About the controllability of $y_t - \varepsilon y_{xx} + M y_x = 0$ w.r.t. ε

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Introduction - The transport diffusion equation

Let $T > 0$, $M \neq 0$, $\varepsilon > 0$ and $Q_T := (0,1) \times (0,T)$. This talk is concerned with the null controllability problem for

$$\begin{cases} L_\varepsilon y^\varepsilon := y^\varepsilon_t - \varepsilon y^\varepsilon_{xx} + M y^\varepsilon_x = 0 & \text{in} \quad (0,1)\times(0,T), \\ y^\varepsilon(0,\cdot) = v^\varepsilon(t),\; y^\varepsilon(1,\cdot) = 0 & \text{on} \quad (0,T), \\ y^\varepsilon(\cdot,0) = y_0 & \text{in} \quad (0,1), \end{cases} \tag{1}$$

- Well-poseddness:

$$\forall y^\varepsilon_0 \in H^{-1}(0,L), v^\varepsilon \in L^2(0,T), \quad \exists y^\varepsilon \in L^2(Q_T) \cap \mathcal{C}([0,T]; H^{-1}(0,L))$$

- Null control property: From (Russel'78),

$$\forall T > 0, y_0 \in H^{-1}(0,1), \exists v^\varepsilon \in L^2(0,T) \quad \text{such that}$$

$$y^\varepsilon(\cdot,T) = 0 \quad \text{in } H^{-1}(0,1). \tag{2}$$

- We note the non empty set of null controls by

$$\mathcal{C}(y_0, T, \varepsilon, M) := \{(y,v) : v \in L^2(0,T); y \text{ solves } (1) \text{ and satisfies } (2)\}$$

Cost of control

For any $\varepsilon > 0$, we define the cost of control by the following quantity :

$$K(\varepsilon, T, M) := \sup_{\|y_0\|_{L^2(0,L)}=1} \left\{ \min_{u \in \mathcal{C}(y_0, T, \varepsilon, M)} \|u\|_{L^2(0,T)} \right\},$$

and denote by T_M the minimal time for which the cost $K(\varepsilon, T, M)$ is uniformly bounded with respect to ε. In other words, (1) is uniformly controllable with respect to ε if and only if $T \geq T_M$.

Theorem [Coron-Guerrero'2006]

$$T_M \in [1, 4.3]\frac{1}{M} \quad \text{if} \quad M > 0, \qquad [2, 57.2]\frac{1}{|M|} \quad \text{if} \quad M < 0.$$

Theorem [Glass'2009]

$$T_M \in [1, 4.2]\frac{1}{M} \quad \text{if} \quad M > 0, \qquad [2, 6.1]\frac{1}{|M|} \quad \text{if} \quad M < 0.$$

Theorem [Lissy'2015]

$$T_M \in [1, 2\sqrt{3}]\frac{1}{M} \quad \text{if} \quad M > 0, \qquad [2\sqrt{2}, 2(1+\sqrt{3})]\frac{1}{|M|} \quad \text{if} \quad M < 0.$$

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Remarks (1)

• The lower bound $1/|M|$ is expected because the weak limit of the y^ε-system is the transport equation

$$\begin{cases} y^0_t + My^0_x = 0 & \text{in} \quad (0,1)\times(0,T), \\ y^0(0,\cdot) = v(t) & \text{on} \quad (0,T), \\ y^0(\cdot,0) = y_0 & \text{in} \quad (0,1), \end{cases}$$

uniformly controllable if $T \geq 1/|M|$: $\forall v \in L^2(0,T)$, the transport solution y^0 vanishes at any time T larger than $1/|M|$.

• The negative case $M < 0$ is much more singular since the transport term acts against the control. The results are not intuitive at all (singular control problem).

• The upper bounds are obtained using Carleman estimates. The lower bound are obtained using specific initial condition:

$$y_0(x) = K_\varepsilon e^{-\frac{Mx}{2\varepsilon}} \sin(\pi x), \quad (K_\varepsilon = \mathcal{O}(\varepsilon^{-3/2}) \quad \text{s.t.} \quad \|y_0\|_{L^2(0,1)} = 1)$$

For $M > 0$, From Coron-Guerrero'2006,

$$K(\varepsilon, T, M) \geq C_1 \frac{\varepsilon^{-3/2} T^{-1/2} M^2}{1 + M^3 \varepsilon^{-3}} \exp\left(\frac{M}{2\varepsilon}(1 - TM) - \pi^2 \varepsilon T\right)$$

Goals

Goal:

estimate the uniform minimal control time T_M !!!

