Asymptotic and null controllability of an advection-diffusion equation

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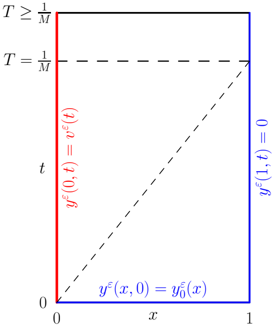

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- Well-posedness:

$$\forall y_0^\varepsilon \in H^{-1}(0,1), v^\varepsilon \in L^2(0,T), \quad \exists!\ y^\varepsilon \in L^2(Q_T) \cap C([0,T]; H^{-1}(0,1))$$

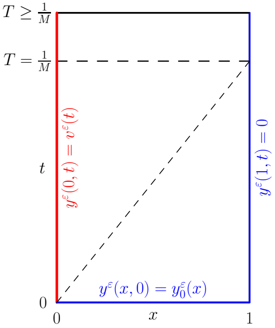
- Null controllability property: From [Fursikov'91],

$$\forall T > 0, y_0^\varepsilon \in H^{-1}(0,1), \exists v^\varepsilon \in L^2(0,T) \quad \text{s.t.} \quad y^\varepsilon(\cdot,T) = 0 \quad \text{in } H^{-1}(0,1)$$

- Main concern: Behavior of the controls v^ε as $\varepsilon \to 0$

- Remark: y^ε exhibits internal and boundary layers as $\varepsilon \to 0$ and make non trivial the analysis of the direct and control problems !

Introduction - The advection-diffusion equation



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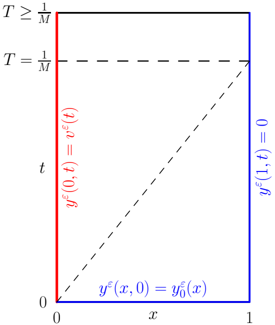

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This 1d scalar linear eq. is an embedded system of the Navier-Stokes system with non-characteristic boundary condition and viscosity coefficient equals to ε as discussed in [Cheng-Temam-Wang' 2000][1]

and appears in many contexts, notably

- Asymptotic analysis of the direct problem:
  L. Bobisud: *Second-order linear parabolic equations with a small parameter*, Arch. Rational Mech. Anal., (1967).
- Numerical approximation of the solution:
  P. Deuring, R. Eymard, and M. Mildner : L^2*-stability independent of diffusion for a finite element-finite volume discretization of a linear convection-diffusion equation*, SIAM J. Numer. Anal (2015).
- Controllability:
  J.-M. Coron, S. Guerrero: *Singular optimal control: a linear 1-D parabolic-hyperbolic example*, Asymptot. Anal., (2005).

[1] New approximation algorithms for a class of partial differential equations displaying boundary layer behavior, Methods Appl. Anal. (2000)

Cost of control

- We note the non empty set of null controls by

$$\mathcal{C}(y_0^\varepsilon, T, \varepsilon, M) := \Big\{ v \in L^2(0,T); y^\varepsilon = y^\varepsilon(v^\varepsilon) \text{ solves } (9) \text{ and satisfies } y^\varepsilon(\cdot, T) = 0 \Big\}$$

and define the cost of control by the following quantity :

$$K(\varepsilon, T, M) := \sup_{\|y_0^\varepsilon\|_{L^2(0,1)}=1} \left\{ \min_{v \in \mathcal{C}(y_0^\varepsilon, T, \varepsilon, M)} \|v\|_{L^2(0,T)} \right\}.$$

$K(\varepsilon, T, M)$ is the norm of the (linear) operator $y_0^\varepsilon \to v_{HUM}$ where v_{HUM} is the control of minimal L^2-norm

- We denote

$$T_M := \inf \Big\{ T > 0; \sup_{\varepsilon > 0} K(\varepsilon, T, M) < \infty \Big\}$$

- Remark $K(\varepsilon, T, 0) \sim_{\varepsilon \to 0^+} \varepsilon^{-1/2} e^{\frac{\kappa}{\varepsilon T}}$, $\kappa \in (1/2, 3/4)$ so that $T_0 = \infty$ [Miller'06].
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With respect to the null controllability issue

- There is a kind of competition between the transport and the diffusion terms: as $\varepsilon \to 0$, the transport term becomes dominant, pushes the solution out of $(0,1)$ and makes $\|y^\varepsilon(\cdot,T)\|_2$ small for all $T \geq 1/|M|$. However, as $\varepsilon \to 0$, the diffusion term, which is the main tools to control to zero the solution, is small.

  Intuitively, one have to wait enough time, from $t = 1/|M|$, to control uniformly w.r.t. ε the remainder $y^\varepsilon(\cdot, 1/|M|)$.
- The negative case $M < 0$ is the "most singular" since then the transport term pushes the solution y^ε from the right to the left line $x = 0$ where the control acts. The control requires more "energy" to act on the whole spatial domain.

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Main objective : Determine the behavior of the cost $K(\varepsilon, T, M)$ as $\varepsilon \to 0$

Outline :

- Part 1: Facts on the diffusion-advection eq. and literature.
- Part 2: Numerical attempt to estimate $K(\varepsilon, T, M)$.
- Part 3: Asymptotic analysis of the corresponding optimality system

Remark

- *By duality, the controllability property of (9) is related to the existence of a constant* $C > 0$ *such that*

$$\|\varphi(\cdot,0)\|_{L^2(0,1)} \leq C\|\varepsilon\varphi_x(0,\cdot)\|_{L^2(0,T)}, \quad \forall \varphi_T \in H_0^1(0,1)\cap H^2(0,1) \tag{2}$$

where φ *solves the adjoint system*

$$\begin{cases} L_\varepsilon^\star\varphi := \varphi_t + \varepsilon\varphi_{xx} + M\varphi_x = 0 & \text{in} \quad Q_T, \\ \varphi(0,\cdot) = \varphi(1,\cdot) = 0 & \text{on} \quad (0,T), \\ \varphi(\cdot,T) = \varphi_T & \text{in} \quad (0,1). \end{cases}$$

- *The quantity*

$$C_{obs}(\varepsilon,T,M) = \sup_{\varphi_T\in H_0^1(0,1)} \frac{\|\varphi(\cdot,0)\|_{L^2(0,1)}}{\|\varepsilon\varphi_x(0,\cdot)\|_{L^2(0,T)}}$$

is the smallest constant for which (2) holds true and

$$K(\varepsilon,T,M) = C_{obs}(\varepsilon,T,M).$$

Theorem (Coron-Guerrero, 2005)

Let $T > 0$, $M \in \mathbb{R}^\star$, $y_0 \in L^2(0,1)$ independent of ε. Let $(v^\varepsilon)_{(\varepsilon)}$ be a sequence of functions in $L^2(0,T)$ such that for some $v \in L^2(0,T)$

$$v^\varepsilon \rightharpoonup v \quad in \quad L^2(0,T), \quad as \quad \varepsilon \to 0^+.$$

For $\varepsilon > 0$, let us denote by $y^\varepsilon \in C([0,T]; H^{-1}(0,1))$ the weak solution of

$$\begin{cases} y_t^\varepsilon - \varepsilon y_{xx}^\varepsilon + M y_x^\varepsilon = 0 & Q_T, \\ y^\varepsilon(0,\cdot) = v^\varepsilon, \; y^\varepsilon(1,\cdot) = 0 & (0,T), \\ y^\varepsilon(\cdot,0) = y_0 & (0,1). \end{cases}$$

Let $y \in C([0,T]; L^2(0,1))$ be the weak solution of

$$\begin{cases} y_t + M y_x = 0 & Q_T, \\ y(0,\cdot) = v \quad if \quad M > 0 & (0,T), \\ y(1,\cdot) = 0 \quad if \quad M < 0 & (0,T), \\ y(\cdot,0) = y_0 & (0,1). \end{cases}$$

Then, $y^\varepsilon \rightharpoonup y \quad in \quad L^2(Q_T) \quad as \quad \varepsilon \to 0^+$.

