol{W_h^N}}{2} = \boldsymbol{w_{0h}}, \ \frac{\boldsymbol{W_h^N} - \boldsymbol{W_h^{N-1}}}{\Delta t} = \boldsymbol{w_{1h}}. \end{cases} \tag{39}$$

Discrete control - Main result I

$$\mathcal{C} = \{(\theta, \alpha, \nu) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+_\star, \lim_{h \to 0} \cos^2(\pi h/2)(\nu^2(1 - 4\alpha) + 4\theta) = 1\}. \tag{40}$$

Theorem

Let $(\theta, \alpha, \nu) \in \mathcal{C}$. Given any $T > 2\max(1, \nu^2)$ and $(\mathbf{y_{0h}}, \mathbf{y_{1h}}) \in \mathbb{R}^{2J}$, the functional $\mathcal{J}_h$ defined by (38) has a unique minimizer $(\hat{\mathbf{w}}_{\mathbf{0h}}, \hat{\mathbf{w}}_{\mathbf{1h}}) \in \mathbb{R}^{2J}$. Let $v_h = (v_h^n)_n$ defined as follows :

$$\begin{cases} v_h^n - \theta_\nu h^2 \dfrac{v_h^{n+1} - 2v_h^n + v_h^{n-1}}{\Delta t^2} = \dfrac{\hat{W}_J^n}{h} - \theta_\nu h \dfrac{\rho^n \frac{\hat{W}_J^{n+1} - \hat{W}_J^n}{\Delta t} - \rho^{n-1} \frac{\hat{W}_J^n - \hat{W}_J^{n-1}}{\Delta t}}{\Delta t}, & 0 \le n \le N, \\ \theta_\nu (v_h^1 - v_h^0) = 0, \quad \theta_\nu (v_h^N - v_h^{N-1}) = 0, & \end{cases} \tag{41}$$

where $\hat{W}_h^n$ is the solution of (39) with initial data $(\hat{\mathbf{w}}_{\mathbf{0h}}, \hat{\mathbf{w}}_{\mathbf{1h}})$. Then, $v_h = (v_h^n)_n$ is a control for (33).

Remark: Analogue to the optimality condition $v = \frac{\partial \varphi}{\partial \nu}$

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where $\hat{\boldsymbol{W}}_{\boldsymbol{h}}^{\boldsymbol{n}}$ is the solution of (39) with initial data $(\hat{\boldsymbol{w}}_{\boldsymbol{0h}}, \hat{\boldsymbol{w}}_{\boldsymbol{1h}})$. Then, $v_h = (v_h^n)_n$ is a control for (33).

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Discrete control - Main result II

Theorem

Let $(\theta, \alpha, \nu) \in \mathcal{C}$ and $T > 2\max(1, \nu^2)$. Let $(\mathbf{Y_{0h}}, \mathbf{Y_{1h}})$ be a sequence of discretizations of the initial data (y_0, y_1). Assume that $(a_{k,h}, b_{k,h})_k$, the Fourier coefficients of the discrete initial data verify

$$(a_{k,h})_k \rightharpoonup (a_k)_k, \ \left(\frac{b_{k,h}}{\sqrt{\lambda_{k,h}^{\theta,\alpha}}}\right)_k \rightharpoonup \left(\frac{b_k}{k\pi}\right)_k \text{ in } l^2 \text{ when } h \to 0, \tag{42}$$

where (a_k, b_k) are the Fourier coefficients of the continuous initial data.

Let $(v_h)_h$ be the sequence of controls given by Theorem 9. Then $(Q(v_h))_h$, $(hP(v_h)')_h$ are uniformly bounded in $L^2(0,T)$, $(h^2 P(v_h)')_h$ is uniformly bounded in $L^\infty(0,T)$ and there exists a subsequence v_{h_j} and $v \in L^2(0,T)$ such that

$$\begin{aligned} Q(v_{h_j}) &\rightharpoonup v \ \in \ L^2([0,T]) \text{ when } h_j \to 0, \\ hP(v_{h_j})' &\rightharpoonup 0 \ \in \ L^2([0,T]) \text{ when } h_j \to 0, \\ h^2 P(v_{h_j})' &\rightharpoonup 0 \ \in \ L^\infty([0,T]) \text{ when } h_j \to 0. \end{aligned} \tag{43}$$

Moreover, the limit v is the L^2 minimal control of the continuous system.
If the convergence in (42) is strong, then the above convergences are strong too.