We can try the following two approaches :

- Numerical estimation of $K(\varepsilon, T, M)$ with respect to ε and $T \geq \frac{1}{M}$ (for $M > 0$ and $M < 0$)
- Asymptotic analysis with respect to the parameter ε of the corresponding optimality system.

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Attempt 1 : Numerical estimation of $K(\varepsilon, T, M)$

Reformulation of the cost of control

$$K^2(\varepsilon, T, M) = \sup_{y_0 \in L^2(0,1)} \frac{(\mathcal{A}_\varepsilon y_0, y_0)_{L^2(0,1)}}{(y_0, y_0)_{L^2(0,1)}}$$

where $\mathcal{A}_\varepsilon : L^2(0,1) \to L^2(0,1)$ is the control operator defined by $\mathcal{A}_\varepsilon y_0 := -\hat{\varphi}(0)$ where $\hat{\varphi}$ solves the adjoint system

$$\begin{cases} -\varphi_t - \varepsilon\varphi_{xx} - M\varphi_x = 0 & \text{in} \quad (0,1)\times(0,T), \\ \varphi(0,\cdot) = \varphi(L,\cdot) = 0 & \text{on} \quad (0,T), \\ \varphi(\cdot, T) = \varphi_T & \text{in} \quad (0,1), \end{cases} \tag{3}$$

associated to the initial condition $\varphi_T \in H_0^1(0,1)$, solution of the extremal problem

$$\inf_{\varphi_T \in H_0^1(0,L)} J^\star(\varphi_T) := \frac{1}{2}\int_0^T (\varepsilon\varphi_x(0,\cdot))^2 dt + (y_0, \varphi(\cdot,0))_{L^2(0,T)}.$$

REFORMULATION - $K(\varepsilon, T, M)$ is solution of the generalized eigenvalue problem :

$$\sup\left\{\sqrt{\lambda} \in \mathbb{R} : \exists\, y_0 \in L^2(0,1), y_0 \neq 0, \text{ s.t. } \mathcal{A}_\varepsilon y_0 = \lambda y_0 \quad \text{in} \quad L^2(0,1)\right\}.$$

The generalized eigenvalue problem by the power iterated method

In order to get the largest eigenvalue of the operator $\mathcal{A}_\varepsilon$, we may employ the power iterate method (Chatelain'89):

$$\begin{cases} y_0^0 \in L^2(0,1) \quad \text{given such that} \quad \|y_0^0\|_{L^2(0,1)} = 1, \\ \tilde{y}_0^{k+1} = \mathcal{A}_\varepsilon y_0^k, \quad k \geq 0, \\ y_0^{k+1} = \dfrac{\tilde{y}_0^{k+1}}{\|\tilde{y}_0^{k+1}\|_{L^2(0,1)}}, \quad k \geq 0. \end{cases} \tag{4}$$

The real sequence $\{\|\tilde{y}_0^k\|_{L^2(0,1)}\}_{k>0}$ converges to the eigenvalue with largest module of the operator $\mathcal{A}_\varepsilon$:

$$\sqrt{\|\tilde{y}_0^k\|_{L^2(0,1)}} \to K(\varepsilon, T, M) \quad \text{as} \quad k \to \infty. \tag{5}$$

The L^2 sequence $\{y_0^k\}_k$ then converges toward the corresponding eigenvector.

The first step requires to compute the image of $\mathcal{A}_\varepsilon$: this is done by determining the control of minimal L^2 norm by minimizing $J^\star$ with y_0^k as initial condition for (1).

Computation of the control of minimal L^2-norm

For a fixed initial data $y^0 \in L^2(0,1)$ and ε small, the numerical approximation of controls of minimal L^2-norm is a VERY SERIOUS CHALLENGE :

- the minimization of $J^\star$ is ill-posed : the infimum φ_T lives in a huge dual space !!! this implies that the minimizer φ_T is highly oscillating at time T leading to highly oscillation of the control $\varepsilon\varphi_{,x}$.
- Tychonoff like regularization

$$\inf_{\varphi_T \in H_0^1(0,1)} J_\beta^\star(\varphi_T) := J^\star(\varphi_T) + \beta\|\varphi_T\|_{H_0^1(0,1)} \longrightarrow \|y^\varepsilon(\cdot,T)\|_{H^{-1}(0,1)} \leq \beta \quad (6)$$

  is meaningless here because the uncontrolled solution $y^\varepsilon(\cdot,T)$ goes to zero with ε for $T \geq 1/M$.
- Boundary layers occurs for y^ε and φ^ε on the boundary and requires fine discretization.