Corollary

For $T < \frac{1}{|M|}$, $\lim_{\varepsilon\to 0} K(\varepsilon, T, M) \to \infty$. Consequently, $T_M \geq \frac{1}{|M|}$.

PROOF. Assume that $K(\varepsilon, T, M) \not\to +\infty$. There exists $(\varepsilon_n)_{(n\in\mathbb{N})}$ positive tending to 0 such that $(K(\varepsilon_n, T, M))_{(n\in\mathbb{N})}$ is bounded.

Let v^{ε_n} the optimal control driving y_0 to 0 at time T and y^{ε_n} the corresponding solution. Let $T_0 \in (T, 1/|M|)$. We extend y^{ε_n} and v^{ε_n} by 0 on (T, T_0). From the inequality

$$\|v^{\varepsilon_n}\|_{L^2(0,T_0)} = \|v^{\varepsilon_n}\|_{L^2(0,T)} \leq K(\varepsilon^n, T, M)\|y_0\|_{L^2(0,1)},$$

we deduce that $(v^{\varepsilon_n})_{(n\in\mathbb{N})}$ is bounded in $L^2(0, T_0)$, so we extract a subsequence $(v^{\varepsilon_n})_{(n\in\mathbb{N})}$ such that $v^{\varepsilon_n} \rightharpoonup v$ in $L^2(0, T_0)$. We deduce that $y^{\varepsilon_n} \rightharpoonup y$ in $L^2(Q_{T_0})$ solution of the transport equation. Necessarily, $y \equiv 0$ on $(0, 1) \times (T, T_0)$. Contradiction.

Exponential blow up of the control cost for $M > 0$

Theorem (Coron-Guerrero'2005)

- *Assume $M > 0$, $T < 1/M$. $\exists c, C > 0$ such that $K(\varepsilon, T, M) \geq Ce^{c/\varepsilon}$ $\forall \varepsilon > 0$.*

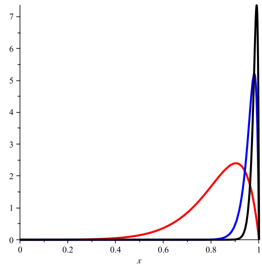
The lower bound are obtained using specific initial condition:

$$y_0^\varepsilon(x) = K_\varepsilon e^{\frac{Mx}{2\varepsilon}} \sin(\pi x),$$

$$K_\varepsilon = \mathcal{O}(\varepsilon^{-3/2} e^{\frac{-M}{2\varepsilon}}) \quad \text{s.t.} \quad \|y_0^\varepsilon\|_{L^2(0,1)} = 1$$

leading, for $M > 0$, to

$$K(\varepsilon, T, M) \geq C_1 \frac{\varepsilon^{-3/2} T^{-1/2} M^2}{1 + M^3 \varepsilon^{-3}} \exp\left(\frac{M}{2\varepsilon}(1 - TM) - \pi^2 \varepsilon T\right)$$



y_0^ε for $\varepsilon = 5 \times 10^{-2}$, $\varepsilon = 10^{-2}$, $\varepsilon = 5 \times 10^{-3}$.

$\Longrightarrow$ Surprising result since $\forall \varepsilon > 0, t > 0, \|y^\varepsilon(\cdot, t)\|_2 = \mathcal{O}(e^{-t/\varepsilon})$!!!

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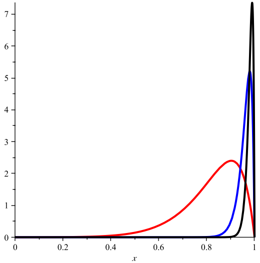

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Theorem (Coron-Guerrero'2005)

- *If $M < 0$, then $K(\varepsilon, T, M) \geq Ce^{c/\varepsilon}$, $c, C > 0$, when $\varepsilon \to 0$ for $T < \frac{2}{|M|}$.*

With again $y_0(x) = K_\varepsilon e^{\frac{Mx}{2\varepsilon}} \sin(\pi x)$,

$$K(\varepsilon, T, M) \geq C_1 \frac{\varepsilon^{-3/2} T^{-1/2} M^2}{1 + |M|^3 \varepsilon^{-3}} \exp\left(\frac{|M|}{2\varepsilon}(2 - T|M|) - \pi^2 \varepsilon T \right)$$

The negative case $M < 0$ is more singular, since the transport term acts "against" the control function.

Lemma

Let $\alpha \in [0,1)$. The free solution (i.e. $v^\varepsilon \equiv 0$) satisfies

$$\|y^\varepsilon(\cdot,t)\|_{L^2(0,1)} \le \|y_0^\varepsilon\|_{L^2(0,1)} e^{-\frac{M\alpha^2}{4\varepsilon(1-\alpha)}}, \quad \forall t \ge \frac{1}{|M|(1-\alpha)}.$$

PROOF. Assume $M > 0$. $z^\varepsilon(x,t) = e^{\frac{-M\alpha x}{2\varepsilon}} y^\varepsilon(x,t)$ solves

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$M \neq 0$ - Direct problem - Behavior of $\|y^\varepsilon(\cdot,T)\|_{L^2(0,1)}$ as $\varepsilon \to 0$ for $T > 1/|M|$

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Corollary (Cost of approximate control)

$$\forall \delta > 0, \forall T > \frac{1}{|M|}, \quad \exists \varepsilon_0(\delta) > 0 \quad s.t. \quad \forall \varepsilon < \varepsilon_0, \quad K_\delta(\varepsilon, T, M) = 0 \tag{3}$$

where

$$\mathcal{C}_\delta(y_0^\varepsilon, T, \varepsilon, M) := \left\{ v \in L^2(0,T); y = y(v) \text{ solves } (9) \text{ and satisfies } \|y(\cdot, T)\|_{L^2(0,1)} \leq \delta \right\}$$

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$$\|y^\varepsilon(\cdot, t)\|_{L^2(0,1)} \leq \|y_0^\varepsilon\|_{L^2(0,1)} e^{-\frac{M\alpha^2}{4\varepsilon(1-\alpha)}}, \quad \forall t \geq \frac{1}{|M|(1-\alpha)}.$$

Corollary (Cost of approximate control)

$$\forall \delta > 0, \forall T > \frac{1}{|M|}, \quad \exists \varepsilon_0(\delta) > 0 \quad s.t. \quad \forall \varepsilon < \varepsilon_0, \quad K_\delta(\varepsilon, T, M) = 0 \tag{3}$$

where

$$\mathcal{C}_\delta(y_0^\varepsilon, T, \varepsilon, M) := \left\{ v \in L^2(0,T); y = y(v) \textit{ solves } (9) \textit{ and satisfies } \|y(\cdot, T)\|_{L^2(0,1)} \leq \delta \right\}$$

and

$$K_\delta(\varepsilon, T, M) := \sup_{\|y_0^\varepsilon\|_{L^2(0,1)}=1} \left\{ \min_{v \in \mathcal{C}_\delta(y_0^\varepsilon, T, \varepsilon, M)} \|v\|_{L^2(0,T)} \right\}.$$

Proposition (Amirat, M 19)

Assume $M > 0$, $v^\varepsilon \equiv 0$, $y_0^\varepsilon = y_0 \in H^3(0,1)$. For $\varepsilon > 0$ small enough, the free solution y^ε satisfies

$$\left\| y^\varepsilon\left(\cdot, \frac{1}{M}\right)\right\|_{L^2(0,1)} \leq c\left(|y_0(0)|\varepsilon^{1/4} + |y_0^{(1)}(0)|\varepsilon^{3/4} + |y_0^{(2)}(0)|\varepsilon^{5/4}\right) + \mathcal{O}(\varepsilon^{3/2}) \quad (4)$$

for some constant $c > 0$, independent of ε.