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Let $(\theta, \alpha, \nu) \in \mathcal{C}$ and $T > 2\max(1, \nu^2)$. Let $(\boldsymbol{Y_{0h}}, \boldsymbol{Y_{1h}})$ be a sequence of discretizations of the initial data (y_0, y_1). Assume that $(a_{k,h}, b_{k,h})_k$, the Fourier coefficients of the discrete initial data verify

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Moreover, the limit v is the L^2 minimal control of the continuous system.
If the convergence in (42) is strong, then the above convergences are strong too.

The homogeneous system

$$(\mathcal{A}^{\theta,\alpha}_{h,\Delta t})\begin{cases}(1+\theta_\nu h^2\Delta_h)\Delta_{\Delta t}u^n_j=\Delta_h u^n_j, & 1\le j\le J,\quad 0\le n\le N,\\ u^n_0=u^n_{J+1}=0, & 0\le n\le N,\\ (u^0_j+u^1_j)/2=u_{0j},\quad (u^1_j-u^0_j)/\Delta t=u_{1j}, & 0\le j\le J+1.\end{cases} \tag{44}$$

which takes the following vectorial form

$$\begin{cases} M_0^{\theta_\nu}(\boldsymbol{U_h^{n+1}}-2\boldsymbol{U_h^n}+\boldsymbol{U_h^{n-1}})+\nu^2 K\boldsymbol{U_h^n}=0 \quad n=0,...,N,\\ \frac{\boldsymbol{U_h^0}+\boldsymbol{U_h^1}}{2}=\boldsymbol{u_{0h}},\ \frac{\boldsymbol{U_h^1}-\boldsymbol{U_h^0}}{\Delta t}=\boldsymbol{u_{1h}}.\end{cases} \tag{45}$$

where $\boldsymbol{U_h^n}=(u^n_1,...,u^n_J)^T$.

Definition

The discrete energy $E_n^{\theta,\alpha}$, $n=0,...,N$, associated to the scheme (44) is

$$E_n^{\theta,\alpha}=\frac{1}{2}(K_h\boldsymbol{U_h^{n+1}},\boldsymbol{U_h^n})+\frac{1}{2}\left((I-\theta_\nu K)\frac{\boldsymbol{U_h^{n+1}}-\boldsymbol{U_h^n}}{\Delta t},\frac{\boldsymbol{U_h^{n+1}}-\boldsymbol{U_h^n}}{\Delta t}\right) \tag{46}$$

Stability and consistency

$$\boldsymbol{\mathcal{S}} = \left\{(\theta, \alpha, \nu) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+_\star,\ \cos^2\left(\frac{\pi h}{2}\right)(\nu^2(1-4\alpha)+4\theta) \leq 1, \forall h > 0\right\}. \quad (47)$$

Stability -The scheme $(\mathcal{A}^{\theta,\alpha}_{h,\Delta t})$ is stable if and only if $(\theta, \alpha, \nu) \in \boldsymbol{\mathcal{S}}$.

Proof. $(\mathcal{A}^{\theta,\alpha}_{h,\Delta t})$ is stable if $E_0^{\theta,\alpha}$ is a positive quadratic form, i.e. if the matrix $M_1^{\theta_\nu}$ is positive definite (the matrix K_h is positive definite). The eigenvalues $0 < \lambda_1^K < \lambda_2^K < ... < \lambda_J^K$ of K are $\lambda_j^K = 4\sin^2(j\pi h/2)$. The eigenvalues of $M_1^{\theta_\nu} = I - (\theta_\nu + \nu^2/4)K$ are

$$\lambda_j^{M_1^{\theta_\nu}} = 1 - 4\left(\theta_\nu + \frac{\nu^2}{4}\right)\sin^2\left(\frac{j\pi h}{2}\right), \quad 1 \leq j \leq J. \quad (48)$$

Consistency - $\forall \theta, \alpha \geq 0$, the error of consistency associated to $(\mathcal{A}^{\theta,\alpha}_{h,\Delta t})$ is of order

$$(\theta - 1/12)O(h^2) + (\alpha - 1/12)O(\Delta t^2) + O(h^4) + O(\Delta t^4) + O(h^2\Delta t^2). \quad (49)$$

Convergence - $\forall (\theta, \alpha, \nu) \in \boldsymbol{\mathcal{S}}$, $(\mathcal{A}^{\theta,\alpha}_{h,\Delta t})$ is convergent of order 2.