We use the variational approach developed in [Fernandez-Cara-Munch, 2013], [De Souza-Munch, 2015] leading to convergent approximation with respect to the discretization parameter (ε being fixed).

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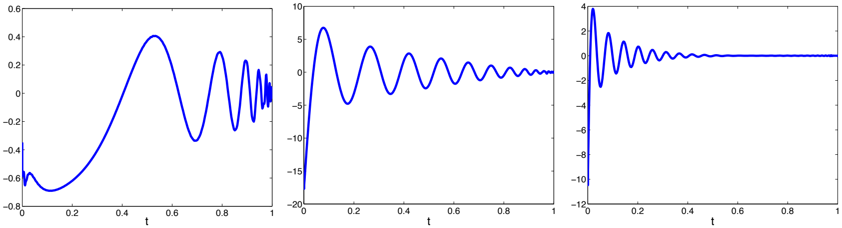
Picture of controls with respect to ε, y_0 fixed

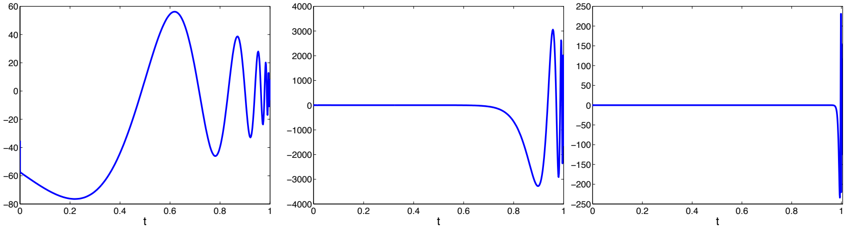
$$y_0(x) = \sin(\pi x); \quad T = 1; \quad M = 1$$



Control of minimal $L^2(0, T)$-norm $v\varepsilon(t) \in [0, T]$ for $\varepsilon = 10^{-1}, 10^{-2}$ and 10^{-3}

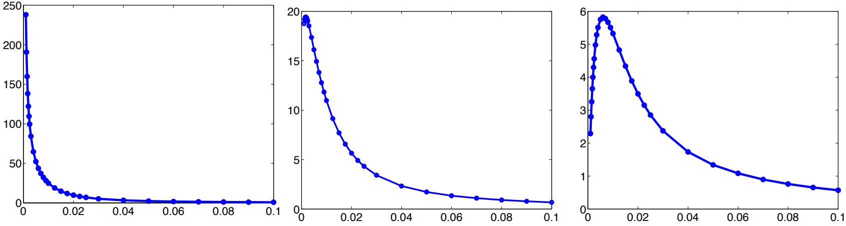
Picture of controls with respect to ε, y_0 fixed

$$y_0(x) = \sin(\pi x); \quad T = 1; \quad M = -1$$



Control of minimal $L^2(0, T)$-norm $v\varepsilon(t) \in [0, T]$ for $\varepsilon = 10^{-1}, 10^{-2}$ and 10^{-3}

Cost of control $K(\varepsilon, T, M)$ w.r.t. ε - $M = 1$



Cost of control w.r.t. ε for $T = 0.95\frac{1}{M}$, $T = \frac{1}{M}$ and $T = 1.05\frac{1}{M}$

In agreement with Coron-Guerrero'2006,

$$K(\varepsilon, T, M) \geq C_1 \frac{\varepsilon^{-3/2} T^{-1/2} M^2}{1 + M^3 \varepsilon^{-3}} \exp\left(\frac{M}{2\varepsilon}(1 - TM) - \pi^2 \varepsilon T\right) \tag{7}$$