Theorem (Coron-Guerrero'2005)

- *Assume* $M > 0$ *and* $T \geq \frac{4.3}{M}$. $\exists c, C > 0$ *s.t.* $K(\varepsilon, T, M) \leq Ce^{-c/\varepsilon}\ \forall \varepsilon > 0$.
- *Assume* $M < 0$ *and* $T \geq \frac{57.2}{M}$. $\exists c, C > 0$ *s.t.* $K(\varepsilon, T, M) \leq Ce^{-c/\varepsilon}\ \forall \varepsilon > 0$.

Estimates for T_M (Upper and lower bounds)

Theorem (Coron-Guerrero'2005)

$$T_M \in [1, 4.3]\frac{1}{M} \quad \textit{if} \quad M > 0, \qquad [2, 57.2]\frac{1}{|M|} \quad \textit{if} \quad M < 0.$$

Theorem (Glass'2009)

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Theorem (Lissy'2015)

$$T_M \in [1, 2\sqrt{3}]\frac{1}{M} \quad \textit{if} \quad M > 0, \qquad [2\sqrt{2}, 2(1+\sqrt{3})]\frac{1}{|M|} \quad \textit{if} \quad M < 0.$$

$(2\sqrt{3} \approx 3.46)$

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Numerical estimate of the cost $K(\varepsilon, T, M)$ w.r.t. ε !??

Rk: At the numerical level, with δ small, one can not distinguish the δ approximate controllability from the null controllability. Therefore, the numerical estimate is a priori feasible only for "large" ε and T "close" to $1/|M|$.

$$K^2(\varepsilon, T, M) = \sup_{y_0 \in L^2(0,1)} \frac{(\mathcal{A}_\varepsilon y_0, y_0)_{L^2(0,1)}}{(y_0, y_0)_{L^2(0,1)}}$$

where $\mathcal{A}_\varepsilon : L^2(0,1) \to L^2(0,1)$ is the control operator defined by $\mathcal{A}_\varepsilon y_0 := -\hat{\varphi}(0)$ where $\hat{\varphi}$ solves the adjoint system

$$\begin{cases} L_\varepsilon^\star \varphi := \varphi_t + \varepsilon \varphi_{xx} + M\varphi_x = 0 & \text{in} \quad Q_T, \\ \varphi(0,\cdot) = \varphi(1,\cdot) = 0 & \text{on} \quad (0,T), \\ \varphi(\cdot, T) = \varphi_T & \text{in} \quad (0,1), \end{cases}$$

associated to the initial condition $\varphi_T \in H_0^1(0,1)$, solution of the extremal problem

$$\inf_{\varphi_T \in H_0^1(0,1)} J^\star(\varphi_T) := \frac{1}{2} \|\varepsilon \varphi_x(0,\cdot)\|^2_{L^2(0,T)} + (y_0, \varphi(\cdot,0))_{L^2(0,1)}.$$

REFORMULATION - $K(\varepsilon, T, M)$ is solution of the generalized eigenvalue problem :

$$\sup\left\{\sqrt{\lambda} \in \mathbb{R} : \exists\, y_0 \in L^2(0,1), y_0 \neq 0, \text{ s.t. } \mathcal{A}_\varepsilon y_0 = \lambda y_0 \quad \text{in} \quad L^2(0,1)\right\}.$$

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The generalized eigenvalue problem by the power iterated method

In order to get the largest eigenvalue of the operator $\mathcal{A}_\varepsilon$, we may employ the power iterate method (Chatelain'89):

$$\begin{cases} y_0^0 \in L^2(0,1) \quad \text{given such that} \quad \|y_0^0\|_{L^2(0,1)} = 1, \\ \tilde{y}_0^{k+1} = \mathcal{A}_\varepsilon y_0^k, \quad k \geq 0, \\ y_0^{k+1} = \dfrac{\tilde{y}_0^{k+1}}{\|\tilde{y}_0^{k+1}\|_{L^2(0,1)}}, \quad k \geq 0. \end{cases}$$

The real sequence $\{\|\tilde{y}_0^k\|_{L^2(0,1)}\}_{k>0}$ converges to the eigenvalue with largest module of the operator $\mathcal{A}_\varepsilon$:

$$\sqrt{\|\tilde{y}_0^k\|_{L^2(0,1)}} \to K(\varepsilon, T, M) \quad \text{as} \quad k \to \infty.$$

The L^2-sequence $\{y_0^k\}_k$ then converges toward the corresponding eigenvector.

Remark The first step requires to determine the control of minimal L^2 for (9) with initial condition y_0^k.

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Remark The first step requires to determine the control of minimal L^2 for (9) with initial condition y_0^k.

Computation of the control of minimal L^2-norm

For a fixed initial data $y^0 \in L^2(0,1)$ and ε small, the numerical approximation of controls of minimal L^2-norm is a serious challenge :

- the minimization of $J^\star$ is ill-conditionned : the infimum φ_T lives in a huge dual space !!! this implies that the minimizer φ_T is highly oscillating at time T leading to high oscillations of the control $\varepsilon\varphi_{,x}(0,\cdot)$;
- Tychonoff like regularization

$$\inf_{\varphi_T \in H_0^1(0,1)} J_\beta^\star(\varphi_T) := J^\star(\varphi_T) + \beta \|\varphi_T\|_{H_0^1(0,1)} \longrightarrow \|y^\varepsilon(\cdot, T)\|_{H^{-1}(0,1)} \leq \beta$$

  is meaningless here for $T \geq 1/|M|$ because the uncontrolled solution $y^\varepsilon(\cdot, T)$ goes to zero with ε;
- Several singular layers occur for y^ε and φ^ε and requires fine discretization and adapted meshes.

We use the variational approach developed in [Fernandez-Cara-Münch, 2013], [De Souza-Münch, 2015] leading to convergent approximation with respect to the discretization parameter (ε being fixed).

We replace the minimization over $\varphi(\cdot, T)$ by the minimization over φ subjected to the constraint equality $L_\varepsilon^\star \varphi = 0$; a lagrange multiplier $\lambda \in L^2(Q_T)$ is introduced and then the saddle-point problem :

$$\sup_{\lambda \in L^2(Q_T)} \inf_{\varphi \in \Phi} \mathcal{L}(\varphi, \lambda) := \frac{1}{2} \|\varepsilon \rho_0^{-1} \varphi_x(0, \cdot)\|^2_{L^2(0,T)} + (y_0, \varphi(0, \cdot))_{L^2(0,1)} + < \lambda, \rho^{-1} L_\varepsilon^\star \varphi >_{L^2(Q_T)}$$

where ρ, ρ_0 denote appropriate Carleman weights in $L^\infty(Q_{T-\delta})$.