Stability and consistency

$$\boldsymbol{\mathcal{S}} = \left\{(\theta,\alpha,\nu) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+_\star,\ \cos^2\left(\frac{\pi h}{2}\right)(\nu^2(1-4\alpha)+4\theta) \leq 1, \forall h > 0\right\}. \quad (47)$$

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$$\boldsymbol{\mathcal{S}} = \left\{(\theta,\alpha,\nu) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+_\star,\ \cos^2\left(\frac{\pi h}{2}\right)(\nu^2(1-4\alpha)+4\theta) \leq 1, \forall h > 0\right\}. \quad (47)$$

Stability -The scheme $(\mathcal{A}_{h,\Delta t}^{\theta,\alpha})$ is stable if and only if $(\theta,\alpha,\nu) \in \boldsymbol{\mathcal{S}}$.

Proof. $(\mathcal{A}_{h,\Delta t}^{\theta,\alpha})$ is stable if $E_0^{\theta,\alpha}$ is a positive quadratic form, i.e. if the matrix $M_1^{\theta_\nu}$ is positive definite (the matrix K_h is positive definite). The eigenvalues $0 < \lambda_1^K < \lambda_2^K < ... < \lambda_J^K$ of K are $\lambda_j^K = 4\sin^2(j\pi h/2)$. The eigenvalues of $M_1^{\theta_\nu} = I - (\theta_\nu + \nu^2/4)K$ are

$$\lambda_j^{M_1^{\theta_\nu}} = 1 - 4\left(\theta_\nu + \frac{\nu^2}{4}\right)\sin^2\left(\frac{j\pi h}{2}\right), \quad 1 \leq j \leq J. \quad (48)$$

Consistency - $\forall\theta,\alpha \geq 0$, the error of consistency associated to $(\mathcal{A}_{h,\Delta t}^{\theta,\alpha})$ is of order

$$(\theta - 1/12)O(h^2) + (\alpha - 1/12)O(\Delta t^2) + O(h^4) + O(\Delta t^4) + O(h^2\Delta t^2). \quad (49)$$

Convergence - $\forall(\theta,\alpha,\nu) \in \boldsymbol{\mathcal{S}}$, $(\mathcal{A}_{h,\Delta t}^{\theta,\alpha})$ is convergent of order 2.

Uniform discrete observability inequality

Theorem

Let $(\theta, \alpha, \nu) \in \mathcal{C}$ and $T > 2\max(1, \nu^2)$. There exist two constants $C_1, C_2 > 0$ independent of h and Δt such that

$$C_1 E_0^{\theta,\alpha} \leq \Delta t \sum_{n=0}^{N-1} \left(\left| \frac{u_J^n}{h} \right|^2 + \theta_\nu \left| \frac{u_J^{n+1} - u_J^n}{\Delta t} \right|^2 \right) \leq C_2 E_0^{\theta,\alpha}. \qquad (50)$$

The left part of (50) is a uniform discrete observability inequality, discrete version of the inequality

$$C_1 E(0) \leq \int_0^T [(u_x(1,t))^2 + \theta_\nu (u_x'(1,t))^2] dt \qquad (51)$$

with $\theta_\nu := \theta - \alpha\nu^2 = \frac{(1-\nu^2)}{4}$, $\nu = \frac{\Delta t}{h}$

Uniform discrete observability inequality - 2

$$\boldsymbol{U_h^n} = \sum_{k=1}^{J}\left[a_{k,h}\cos\left(\sqrt{\lambda_{k,h}^{\theta,\alpha}}n\Delta t\right) + \frac{b_{k,h}}{\sqrt{\lambda_{k,h}^{\theta,\alpha}}}\sin\left(\sqrt{\lambda_{k,h}^{\theta,\alpha}}n\Delta t\right)\right]\boldsymbol{\phi_{k,h}}, \tag{52}$$