Corresponding worst initial condition

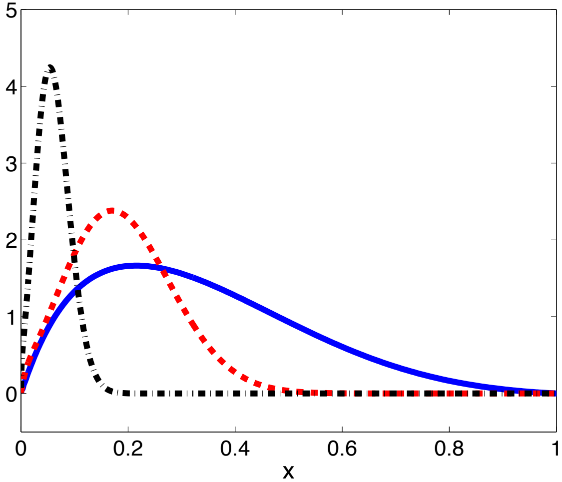
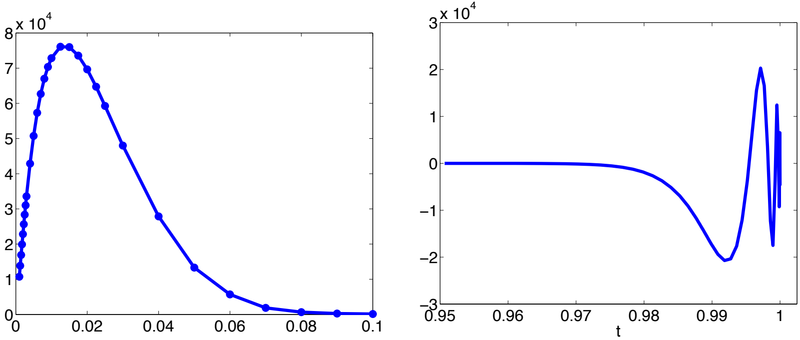


Figure: $T = 1$ - $M = 1$ - The optimal initial condition y_0 in $(0, 1)$ for $\varepsilon = 10^{-1}$ (full line), $\varepsilon = 10^{-2}$ (dashed line) and $\varepsilon = 10^{-3}$ (dashed-dotted line).

$\Longrightarrow$ y_0 is closed to $e^{-\frac{Mx}{2\varepsilon}} \sin(\pi x)$

Cost of control $K(\varepsilon, T, M)$ w.r.t. ε - $M = -1$



Left: Cost of control w.r.t. ε for $T = \frac{1}{|M|}$; **Right**: Corresponding control v^{ε} in the neighborhood of T for $\varepsilon = 10^{-3}$

Corresponding worst initial condition for $M = -1$

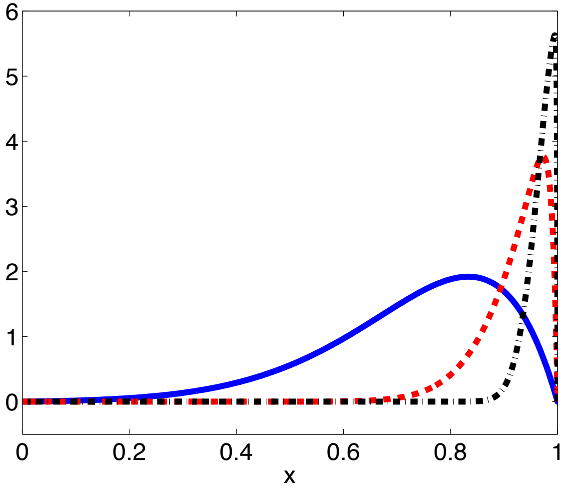


Figure: $T = 1$ - $M = -1$ - The optimal initial condition y_0 in $(0, 1)$ for $\varepsilon = 10^{-1}$ (full line), $\varepsilon = 10^{-2}$ (dashed line) and $\varepsilon = 10^{-3}$ (dashed-dotted line).

Attempt 2 : Asymptotic analysis w.r.t. ε

We take $M > 0$.

Optimality system :

$$\begin{cases} L_\varepsilon y^\varepsilon = 0, \quad L_\varepsilon^\star \varphi^\varepsilon = 0, & x \in (0,1), t \in (0,T), \\ y^\varepsilon(\cdot,0) = y_0^\varepsilon, & x \in (0,1), \\ v^\varepsilon(t) = y^\varepsilon(0,t) = \varepsilon \varphi_x^\varepsilon(0,t), & t \in (0,T), \\ y^\varepsilon(1,t) = 0,. & t \in (0,T), \\ \varphi^\varepsilon(0,t) = \varphi^\varepsilon(1,t) = 0, & t \in (0,T), \\ -\beta \varphi_{xx}^\varepsilon(\cdot,T) + y^\varepsilon(\cdot,T) = 0, & x \in (0,1). \end{cases} \tag{8}$$

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Direct problem - Asymptotic expansion

$$\begin{cases} y_t^\varepsilon - \varepsilon y_{xx}^\varepsilon + M y_x^\varepsilon = 0, & (x,t) \in (0,1) \times (0,T), \\ y^\varepsilon(0,t) = v^\varepsilon(t), & t \in (0,T), \\ y^\varepsilon(1,t) = 0, & t \in (0,T), \\ y^\varepsilon(x,0) = y_0(x), & x \in (0,1), \end{cases} \tag{9}$$

y_0 and v^ε are given functions.