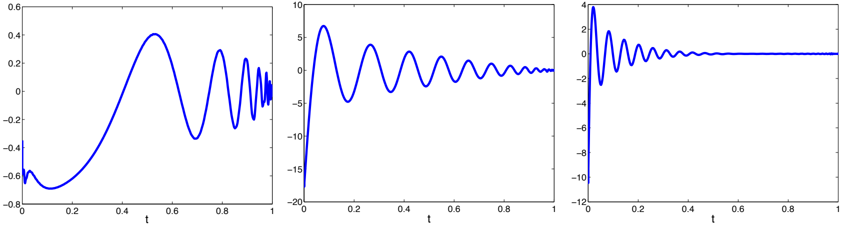
Remarks : • a conformal approximation of Φ leads to strong convergent approximation of the controls;

• This leads to a space-time approach well-suited for mesh adaptivity.

A FreeFem++ code associated to the space-time variational formulation

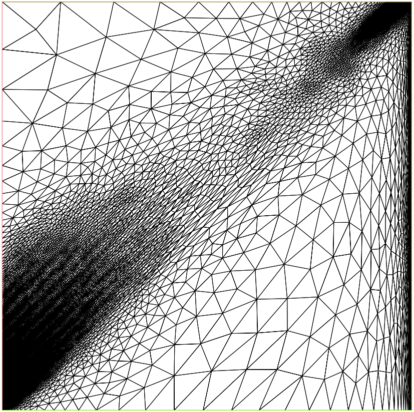
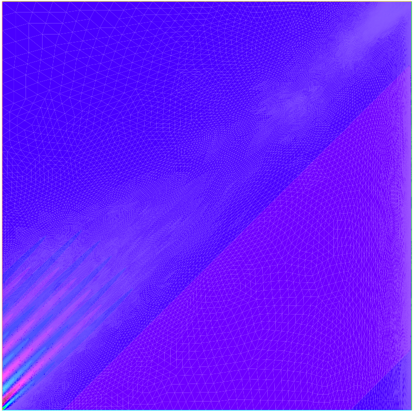
```
border bas(s=0,1){x=s; y=0;label=Ntop;};  border droit(s=0,T){x=1;y=s;label=Nright;}
border haut(s=1,0){x=s;y=T;label=Nhaut;} border gauche(s=T,0){x=0;y=s;label=Ngauche;}
mesh Th=buildmesh(bas(50)+droit(50)+haut(50)+gauche(50)) ;

fespace Vh(Th,P3); fespace Ph(Th,P3);
real eps=1.e-3, M=1, r1=1.e-6, r2=1.e-6, alpha1=5.e-2, alpha2=5.e-2;

Vh phi,p,phit,pt; Ph l1,l2,l1t,l2t;  Vh y0 = sin(pi*x)*(1-y);

problem transport([phi,p,l1,l2],[phit,pt,l1t,l2t])=
// Initial conjugate cost
 int1d(Th,Ngauche)  (eps*eps*dx(phi)*dx(phit))+int1d(Th,Nbas)(y0*phit)

 // bilinear adjoint- direct solution terms
 + int2d(Th)((dy(phi)+dx(p)+M*dx(phi))*l1t)
 + int2d(Th)((dy(phit)+dx(pt)+M*dx(phit))*l1)
 + int2d(Th)((p-eps*dx(phi))*l2t)
 + int2d(Th)((pt-eps*dx(phit))*l2)

 // Augmentation terms
 + int2d(Th)(r1*(dy(phi)+dx(p)+M*dx(phi))*  (dy(phit)+dx(pt)+M*dx(phit)))
 + int2d(Th)(r2*  (eps*dx(phi)-p)  *  (eps*dx(phit)-pt))

 // stabilized terms
 -int2d(Th)(alpha1*(dy(l1)+M*dx(l1)-eps*dx(l2))*(dy(l1t)+M*dx(l1t)-eps*dx(l2t)))
 - int2d(Th)(alpha2*(dx(l1)-l2)*(dx(l1t)-l2t))

 // boundary conditions for the adjoint and lagrange multiplier solutions
 + on(Nbas,l1=y0)+on(Ndroit,Ngauche,phi=0.)+on(Ndroit, Nhaut, l1=0.);
```

$$y_0(x) = \sin(\pi x); \quad T = 1; \quad M = 1$$



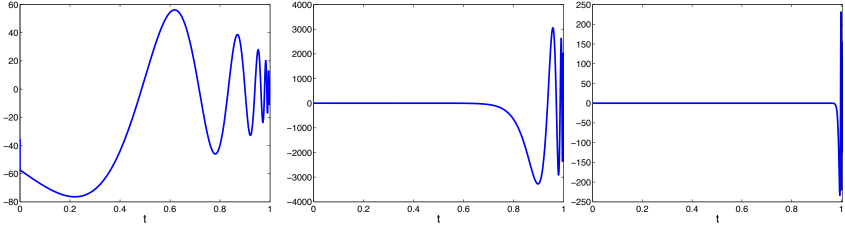
Control of minimal $L^2(0,T)$-norm $v^\varepsilon(t) \in [0,T]$ for $\varepsilon = 10^{-1}, 10^{-2}$ and 10^{-3}.

One adapted mesh over the space-time Q_T

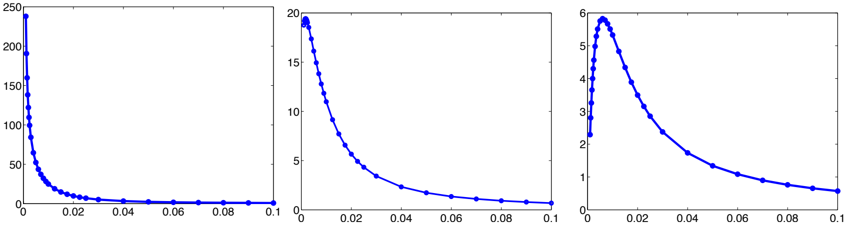
$y_0(x) = \sin(\pi x)$ - $M = 1$ - $\varepsilon = 10^{-3}$.

$$y_0(x) = \sin(\pi x); \quad T = 1; \quad M = -1$$



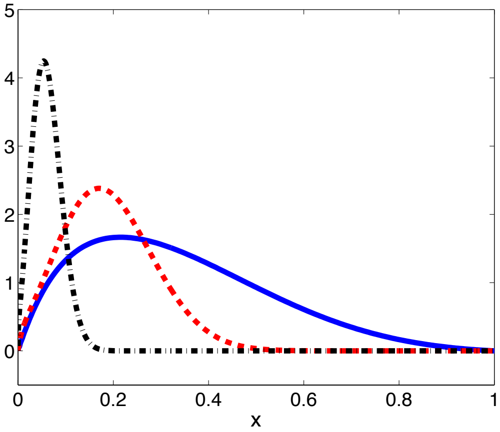
Control of minimal $L^2(0,T)$-norm $v^\varepsilon(t) \in [0,T]$ for $\varepsilon = 10^{-1}, 10^{-2}$ and 10^{-3}.

Cost of control $K(\varepsilon, T, M)$ w.r.t. ε - $M = 1$.



Cost of control w.r.t. $\varepsilon \in [10^{-3}, 10^{-1}]$ for $T = 0.95\frac{1}{M}$, $T = \frac{1}{M}$ and $T = 1.05\frac{1}{M}$

Corresponding worst initial condition



$T = 1$ - $M = 1$ - The optimal initial condition y_0 in $(0,1)$ for $\varepsilon = 10^{-1}$, $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$.