with

$$\begin{cases} \lambda_{k,h}^{\theta,\alpha} = \left[\dfrac{2}{\Delta t}\arcsin\left(\dfrac{\nu\sin(k\pi h/2)}{\sqrt{1-4(\theta-\alpha\nu^2)\sin^2(k\pi h/2)}}\right)\right]^2, \quad \forall k = 1,\dots,J, \\ \boldsymbol{\phi_{k,h}} = (\phi_{k,j})_{(1\le j\le J)}, \quad \phi_{k,j} = \sin(k\pi jh). \end{cases} \tag{53}$$

or equivalently

$$\boldsymbol{U_h^n} = \sum_{|k|\le J, k\neq 0} c_{k,h}e^{i\mu_{k,h}n\Delta t}\boldsymbol{\phi_{k,h}}, \tag{54}$$

with

$$\mu_{-k,h} = -\mu_{k,h};\ \mu_{k,h} = \sqrt{\lambda_{k,h}^{\theta,\alpha}};\ c_{k,h} = \frac{a_{k,h} - ib_{k,h}/\mu_{k,h}}{2};\ c_{-k,h} = \overline{c_{k,h}}. \tag{55}$$

Uniform discrete observability inequality - 3

We have to estimate the quantity :

$$C_1 E_0^{\theta,\alpha} \leq \Delta t \sum_{n=0}^{N-1} \left(\left| \frac{u_J^n}{h} \right|^2 + \theta_\nu \left| \frac{u_J^{n+1} - u_J^n}{\Delta t} \right|^2 \right) \leq C_2 E_0^{\theta,\alpha}. \tag{56}$$

$$\Delta t \sum_{n=0}^{N-1} \left| \frac{u_J^n}{h} \right|^2 = \Delta t \sum_{n=0}^{N-1} \left| \sum_{|k| \leq J, k \neq 0} c_{k,h} e^{i\mu_{k,h} n \Delta t} \frac{\sin(k\pi J h)}{h} \right|^2. \tag{57}$$

Theorem (Discrete ingham inequality, Negreanu'03) [7] Let $\Delta t > 0$ and $\{\mu_k\}$ be a sequence of reel numbers satisfying for some γ and $0 \leq p < 1/2$ the conditions :

$$\begin{aligned} &\mu_{k+1} - \mu_k \geq \gamma > 0, \quad \forall\, k \in \mathbb{Z}, \\ &|\mu_k - \mu_l| \leq \frac{2\pi - (\Delta t)^p}{\Delta t}, \quad \forall\, k, l \in \mathbb{Z}. \end{aligned} \tag{58}$$

Then, for every $T > 2\pi/\gamma$, there exist two positives constants $C_1(T,\gamma)$ and $C_2(T,\gamma)$ such that

$$C_1(T,\gamma) \sum_{k=-J}^{J} |c_k|^2 \leq \Delta t \sum_{n=0}^{N-1} \sum_{k=-J}^{J} \left| c_k e^{in\Delta t \mu_k} \right|^2 \leq C_2(T,\gamma) \sum_{k=-J}^{J} |c_k|^2 \tag{59}$$

for every complex sequence $(c_k)_{k \in Z} \in l^2$.

[7] M. Negreanu, Discrete Ingham inequalities and applications (2003)

Uniform discrete observability inequality - 3

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[7] M. Negreanu, Discrete Ingham inequalities and applications (2003)

Uniform discrete observability inequality - 4

Lemma

Let $(\theta, \alpha, \nu) \in \mathcal{C}$. *Then,*

$$\sqrt{\lambda_{j,h}^{\theta,\alpha}} - \sqrt{\lambda_{j-1,h}^{\theta,\alpha}} \geq \pi \min(1, \nu^{-2}) \quad \forall\, j = 2, ..., J;$$
$$|\sqrt{\lambda_{j,h}^{\theta,\alpha}} - \sqrt{\lambda_{k,h}^{\theta,\alpha}}| \leq 2\pi \Delta t^{-1} + O(1) \quad \forall\, j, k = 1, ..., J. \tag{60}$$