We assume that

$$v^\varepsilon = \sum_{k=0}^{m} \varepsilon^k v^k,$$

the functions $v^0, v^1, \cdots, v^m$ being known.

We construct an asymptotic approximation of the solution y^ε of (9) by using the matched asymptotic expansion method.

Let us consider two formal asymptotic expansions of y^ε:
– the outer expansion

$$\sum_{k=0}^{m} \varepsilon^k y^k(x,t), \quad (x,t) \in (0,T),$$

– the inner expansion

$$\sum_{k=0}^{m} \varepsilon^k Y^k(z,t), \quad z = \frac{1-x}{\varepsilon} \in (0,\varepsilon^{-1}),\ t \in (0,T).$$

Direct problem - Outer expansion

Putting $\sum_{k=0}^{m} \varepsilon^k y^k(x,t)$ into equation $(9)_1$, the identification of the powers of ε yields

$$\begin{aligned} \varepsilon^0 : &\quad y_t^0 + M y_x^0 = 0, \\ \varepsilon^k : &\quad y_t^k + M y_x^k = y_{xx}^{k-1}, \qquad \text{for any } 1 \leq k \leq m. \end{aligned}$$

Taking the initial and boundary conditions into account we define y^0 and y^k $(1 \leq k \leq m)$ as functions satisfying the transport equations, respectively,

$$\begin{cases} y_t^0 + M y_x^0 = 0, & (x,t) \in (0,1) \times (0,T), \\ y^0(0,t) = v^0(t), & t \in (0,T), \\ y^0(x,0) = y_0(x), & x \in (0,1), \end{cases} \tag{10}$$

and

$$\begin{cases} y_t^k + M y_x^k = y_{xx}^{k-1}, & (x,t) \in (0,1) \times (0,T), \\ y^k(0,t) = v^k(t), & t \in (0,T), \\ y^k(x,0) = 0, & x \in (0,1). \end{cases} \tag{11}$$

Direct problem - Outer expansion

The solution of (10) is given by

$$y^0(x,t) = \begin{cases} y_0(x - Mt) & x > Mt, \\ v^0\left(t - \dfrac{x}{M}\right), & x < Mt. \end{cases}$$

Using the method of characteristics we find that, for any $1 \leq k \leq m$,

$$y^k(x,t) = \begin{cases} \displaystyle\int_0^t y_{xx}^{k-1}(x + (s-t)M, s)ds, & x > Mt, \\ v^k\left(t - \dfrac{x}{M}\right) + \displaystyle\int_0^{x/M} y_{xx}^{k-1}\left(sM, t - \dfrac{x}{M} + s\right)ds, & x < Mt. \end{cases}$$

Remark

$$y^1(x,t) = \begin{cases} t\, y_0''(x - Mt), & x > Mt, \\ v^1\left(t - \dfrac{x}{M}\right) + \dfrac{x}{M^3}(v^0)''\left(t - \dfrac{x}{M}\right), & x < Mt, \end{cases}$$

$$y^2(x,t) = \begin{cases} \dfrac{t^2}{2} y_0^{(4)}(x - Mt), & x > Mt, \\ v^2\left(t - \dfrac{x}{M}\right) + \dfrac{x}{M^3}(v^1)''\left(t - \dfrac{x}{M}\right) \\ \quad - \dfrac{2x}{M^5}(v^0)^{(3)}\left(t - \dfrac{x}{M}\right) + \dfrac{x^2}{2M^6}(v^0)^{(4)}\left(t - \dfrac{x}{M}\right), & x < Mt. \end{cases}$$

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Direct problem - Inner expansion

Now we turn back to the construction of the inner expansion. Putting $\sum_{k=0}^{m} \varepsilon^k Y^k(z,t)$ into equation $(9)_1$, the identification of the powers of ε yields

$$\begin{aligned} &\varepsilon^{-1}: \quad Y^0_{zz}(z,t) + MY^0_z(z,t) = 0, \\ &\varepsilon^{k-1}: \quad Y^k_{zz}(z,t) + MY^k_z(z,t) = Y^{k-1}_t(z,t), \quad \text{for any } 1 \leq k \leq m. \end{aligned}$$