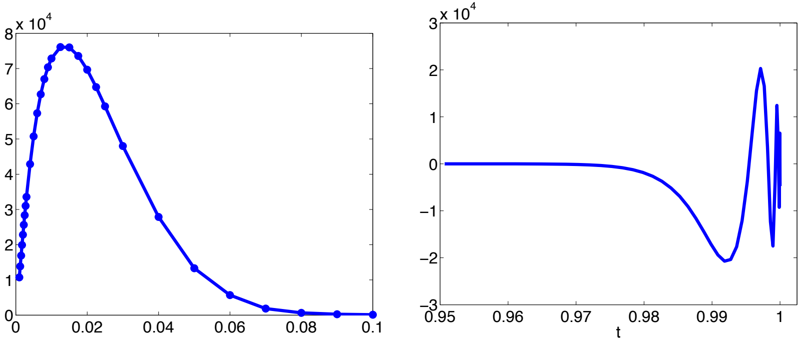
$\Longrightarrow y_0$ is close to $e^{-\frac{Mx}{2\varepsilon}} \sin(\pi x)/\|e^{-\frac{Mx}{2\varepsilon}} \sin(\pi x)\|_{L^2(0,1)}$

ε	$T = 1.$
10^{-3}	18.7555
1.25×10^{-3}	19.1953
1.5×10^{-3}	19.3883
1.75×10^{-3}	19.4234
2×10^{-3}	19.3540
2.25×10^{-3}	19.2093
2.5×10^{-3}	19.0163
3×10^{-3}	18.5275
4×10^{-3}	17.3600
5×10^{-3}	16.1269
6×10^{-3}	14.9392
7×10^{-3}	13.8166
8×10^{-3}	12.7839
9×10^{-3}	11.8380
10^{-2}	10.9763
10^{-1}	0.6808

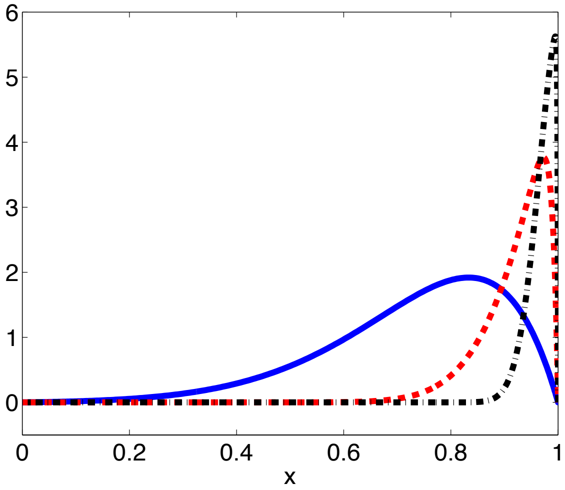
ε	$T = 1.$
10^{-3}	10718.0955
1.25×10^{-3}	13839.4039
1.5×10^{-3}	16903.9918
1.75×10^{-3}	19898.1360
2×10^{-3}	22812.2634
2.25×10^{-3}	25638.7601
2.5×10^{-3}	28375.3693
3×10^{-3}	33575.9482
4×10^{-3}	42871.1424
5×10^{-3}	50751.4443
6×10^{-3}	57316.7716
7×10^{-3}	62692.7273
8×10^{-3}	66997.3602
9×10^{-3}	70350.3966
10^{-2}	72862.0738
10^{-1}	123.3069

Cost $K(\varepsilon, T, M)$ w.r.t ε for $M = 1$(Left) and $M = -1$ (Right).

Cost of control $K(\varepsilon, T, M)$ w.r.t. ε - $M = -1$



Left: Cost of control w.r.t. ε for $T = \frac{1}{|M|}$; **Right**: Corresponding control v^{ε} in the neighborhood of T for $\varepsilon = 10^{-3}$



$T = 1$ - $M = -1$ - The optimal initial condition y_0 in $(0, 1)$ for $\varepsilon = 10^{-1}$, $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$.

- $y_0^\varepsilon(x) = K_\varepsilon \sin(\pi x) \exp(-\frac{Mx}{2\varepsilon})$ is a candidate !
- Estimation of the corresponding L^2 minimal control norm ?
- Weak limit of the system for ε-dependent initial condition ?
- The negative case $M < 0$ is out of reach numerically ! (we need to wait at least $T = 2\sqrt{2}/|M|$ for which the free solution satisfies $\|y^\varepsilon(\cdot, T)\|_2 \approx e^{-|M|/\varepsilon}$)

Second Attempt : Asymptotic analysis w.r.t. ε of the optimality system

We take $M > 0$.

Optimality system :

$$\begin{cases} L_\varepsilon y^\varepsilon = 0, \quad L_\varepsilon^\star \varphi^\varepsilon = 0, & (x,t) \in Q_T, \\ y^\varepsilon(\cdot,0) = y_0^\varepsilon, & x \in (0,1), \\ v^\varepsilon(t) = y^\varepsilon(0,t) = \varepsilon \varphi_x^\varepsilon(0,t), & t \in (0,T), \\ y^\varepsilon(1,t) = 0, . & t \in (0,T), \\ \varphi^\varepsilon(0,t) = \varphi^\varepsilon(1,t) = 0, & t \in (0,T), \\ -\beta(\varepsilon)\varphi_{xx}^\varepsilon(\cdot,T) + y^\varepsilon(\cdot,T) = 0, & x \in (0,1). \end{cases}$$

$\beta(\varepsilon) \geq 0$- Regularization parameter

J.-L. Lions *Perturbations singulières dans les problèmes aux limites et en contrôle optimal.* Lecture Notes in Mathematics. Springer 1973.

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$\beta(\varepsilon) \geq 0$- Regularization parameter

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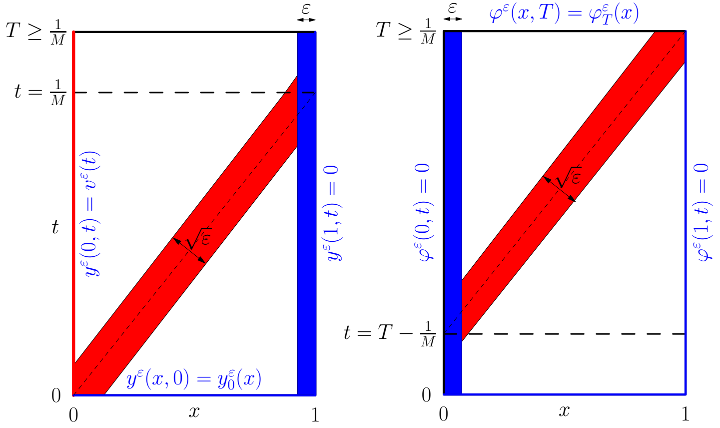
The situation is tricky because (assume $M > 0$)

- y^ε exhibits a boundary layer of size $\mathcal{O}(\varepsilon)$ at $x = 1$ and an inner layer of size $\mathcal{O}(\sqrt{\varepsilon})$ along the characteristic $\{(x,t) \in Q_T, x - Mt = 0\}$;
- φ^ε exhibits a boundary layer of size $(\mathcal{O}(\varepsilon))$ at $x = 0$ and an inner layer of size $(\mathcal{O}(\sqrt{\varepsilon}))$ along the characteristic $\{(x,t) \in Q_T, x - M(t-T) - 1 = 0\}$;



Singular layers zone for y^ε (left) and φ^ε(right) in the case $M > 0$.

$$\begin{cases} y_t^\varepsilon - \varepsilon y_{xx}^\varepsilon + M y_x^\varepsilon = 0, & (x,t) \in (0,1) \times (0,T), \\ y^\varepsilon(0,t) = v^\varepsilon(t) = v(t), \quad y^\varepsilon(1,t) = 0, & t \in (0,T), \\ y^\varepsilon(x,0) = y_0(x), & x \in (0,1), \end{cases} \tag{5}$$

v being known.