Proposition

Let $(\theta, \alpha, \nu) \in \mathcal{C}$. For every $T > 2\max(1, \nu^2)$, there exist two constants $c, C > 0$ independent of Δt and h such that

$$c(T)\min(1,\nu^{-2}) \sum_{|k|\le J, k\neq 0} |c_{k,h}|^2 \left|\frac{\sin(k\pi h)}{h}\right|^2 \le \Delta t \sum_{n=0}^{N-1}\left(\left|\frac{u_J^n}{h}\right|^2 + \theta_\nu \left|\frac{u_J^{n+1}-u_J^n}{\Delta t}\right|^2\right)$$

$$\le C(T)\max(1,\nu^{-2}) \sum_{|k|\le J, k\neq 0} |c_{k,h}|^2 \left|\frac{\sin(k\pi h)}{h}\right|^2. \qquad (61)$$

Uniform discrete observability inequality - 6

Theorem

Let $(\theta, \alpha, \nu) \in \mathcal{C}$. Then, we have the following inequalities :

$$\min(1, \nu^{-2}) \sum_{|k| \leq J, k \neq 0} |c_{k,h}|^2 \left| \frac{\sin(k\pi h)}{h} \right|^2 \leq 2E_0^{\theta,\alpha} \leq \max(1, \nu^{-2}) \sum_{|k| \leq J, k \neq 0} |c_{k,h}|^2 \left| \frac{\sin(k\pi h)}{h} \right|^2. \tag{62}$$

Lack of discrete observability w.r.t. $(\Delta t, h)$ - 7

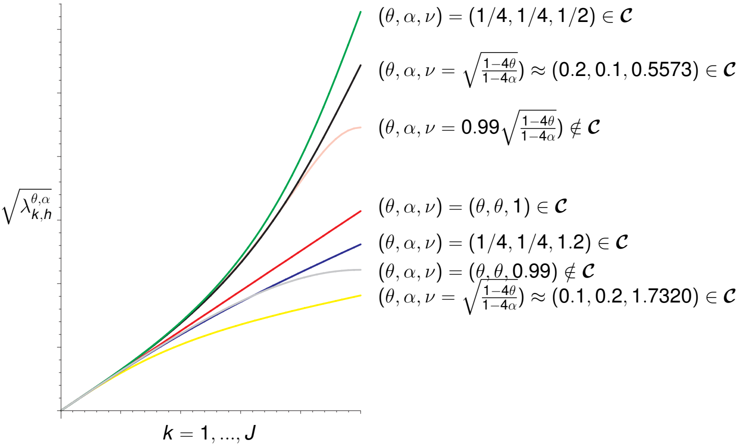

Figure: Evolution of $\sqrt{\lambda_{k,h}^{\theta,\alpha}}$, $k = 1, ..., J$ for different values of θ, α and ν.

Lack of discrete observability w.r.t. $(\Delta t, h)$ - 8

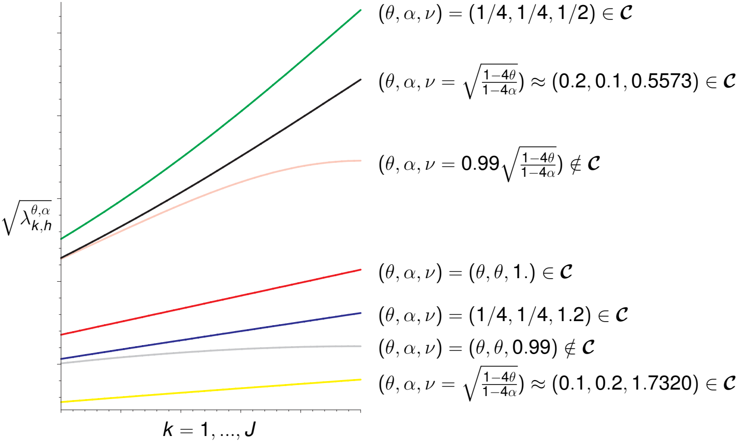

Figure: Evolution of $\sqrt{\lambda_{k,h}^{\theta,\alpha}}$, $k = \frac{3}{4}J, ..., J$ for different values of θ, α and ν: zoom on the high frequencies.

Positive Commutation diagram

If $(\theta, \alpha, \nu) \in \mathcal{C}$ then $h^2\theta_\nu = \frac{1}{4}(h^2 - \Delta t^2)$

$$(\overline{\mathcal{S}}_{h,\Delta t}) \begin{cases} \Delta_{\Delta t} y_{h,\Delta t} + \frac{1}{4}(h^2 - \Delta t^2)\Delta_h \Delta_{\Delta t} y_{h,\Delta t} - \Delta_h y_{h,\Delta t} = 0, \\ + \text{Initial conditions and Boundary terms} \end{cases} \tag{63}$$

produces a discrete uniformly bounded and converging control under the condition $\Delta t < h\sqrt{T/2}$.