We impose that $Y^k(0,t) = 0$ for any $0 \leq k \leq m$ and use the asymptotic matching conditions

$$\begin{aligned} &Y^0(z,t) \sim y^0(1,t), \quad \text{as } z \to +\infty, \\ &Y^1(z,t) \sim y^1(1,t) - y^0_x(0,t)z, \quad \text{as } z \to +\infty, \\ &Y^2(z,t) \sim y^2(1,t) - y^1_x(0,t)z + \frac{1}{2} y^0_{xx}(0,t)z^2, \quad \text{as } z \to +\infty, \\ &\dots \\ &Y^m(z,t) \sim y^m(1,t) - y^{m-1}_x(0,t)z + \frac{1}{2} y^{m-2}_{xx}(0,t)z^2 + \cdots + \frac{1}{m!} (y^0)^{(m)}_x(1,t)(-z)^m, \\ &\text{as } z \to +\infty. \end{aligned}$$

Lemma

$$Y^0(z,t) = y^0(1,t)\left(1 - e^{-Mz}\right), \quad (z,t) \in (0,+\infty) \times (0,T).$$

For any $1 \leq k \leq m$*, the solution of reads*

$$Y^k(z,t) = Q^k(z,t) + e^{-Mz}P^k(z,t), \quad (z,t) \in (0,+\infty) \times (0,t), \tag{12}$$

where

$$P^k(z,t) = -\sum_{i=0}^{k} \frac{1}{i!}\frac{\partial^i y^{k-i}}{\partial x^i}(1,t)z^i, \quad Q^k(z,t) = \sum_{i=0}^{k} \frac{(-1)^i}{i!}\frac{\partial^i y^{k-i}}{\partial x^i}(1,t)z^i.$$

Asymptotic approximation

Let $\mathcal{X} : \mathbb{R} \to [0, 1]$ denote a C^2 cut-off function satisfying

$$\mathcal{X}(s) = \begin{cases} 1, & s \geq 2, \\ 0, & s \leq 1, \end{cases} \tag{13}$$

We define, for $\gamma \in (0, 1)$, the function

$$\mathcal{X}_\varepsilon(x) = \mathcal{X}\left(\frac{1-x}{\varepsilon^\gamma}\right),$$

then introduce the function

$$w_m^\varepsilon(x,t) = \mathcal{X}_\varepsilon(x) \sum_{k=0}^{m} \varepsilon^k y^k(x,t) + (1 - \mathcal{X}_\varepsilon(x)) \sum_{k=0}^{m} \varepsilon^k Y^k\left(\frac{1-x}{\varepsilon}, t\right), \tag{14}$$

defined to be an asymptotic approximation at order m of the solution y^ε of (9).

Asymptotic approximation- Convergence

Assume that $y_0 \in C^{2m+1}[0,1]$, $v^0 \in C^{2m+1}[0,T]$ and the following C^{2m+1}-matching conditions

$$M^p(y_0)^{(p)}(0) + (-1)^{p+1}(v^0)^{(p)}(0) = 0, \quad 0 \leq p \leq 2m+1. \tag{15}$$

Then $y^0 \in C^{2m+1}([0,1] \times [0,T])$.
Assume that $v^k \in C^{2(m-k)+1}[0,T]$, and the following $C^{2(m-k)+1}$-matching conditions

$$(v^k)^{(p)}(0) = \sum_{i+j=p-1} (-1)^i M^i \frac{\partial^{p+1} y^{k-1}}{\partial x^{i+2} \partial t^j}(0,0), \quad 0 \leq p \leq 2(m-k)+1. \tag{16}$$

Then $y^k \in C^{2(m-k)+1}([0,1] \times [0,T])$.

Lemma

Let w_m^ε be the function defined by (14). Then there is a constant c independent of ε such that

$$\|L_\varepsilon(w_m^\varepsilon)\|_{C([0,T];L^2(0,1))} \leq c\varepsilon^{\frac{(2m+1)\gamma}{2}}.$$

Proposition

Let y^ε be the solution of problem (9) and let w_m^ε be the function defined by (12). Then there is a constant c independent of ε such that

$$\|y^\varepsilon - w_m^\varepsilon\|_{C([0,T];L^2(0,1))} \leq c\varepsilon^{\frac{2m+1}{2}\gamma}.$$

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Proposition

Let $m \in \mathbb{N}$, $T > \frac{1}{M}$ and $a \in]0, T - \frac{1}{M}[$. Assume regularity and matching conditions on the initial condition y_0 and functions v^k, $0 \leq k \leq m$. Assume moreover that

$$v^k(t) = 0, \quad 0 \leq k \leq m, \forall t \in [a, T].$$

Then, the solution y^ε of problem (9) *satisfies the following property*

$$\|y^\varepsilon(\cdot, T)\|_{L^2(0,1)} \leq c\varepsilon^{\frac{(2m+1)\gamma}{2}}, \quad \forall \gamma \in (0, 1)$$

for some constant $c > 0$ independent of ε. The function $v^\varepsilon \in C([0, T])$ defined by $v^\varepsilon := \sum_{k=0}^{m} \varepsilon^k v^k$ is an approximate null control for (1).