We construct an asymptotic approximation of the solution y^ε of (5) by using the matched asymptotic expansion method[Cole96, Echkauss81]. We consider two formal asymptotic expansions of y^ε:

– the outer expansion

$$\sum_{k=0}^{m} \varepsilon^k y^k(x,t), \quad (x,t) \in (0,T)$$

– the inner expansion (boundary layer at $x = 1$)

$$\sum_{k=0}^{m} \varepsilon^k Y^k(z,t), \quad z = \frac{1-x}{\varepsilon} \in (0, \varepsilon^{-1}), \; t \in (0,T)$$

Direct problem - Matched asymptotic expansion method - With compatibility conditions

$$\begin{cases} y_t^\varepsilon - \varepsilon y_{xx}^\varepsilon + M y_x^\varepsilon = 0, & (x,t) \in (0,1)\times(0,T), \\ y^\varepsilon(0,t) = v^\varepsilon(t) = v(t), \quad y^\varepsilon(1,t) = 0, & t \in (0,T), \\ y^\varepsilon(x,0) = y_0(x), & x \in (0,1), \end{cases} \tag{5}$$

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– the outer expansion

$$\sum_{k=0}^{m} \varepsilon^k y^k(x,t), \quad (x,t) \in (0,T) \Longrightarrow y_t^k + M y_x^k = y_{xx}^{k-1}$$

– the inner expansion (boundary layer at $x = 1$)

$$\sum_{k=0}^{m} \varepsilon^k Y^k(z,t), \quad z = \frac{1-x}{\varepsilon} \in (0, \varepsilon^{-1}),\ t \in (0,T) \Longrightarrow Y_{zz}^k + M Y_z^k = Y_t^{k-1}$$

$$y^0(x,t) = \begin{cases} y_0(x - Mt) & x > Mt, \\ v\left(t - \dfrac{x}{M}\right), & x < Mt. \end{cases}$$

Using the method of characteristics we find that, for any $1 \leq k \leq m$,

$$y^k(x,t) = \begin{cases} \displaystyle\int_0^t y_{xx}^{k-1}(x + (s-t)M, s)ds, & x > Mt, \\ \displaystyle\int_0^{x/M} y_{xx}^{k-1}(sM, t - \frac{x}{M} + s)ds, & x < Mt. \end{cases}$$

For instance,

$$y^1(x,t) = \begin{cases} t\, y_0''(x - Mt), & x > Mt, \\ \dfrac{x}{M^3} v''\left(t - \dfrac{x}{M}\right), & x < Mt, \end{cases}$$

$$y^2(x,t) = \begin{cases} \dfrac{t^2}{2} y_0^{(4)}(x - Mt), & x > Mt, \\ -\dfrac{2x}{M^5}(v)^{(3)}\left(t - \dfrac{x}{M}\right) + \dfrac{x^2}{2M^6} v^{(4)}\left(t - \dfrac{x}{M}\right), & x < Mt. \end{cases}$$

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Lemma

$$Y^0(z,t) = y^0(1,t)\left(1 - e^{-Mz}\right), \quad (z,t) \in (0,+\infty) \times (0,T).$$

For any $1 \leq k \leq m$*, the solution reads*

$$Y^k(z,t) = Q^k(z,t) + e^{-Mz} P^k(z,t), \quad (z,t) \in (0,+\infty) \times (0,T),$$

where

$$P^k(z,t) = -\sum_{i=0}^{k} \frac{1}{i!} \frac{\partial^i y^{k-i}}{\partial x^i}(1,t) z^i, \quad Q^k(z,t) = \sum_{i=0}^{k} \frac{(-1)^i}{i!} \frac{\partial^i y^{k-i}}{\partial x^i}(1,t) z^i.$$

$\mathcal{X}_\varepsilon$ - a cut-off function

Theorem (Amirat, M)

Let $\gamma \in (0,1)$. Let y^ε be the solution of problem (5) and let w_0^ε be the function defined as follows

$$w_0^\varepsilon(x,t) = \mathcal{X}_\varepsilon(x) y^0(x,t) + (1-\mathcal{X}_\varepsilon(x))\, Y^0\left(\frac{1-x}{\varepsilon}, t\right).$$

Assume that $y_0 \in H^2(0,1)$, $v^0 \in H^2(0,T)$, and the matching conditions

$$v^0(t=0) = y_0(x=0), \qquad M(v^0)'(t=0) + y_0'(x=0) = 0. \tag{6}$$

Then there is a constant c_0 independent of ε such that

$$\|y^\varepsilon - w_0^\varepsilon\|_{C([0,T];L^2(0,1))} \le c_0 \varepsilon^{\frac{1}{2}\gamma}.$$

Theorem (Amirat, M, 19)

Let y^ε be the solution of problem (5) and let w_m^ε be the function defined as follows

$$w_m^\varepsilon(x,t) = \mathcal{X}_\varepsilon(x)\sum_{k=0}^{m}\varepsilon^k y^k(x,t) + (1-\mathcal{X}_\varepsilon(x))\sum_{k=0}^{m}\varepsilon^k Y^k\left(\frac{1-x}{\varepsilon},t\right). \qquad (7)$$

Assume that $y_0 \in C^{2m+1}[0,1]$, $v \in C^{2m+1}[0,T]$, and that the C^{2m+1} - matching conditions are satisfied

$$v^{(p)}(0) = \sum_{i+j=p-1}(-1)^i M^i \frac{\partial^{p+1} y_0}{\partial x^{i+2}\partial t^j}(0,0), \quad 0 \le p \le 2m+1.$$

Then there is a constant c_m independent of ε such that (for any $\gamma \in (0,1/2]$)

$$\|y^\varepsilon(\cdot,t) - w_m^\varepsilon(\cdot,t)\|_{L^2(0,1)} \le c_m \varepsilon^{\frac{2m+1}{2}\gamma} + c_m\left(\varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{(2m+3)\gamma}{2}}\right)e^{-\frac{M^2}{2\varepsilon^\gamma}t}, \quad \forall t \in [0,T].$$

Proposition

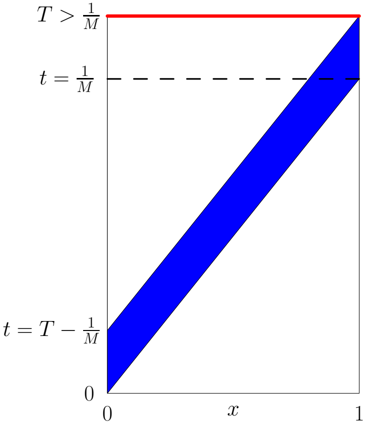
Let $m \in \mathbb{N}$, $T > \frac{1}{M}$ and $a \in]0, T - \frac{1}{M}[$. Assume regularity and matching conditions on the initial condition y_0 and function v. Assume moreover that

$$v(t) = 0, \quad \forall t \in [a, T].$$

Then, the solution y^ε of problem (5) satisfies the following property

$$\|y^\varepsilon(\cdot, T)\|_{L^2(0,1)} \leq c_m \, \varepsilon^{\frac{(2m+1)\gamma}{2}}, \quad \forall \gamma \in (0, 1)$$

for some constant $c_m > 0$ independent of ε.
The function $v \in C([0, T])$ is an approximate null control for (9).



Convergence w.r.t m under conditions on y_0 and the v.

(i) The initial condition y_0 belongs to $C^\infty[0,1]$ and there is $b \in \mathbb{R}$ such that

$$\|y_0^{(k)}\|_{L^2(0,1)} \leq \left\lfloor \frac{k}{2} \right\rfloor !\ b^{\frac{k}{2}}, \quad \forall k \in \mathbb{N}.$$

(ii) v is a polynomial of degree $\leq p-1$, $p \geq 1$, uniformly bounded in $C^{p-1}[0,T]$.