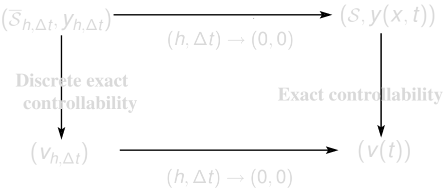

Figure: Commuting diagram associated to the scheme $(\overline{\mathcal{S}}_{h,\Delta t})$ for $\Delta t < h\sqrt{T/2}$.

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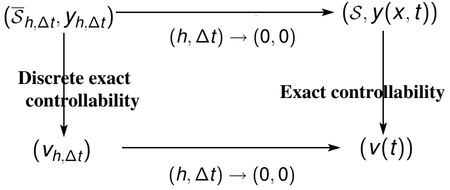

Figure: Commuting diagram associated to the scheme $(\overline{\mathcal{S}}_{h,\Delta t})$ for $\Delta t < h\sqrt{T/2}$.

Negative Commutation diagram

If $(\theta, \alpha, \nu) \notin \mathcal{C}$, in particular $\theta = \alpha$ and $\nu < 1$ ($\Delta t < h$)

$$(\overline{\mathcal{S}}_{h,\Delta t}) \begin{cases} \Delta_{\Delta t} y_{h,\Delta t} - \Delta_h y_{h,\Delta t} = 0, \\ + \text{Initial conditions and Boundary terms} \end{cases} \tag{64}$$

produces a non discrete uniformly bounded and converging control under the condition $\Delta t < h$.

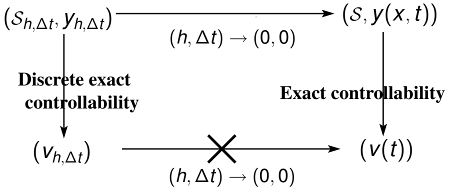

Figure: Non commuting diagram associated to the scheme $(\overline{\mathcal{S}}_{h,\Delta t})$ for $\Delta t < h$.

Numerical example

$$y_0(x) = \begin{cases} 16x & x \in [0, 1/2], \\ 0 & x \in]1/2, 1]. \end{cases} \quad ; \quad y_1(x) = 0. \tag{65}$$

Let us take $T = 2.4$. The control v with minimal L^2-norm is the following discontinuous function :

$$v(t) = \begin{cases} 0 & t \in [0, 0.9] \cup [1.9, T], \\ 8(t - 1.4) & t \in]0.9, 1.9[, \end{cases} \tag{66}$$

leading to $||v||_{L^2(0,T)} = 4/\sqrt{3} \approx 2.3094$. The corresponding initial conditions of the forward problem are

$$\hat{w}_0(x) = 0 \quad ; \quad \hat{w}_1(x) = \begin{cases} -8x & x \in [0, 1/2[, \\ 0 & x \in [1/2, 1]. \end{cases} \tag{67}$$

Usual scheme - Minimizer

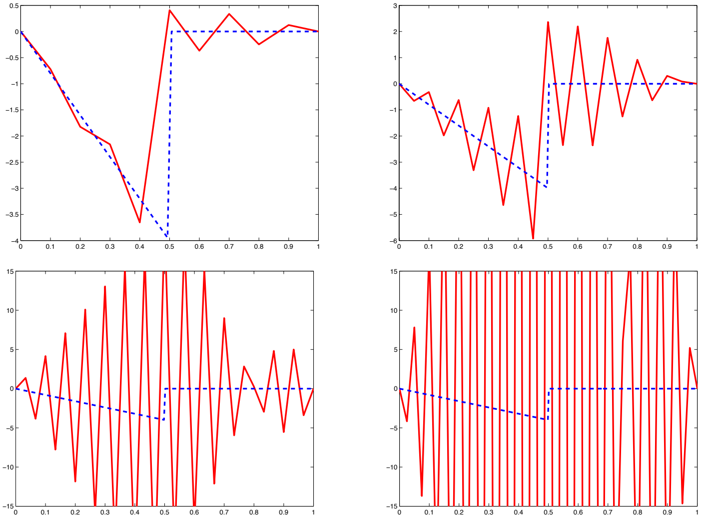

Figure: (FDS) $P(\hat{\mathbf{w}}_{\mathbf{1h}})(x)$ vs. $x \in [0, 1]$, $\nu = 0.98$, $T = 2.4$ and $h = 1/10$ (top left), $h = 1/20$ (top right), $h = 1/30$ (bottom left), $h = 1/40$ (bottom right).