The limit case $T = 1/M$ can be considered as well but requires explicit formula. We consider simply the case $m = 2$ and make use of Remark 1.

Proposition

Let $m = 2$. Let $T = \frac{1}{M}$. Assume regularity and matching conditions on the initial condition y_0 and functions v^k, $0 \le k \le m$.. There exist functions v^k, $0 \le k \le m$ such that the solution y^ε of problem (9) satisfies the following property, for all $\gamma \in (0,1)$

$$\|y^\varepsilon(\cdot, T)\|_{L^2(0,1)} \le \begin{cases} c\varepsilon^{\frac{\gamma}{2}}, & \text{if} \quad y_0(0) \neq 0, \\ c\varepsilon^{\frac{3\gamma}{2}}, & \text{if} \quad y_0(0) = 0, (y_0)^{(1)}(0) + \dfrac{(y_0)^{(3)}(0)}{M} \neq 0, \\ c\varepsilon^{\frac{5\gamma}{2}}, & \text{if} \quad y_0(0) = 0, (y_0)^{(1)}(0) + \dfrac{(y_0)^{(3)}(0)}{M} = 0, \end{cases}$$

for some constant $c > 0$ independent of ε.

The case of initial condition y_0^ε of the form $y_0^\varepsilon(x) = e^{-\frac{Mx}{2\varepsilon}}$

The asymptotic analysis is not valid for $y_0^\varepsilon(x) = f(x)e^{\frac{M\alpha x}{2\varepsilon}}$, $\alpha < 0$. Another expansion is needed

$$\begin{cases} y^\varepsilon(x,t) = e^{\frac{M\alpha}{2\varepsilon}\left(x - \frac{(2-\alpha)Mt}{2}\right)} z^\varepsilon(x,t), \\ L_\varepsilon y^\varepsilon(x,t) = e^{\frac{M\alpha}{2\varepsilon}\left(x - \frac{(2-\alpha)Mt}{2}\right)} \left(z_t^\varepsilon - \varepsilon z_{xx}^\varepsilon + M(1-\alpha) z_x^\varepsilon\right) \end{cases} \tag{17}$$

We then define the approximations

$$\begin{cases} z_m^\varepsilon(x,t) = \mathcal{X}_\varepsilon(x) \sum_{k=0}^{m} \varepsilon^k z^k(x,t) + (1 - \mathcal{X}_\varepsilon(x)) \sum_{k=0}^{m} \varepsilon^k Z^k\left(\frac{1-x}{\varepsilon}, t\right), \\ y_m^\varepsilon(x,t) = e^{\frac{M\alpha}{2\varepsilon}\left(x - \frac{(2-\alpha)Mt}{2}\right)} z_m^\varepsilon(x,t) \end{cases} \tag{18}$$

The main issue is now to find control functions $\overline{v}^k$ satisfying the matching conditions such that $\|L_\varepsilon y_m^\varepsilon\|_{C([0,T],L^2(0,1))}$ goes to zero with ε.

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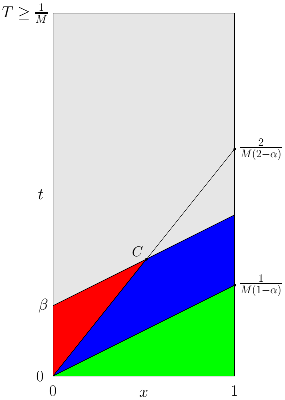
$\|L_\varepsilon y_m^\varepsilon\|_{C([0,T],L^2(0,1))}$



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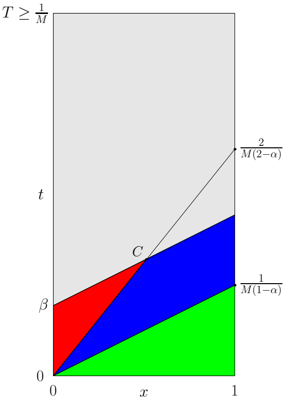