(iii) For any $m \in \mathbb{N}$, the functions v and y_0 satisfy the matching conditions.

Theorem

Assume (i)-(ii)-(iii). $\exists \varepsilon_0 > 0$ *such that for any fixed* $0 < \varepsilon < \varepsilon_0$, *we have*

$$\mathcal{X}_\varepsilon(x) \sum_{k=0}^{\infty} \varepsilon^k y^k(x,t) + (1 - \mathcal{X}_\varepsilon(x)) \sum_{k=0}^{\infty} \varepsilon^k Y^k\left(\frac{1-x}{\varepsilon}, t\right) + \tilde{\theta}^\varepsilon(x)$$
$$= y^\varepsilon(x,t) \quad a.e.\ in\ Q_T.$$

The corrector function $\tilde{\theta}^\varepsilon \in L^2(H_0^1) \cap C(L^2)$ *solves*

$$\begin{cases} L_\varepsilon(\tilde{\theta}^\varepsilon) = f^\varepsilon, & (x,t) \in Q_T, \\ \tilde{\theta}^\varepsilon(0,t) = \tilde{\theta}^\varepsilon(1,t) = 0, & t \in (0,T), \\ \tilde{\theta}^\varepsilon(x,0) = (1 - \mathcal{X}_\varepsilon(x))\, e^{-M\frac{1-x}{\varepsilon}} \sum_{i=0}^{\infty} \frac{y_0^{(i)}(1)}{i!}(1-x)^i, \quad x \in (0,1), & x \in (0,1). \end{cases}$$

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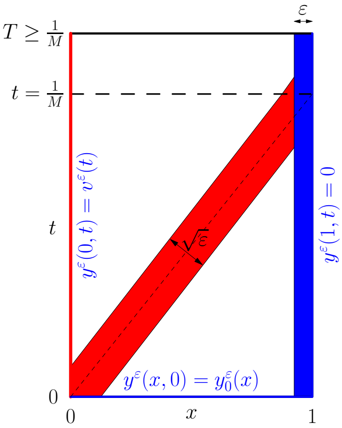

Singular layers zone for y^ε in the case $M > 0$.

Occurence of two interacting singular layers of different sizes !

Very few papers dealing with the asymptotic analysis of PDEs involving two interacting singular layers :
• W. Eckhaus, W. and E.M. de Jager, *Asymptotic solutions of singular perturbation problems for linear differential equations of elliptic type*, Arch. Rational Mech. Anal., 1966.

$$\begin{cases} u^\varepsilon_x(x,y) - \varepsilon\Delta u^\varepsilon(x,y) = 0, & (x,y)\in(0,1)\times(-1,1), \\ u^\varepsilon(0,y) = f(y), \quad u^\varepsilon(x,-1) = u^\varepsilon(x,1) = u^\varepsilon(1,y) = 0, & x\in[0,1], y\in[-1,1], \end{cases} \tag{8}$$

where $f : [-1,1] \to \mathbb{R}$ is a piecewise constant, discontinuous at $y = 0$.

• Larry Bobisud, *Second-order linear parabolic equations with a small parameter*, Arch. Rational Mech. Anal., 1967.

$$\begin{cases} u^\varepsilon_t - \varepsilon u^\varepsilon_{xx} + u^\varepsilon_x + u^\varepsilon = 0, & \text{in} \quad Q_T, \\ u^\varepsilon(0,\cdot) = f, \; u^\varepsilon(1,\cdot) = g, & \text{in} \quad (0,T), \\ u^\varepsilon(\cdot,0) = u_0, & \text{in} \quad (0,1). \end{cases} \tag{9}$$

Assuming $u_0 \in C^4([0,1])$ and $f, g \in C^3([0,T])$, obtention of w^ε such that $\|u^\varepsilon - w^\varepsilon\|_{L^\infty(Q_T)} = \mathcal{O}(\sqrt{\varepsilon})$ by the way of a maximum principle.

We now take into account the boundary layer on the characteristic and consider three formal asymptotic expansions of y^ε:
– the outer expansion

$$\sum_{k=0}^{m} \varepsilon^k y^k(x,t), \quad (x,t) \in Q_T, \quad x - Mt \neq 0$$

– the first inner expansion (on the characteristic $x - Mt = 0$)

$$\sum_{k=0}^{m} \varepsilon^{\frac{k}{2}} W^{k/2}(w,t), \quad w = \frac{x - Mt}{\sqrt{\varepsilon}}, \; t \in (0,T)$$

– the second inner expansion (at $x = 1$)

$$\sum_{k=0}^{m} \varepsilon^{k/2} Y^{k/2}(z,\tau,t), \quad z = \frac{1-x}{\varepsilon}, \; \tau = \frac{\frac{1}{M} - t}{\sqrt{\varepsilon}}$$

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$$\sum_{k=0}^{m} \varepsilon^k y^k(x,t), \quad (x,t) \in Q_T, \quad x - Mt \neq 0 \implies y_t^k + My_x^k = y_{xx}^{k-1}$$

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$$\sum_{k=0}^{m} \varepsilon^{\frac{k}{2}} W^{k/2}(w,t), \quad w = \frac{x - Mt}{\sqrt{\varepsilon}}, \; t \in (0,T) \implies W_t^{k/2} - W_{ww}^{k/2} = 0$$

– the second inner expansion (at $x = 1$)

$$\sum_{k=0}^{m} \varepsilon^{k/2} Y^{k/2}(z,\tau,t), \quad z = \frac{1-x}{\varepsilon}, \; \tau = \frac{\frac{1}{M} - t}{\sqrt{\varepsilon}} \implies Y_{zz}^{k/2} + MY_z^{k/2} = Y_t^{(k-2)/2} - Y_\tau^{(k-1)/2}$$

Example: the first term

- - y^0 solves the transport eq.:

$$\begin{cases} y^0_t + My^0_x = 0, \quad (x,t) \in Q_T \\ y^0(x,0) = y_0, y^0(0,t) = v^0 \end{cases} \Longrightarrow y^0(x,t) = \begin{cases} y_0(x - Mt) & x > Mt, \\ v^0\left(t - \dfrac{x}{M}\right), & x < Mt. \end{cases}$$

- - W^0 solves the heat eq.:

$$\begin{cases} W^0_t(w,t) - W^0_{ww}(w,t) = 0, & (w,t) \in \mathbb{R} \times (0,T), \\ \lim_{w \to +\infty} W^0(w,t) = y^0((Mt)^+, t) = y_0(0), & t \in (0,T), \\ \lim_{w \to -\infty} W^0(w,t) = y^0((Mt)^-, t) = v(t), & t \in (0,T). \end{cases}$$

$$W^0(w,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(w-s)^2}{4t}} g_0(s) ds, \quad w = \frac{x - Mt}{\sqrt{\varepsilon}}$$

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- Y^0 solves the ODE: $z = \frac{1-x}{\varepsilon}$, $\tau = \frac{1/M - t}{\sqrt{\varepsilon}}$

$$\begin{cases} Y_{zz}^0(z,\tau,t) + MY_z^0(z,\tau,t) = 0, & (z,\tau,t) \in \mathbb{R}_\star^+ \times \mathbb{R} \times (0,T), \\ Y^0(0,\tau,t) = 0, \quad \lim\limits_{z \to +\infty} Y^0(z,\tau,t) = p_\varepsilon^0(1,t), & (\tau,t) \in \mathbb{R} \times (0,T). \end{cases}$$

$$Y^0(z,\tau,t) = p_\varepsilon^0(1,t)\left(1 - e^{-Mz}\right), \quad (z,\tau,t) \in \mathbb{R}^+ \times \mathbb{R} \times (0,T).$$