Usual scheme - control

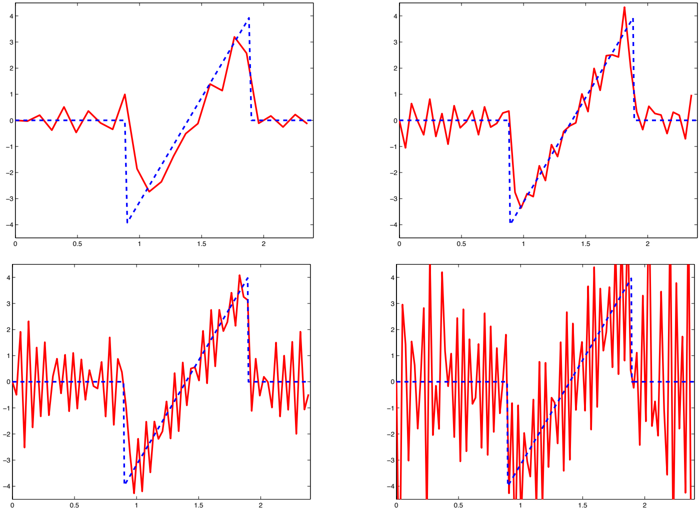

Figure: (FDS): Control $P(\mathbf{v_h})(t)$ vs. $t \in [0, T]$, $\nu = 0.98$, $T = 2.4$ and $h = 1/10$ (top left), $h = 1/20$ (top right), $h = 1/30$ (bottom left), $h = 1/40$ (bottom right).

Usual scheme - control norm

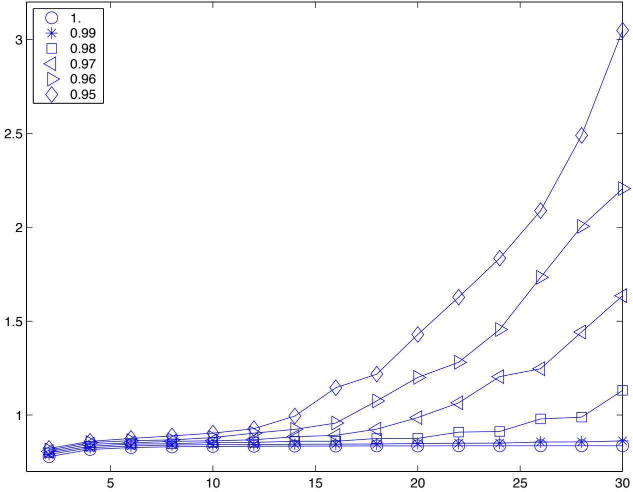

Figure: (FDS) $\log(||Q(\boldsymbol{v_h})||_{L^2(0,T)})$ vs. $1/h$ for different values of ν. $T = 2.4$