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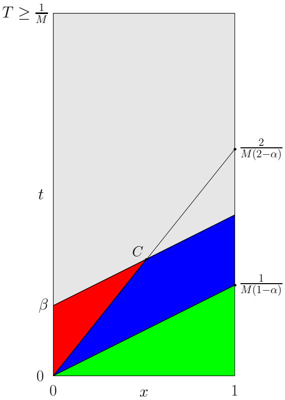

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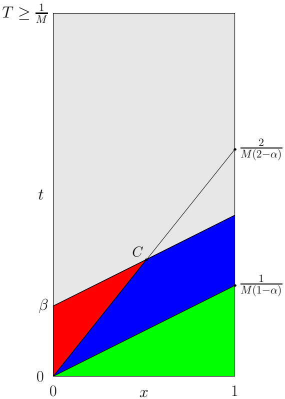

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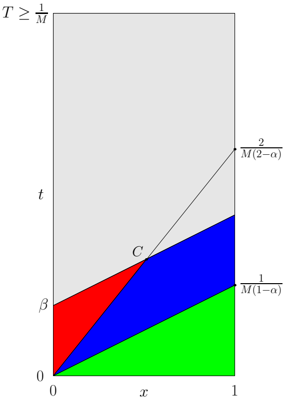

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$\|L_\varepsilon y_m^\varepsilon\|_{C([0,T],L^2(0,1))}$

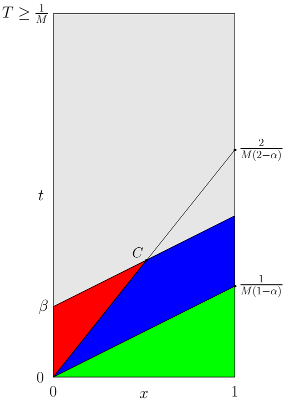

Figure: $(0,1) \times (0,T) = D_\beta^+ \cup (D_\beta^- \cap C_\alpha^-) \cup (D_\beta^- \cap C_\alpha^+)$.

Thus, the corresponding control is given

$$\begin{cases} v^0(t) = e^{-\frac{\gamma M^2 t}{4\varepsilon}}\,\overline{v}^0(t)\,1_{[0,\beta]}(t), \quad \gamma = \alpha(2-\alpha), \\ \overline{v}^0(t) = \dfrac{kC_1 - 2C_2 + kC_2 t}{k^3} e^{kt} + C_3 t + C_4, \quad k := \dfrac{-\eta + \alpha(2-\alpha)}{4\varepsilon} \end{cases} \tag{20}$$

One result

Let $m \geq 0$. Assume that $y_0^\varepsilon \in C^{2m+1}[0, T]$ and that $v^k \in H^{2(m-k)+2}[0, T]$, and the $C^{2(m-k)+1}$-matching conditions

$$(v^k)^{(p)}(0) = \sum_{i+j=p-1} (-1)^i M^i \frac{\partial^{p+1} y^{k-1}}{\partial x^{i+2} \partial t^j}(0,0), \quad 0 \leq p \leq 2(m-k)+1. \qquad (21)$$

THEOREM

Let $M > 0$, $\gamma \in (0,1)$.

The "cost of control"

$$\overline{K}(\varepsilon, T, M, m) = \sup_{y_0 \in L^2(0,1)} \min_{v_\varepsilon \in \overline{\mathcal{C}}} \left\| \sum_{k=0}^{m} \varepsilon^k v^k \right\|_{L^2(0,T)} \qquad (22)$$

with $\overline{\mathcal{C}}(y_0, T, \varepsilon, M) := \left\{ v_\varepsilon = \sum_{k=0}^m \varepsilon^k v^k, \|y_\varepsilon(\cdot, T)\|_{L^2(0,1)} \leq c\varepsilon^{\frac{(2m+1)\gamma}{2}} \right\}$

is bounded uniformly with respect to m and ε as soon as $T > \frac{1}{M}$.

Final remarks

- Instead of imposing regularity assumptions and matching conditions, we may introduce an additional C^2 cut-off $\mathcal{X}$ function to take into account the discontinuity of the solutions y^k on the characteristic line. This allows to deal with the initial optimality system.

  Y. Amirat, A. Münch: *Boundary controls for the equation $y_t - \varepsilon y_{xx} + My_x = 0$: Asymptotic analysis with respect to ε for $M > 0$*.Preprint 2017.

- The negative case is very similar except that the control v_ε lives in the boundary layer. (still in progress !)

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