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Let

$$P^{\varepsilon} = P^0_{\varepsilon} + \sqrt{\varepsilon} P^{1/2}_{\varepsilon} + \varepsilon P^1_{\varepsilon} + \varepsilon^{3/2} P^{3/2}_{\varepsilon}$$

Theorem (Amirat, M '19)

Assume $v \in H^3([0,T])$, $y_0 \in H^3([0,1])$. *Then* $\exists C > 0$ *independent of* ε *s.t.*

$$\left\| y^{\varepsilon}(\cdot,t) - P^{\varepsilon}(\cdot,t) \right\|_{L^2(0,1))} \leq C(\varepsilon^{3/2} + \varepsilon^{1/2} e^{-\frac{M^2}{2\varepsilon^{1/2}} t}) \quad \forall t \in [0,T]$$

and (assuming $y_0(1) = y_0'(1) = 0$*)*

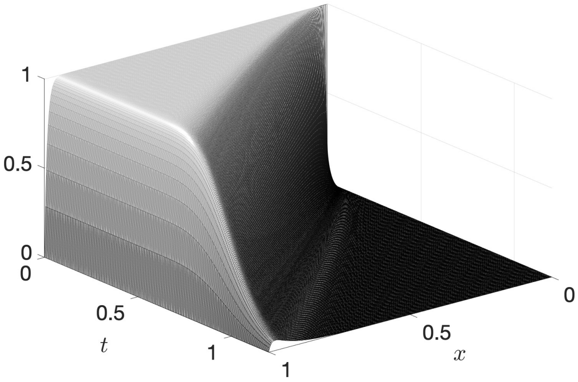
$$\|(y^{\varepsilon} - P^{\varepsilon})_x\|_{L^2(Q_T))} \leq C\varepsilon$$

As an illustration, we consider the simple case $v \equiv 0$ and $y_0 \equiv 1$ for which

$$\begin{cases} P^{\varepsilon}(x,t) = W^0_{\varepsilon}(w,t) - \Big(W^0_{\varepsilon}(M\tau,t) + \varepsilon^{1/2} z W^0_{\varepsilon,w}(M\tau,t) + \\ \qquad \varepsilon \dfrac{z^2}{2} W^0_{\varepsilon,ww}(M\tau,t) + \varepsilon^{3/2} \dfrac{z^3}{6} W^0_{\varepsilon,www}(M\tau,t) \Big) e^{-Mz}, \\ w = \dfrac{x - Mt}{\sqrt{\varepsilon}}, \quad M\tau = \dfrac{1 - Mt}{\sqrt{\varepsilon}}, \quad z = \dfrac{1-x}{\varepsilon}. \end{cases}$$

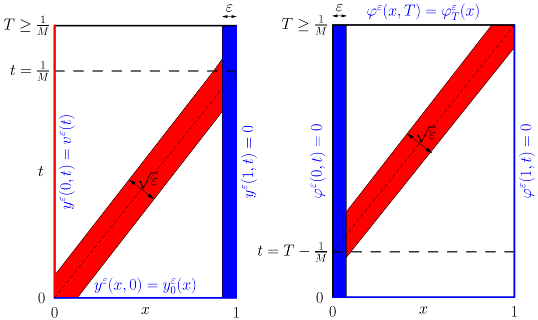
with

$$W^0_{\varepsilon}(w,t) = \frac{y_0(0) - v(0)}{2} erf\left(\frac{w}{2\sqrt{t}}\right) + \frac{y_0(0)+v(0)}{2} + \frac{v(0) - y_0(0)}{2} e^{\frac{Mw}{\sqrt{\varepsilon}} + \frac{M^2 t}{\varepsilon}} erfc\left(\frac{w}{2\sqrt{t}} + \frac{M\sqrt{t}}{\sqrt{\varepsilon}}\right) \tag{11}$$



P^{ε} in $(0,1)\times(0,1.2/M)$; $M=1$, $\varepsilon=10^{-2}$; $v\equiv 0$, $y_0\equiv 1$.

$$\begin{cases} L_\varepsilon y^\varepsilon = 0, \quad L_\varepsilon^\star \varphi^\varepsilon = 0, & (x,t) \in Q_T, \\ y^\varepsilon(\cdot,0) = y_0^\varepsilon, & x \in (0,1), \\ v^\varepsilon(t) = y^\varepsilon(0,t) = \varepsilon \varphi_x^\varepsilon(0,t), & t \in (0,T), \\ y^\varepsilon(1,t) = 0, . & t \in (0,T), \\ \varphi^\varepsilon(0,t) = \varphi^\varepsilon(1,t) = 0, & t \in (0,T), \\ -\beta(\varepsilon)\varphi_{xx}^\varepsilon(\cdot,T) + y^\varepsilon(\cdot,T) = 0, & x \in (0,1), \\ \beta(\varepsilon) = \mathcal{O}(\varepsilon^m) \end{cases} \tag{12}$$



We introduce the following change of unknown

$$y^{\varepsilon}(x,t) = c_{\varepsilon} e^{\frac{-Mx}{2\varepsilon}} e^{-\frac{\gamma M^2 t}{4\varepsilon}} z^{\varepsilon}(x,t), \quad \forall (x,t) \in Q_T,$$

leading to

$$L_{\varepsilon} y^{\varepsilon} := c_{\varepsilon} e^{\frac{-Mx}{2\varepsilon}} e^{-\frac{\gamma M^2 t}{4\varepsilon}} \left(z_t^{\varepsilon} - \varepsilon z_{xx}^{\varepsilon} + 2M z_x^{\varepsilon} - \frac{M^2}{4\varepsilon}(\gamma + 3) z^{\varepsilon} \right).$$

Easier case: Rayleigh plate model of thickness ε

$$\begin{cases} y^\varepsilon_{tt} + \varepsilon\Delta^2 y^\varepsilon - \Delta y^\varepsilon = 0, & \text{in } Q_T, \\ y^\varepsilon = 0, \quad \partial_\nu y^\varepsilon = v^\varepsilon 1_{\Gamma_T}, & \text{on } \Sigma_T, \\ (y^\varepsilon(\cdot,0), y^\varepsilon_t(\cdot,0)) = (y_0, y_1), & \text{on } \Omega. \end{cases}$$

Theorem (Lions 86)

Assume $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and that (Ω, Γ_T, T) satisfies a geometric control condition. For any $\varepsilon > 0$, let v^ε be the control of minimal $L^2(\Gamma_T)$-norm. Then,

$$(-\sqrt{\varepsilon} v^\varepsilon, y^\varepsilon) \rightharpoonup (v, y) \quad in \quad L^2(\Gamma_T) \times L^\infty(0, T; L^2(\Omega)), \quad as \quad \varepsilon \to 0$$

where v is the control of minimal $L^2(\Gamma_T)$-norm for $y \in C(L^2) \cap C^1(H^{-1})$, solution of :

$$\begin{cases} y_{tt} - \Delta y = 0, & in\ Q_T, \\ y = v 1_{\Gamma_T}, & on\ \Sigma_T, \\ (y(\cdot,0), y_t(\cdot,0)) = (y_0, y_1), & in\ \Omega. \end{cases}$$

Assuming regular initial condition, the asymptotic analysis of $\sqrt{\varepsilon} v^\varepsilon$ is easier (and doable !) since controls v^ε and v of minimal L^2-norm are actually more regular ! (joint work with Carlos Castro).

The end

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THANK YOU FOR YOUR ATTENTION