Modified scheme - Minimizer

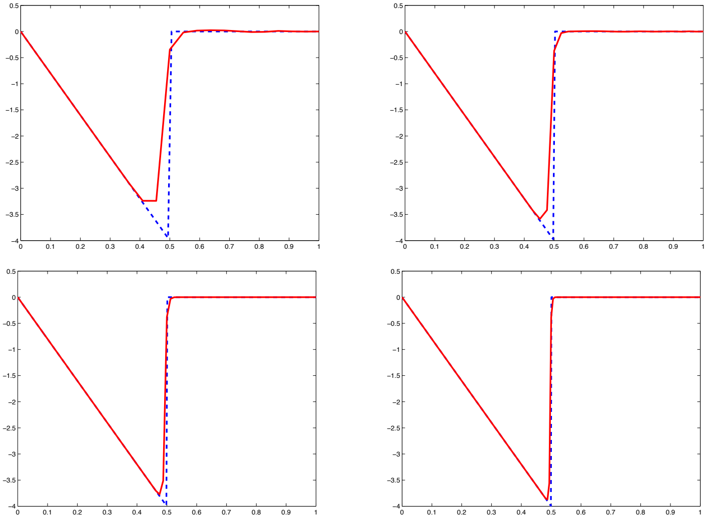

Figure: (MFS) $P(\hat{\boldsymbol{w}}_{\mathbf{1}\boldsymbol{h}})(x)$ vs. $x \in [0,1]$ - $(\theta,\alpha,\nu) \approx (1/20, 1/12, 1.095445)$, $T = 2.4$ and $h = 1/21, 1/41, 1/81, 1/161$.

Modified scheme - control

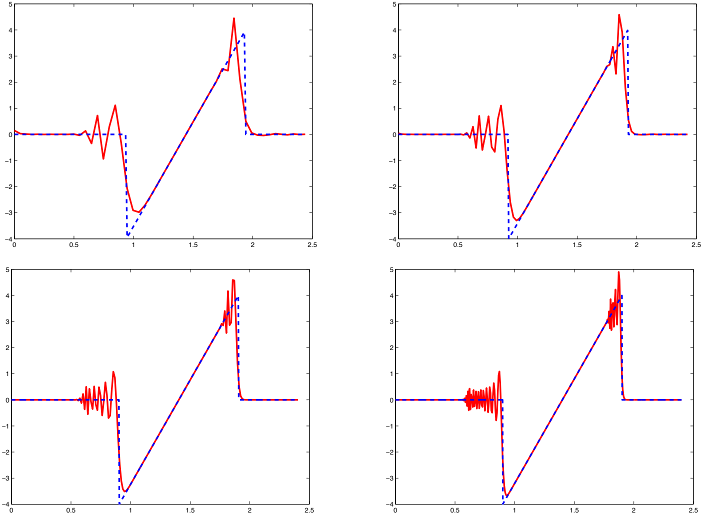

Figure: (MFS) Control $P(\boldsymbol{v_h})(t)$ vs. $t \in [0, T]$ -
$(\theta, \alpha, \nu) \approx (1/20, 1/12, 1.095445)$, $T = 2.4$ and $h = 1/21, 1/41, 1/81, 1/161$.

Remarks

The situation is open when the discrete spectrum is not available.
In practice, we may

- Regularize the discrete cost, i.e. add coercivity [Tychonoff regularization]
  $J^\star_\varepsilon(\varphi_0,\varphi_1) = J^\star(\varphi_0,\varphi_1) + \frac{\varepsilon^2}{2}\|\varphi_0,\varphi_1\|^2_{\boldsymbol{V}}, \qquad \varepsilon > 0$

- Filter out the high frequency components
  Minimize over the "first" discrete modes of φ_0, φ_1

- Use Bi-Grid strategy
  Introduce by [Glowinski,92] and justify by [Zuazua-Ignat, 2009 in 1D] and [Komornik-Loreti 2010] in 2D

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A word on stabilization : lack of uniform exponential decay

$$\begin{cases} y_{tt} - \Delta y + a(x)\,1_\omega y' = 0, & (x,t) \in Q_T := \Omega \times (0,T) \\ y = 0, & (x,t) \in \Sigma_T \\ (y(\cdot,0), y_t(\cdot,0)) = (y_0, y_1) \in H_0^1 \times L^2, & x \in \Omega \end{cases} \quad (68)$$

Theorem

If (Ω, T, ω) satisfy the geometric condition:

$$\exists \alpha, C > 0, \qquad E(t) \le CE(0)e^{-\alpha t}, \quad t > 0$$

but, if $\omega \neq \Omega$, then at the discrete level [Banks-Ito-Wang, 91]

$$\exists \alpha_h, C_h > 0, E_h(t) \le C_h E_h(0) e^{-\alpha_h t}, \quad t > 0, \qquad \lim_{h\to 0} \alpha_h \to 0$$

Theorem (AM, Pazoto (2005))

The semi-discret scheme associated to $(h = (h_1, h_2))$

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preserve the uniforme decay: $\exists \alpha_0 > 0$ such that $\alpha_h \ge \alpha_0, \forall h > 0$

A word on stabilization : lack of uniform exponential decay

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