Boundary Sliding Mode Control of hyperbolic systems

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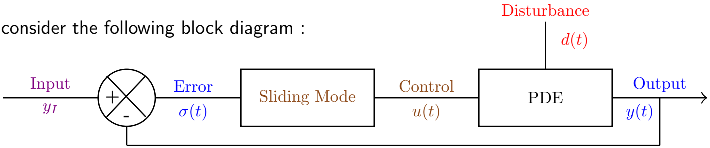
Objective

We consider the following block diagram :



Objective : construct a boundary feedback law $u(\cdot)$ which globally stabilizes the input y_I

Sliding Mode Strategy

The sliding controller u is designed to attain a sliding surface in finite time on which

$\longrightarrow$ the boundary disturbance $d(\cdot)$ is rejected

$\longrightarrow$ the global asymptotic stability is ensured

Question : how is the sliding surface constructed ?

$\longrightarrow$ The sliding surface is derived from the gradient of a Lyapunov function

Outline of the presentation

I. Sliding Mode Control to ODEs

II. Sliding Mode Control to linear PDEs

III. Simulations

IV. Sliding Mode Control to scalar conservation laws

I. Sliding Mode Control to ODEs

Sliding Mode Control to ODEs

We consider the spring-mass system with unknown disturbance $d(\cdot)$:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\dfrac{k}{m}x_1(t) + \dfrac{1}{m}(u(t) + d(t)) \\ x_1(0) = x_1^0,\ x_2(0) = x_2^0 \end{cases} \qquad \text{(ODEs)}$$

The undisturbed case ($d = 0$) : the feedback control law $u(t) = -\nu x_2(t)$ with $\nu > 0$

- The Lyapunov function $V(x_1, x_2) = \frac{1}{2}(kx_1^2 + mx_2^2) \longrightarrow (0,0)$ is stable
- LaSalle's invariance principle $\longrightarrow (0,0)$ is globally asymptotically stable

The disturbed case ($d \neq 0$) : a sliding mode strategy is used

- The sliding variable $\sigma : (x_1, x_2) \in \mathbb{R}^2 \to (1,1).\nabla V(x_1, x_2) = kx_1 + mx_2$
- The sliding surface $\Sigma = \{ X = (x_1, x_2) \in \mathbb{R}^2 \,/\, \sigma(X) = 0 \}$
- The sliding controller $u(t) = -(k+m)x_2(t) - K\mathbf{sign}(\sigma(x_1(t), x_2(t)))$

where $\mathbf{sign}(z) = \begin{cases} -1 & \text{if } z(t) < 0 \\ [-1,1] & \text{if } z(t) = 0 \\ 1 & \text{if } z(t) > 0 \end{cases}$

$\longrightarrow$ Assuming that $K > \|d\|_\infty$ then $(0,0)$ is globally asymptotically stable

Sliding Mode Control to ODEs

Proposition. The solution $(x_1(\cdot), x_2(\cdot))$ attains the sliding surface Σ in finite time and remains on it

Proof. The sliding variable $\sigma : t \to \sigma(x_1(t), x_2(t))$ is a Filippov solution of

$$\begin{cases} \dot\sigma(t) \in -\sigma(t) - K\text{sign}(\sigma(t)) + d(t), \\ \sigma(0) = x_1^0 + x_2^0. \end{cases} \qquad \text{(ODE)}$$

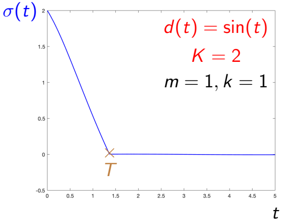
Definition

A Filippov solution of (ODE) is an absolutely continuous map that satisfies (ODE) for almost all t.

Since $d \in L^\infty(\mathbb{R}_+)$ and $K > \|d\|_{L^\infty(\mathbb{R})}$, we have

- $\sigma \in W^{1,\infty}(\mathbb{R})$
- $\exists T \ / \ \forall t > T$, $\sigma(t) = \dot\sigma(t) = 0$.

$$\implies (x_1(t), x_2(t)) \in \Sigma \text{ for any } t > T$$



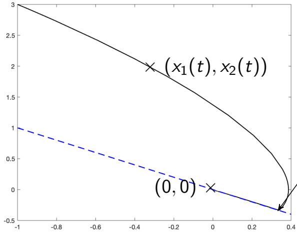
Sliding Mode Control to ODEs

Proposition. The solution $(x_1(\cdot), x_2(\cdot))$ is globally asymptotically stable **on** the sliding surface Σ

Proof. On the sliding surface Σ the spring-mass system

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\dfrac{k}{m}x_1(t) + \dfrac{1}{m}(u(t) + d(t)) \\ x_1(0) = x_1^0, x_2(0) = x_2^0 \end{cases} \quad \text{becomes} \quad \begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\dfrac{k}{m}x_1(t) - \dfrac{k+m}{m}x_2(t) \end{cases}$$

which is globally asymptotically stable.



$(x_1(\cdot), x_2(\cdot))$ reaches the sliding surface Σ at time T

Plotting of $(x_1(\cdot), x_2(\cdot))$ and the sliding surface Σ in the plane phase (x_1, x_2). $K = 2$ and $d(t) = \sin(t)$

II. Sliding Mode Control to PDEs

References

In recent years, some authors have applied a **Sliding Mode Control** to **PDEs** [Barbu, Colli, Cheng, Guo, Jin, Krstic, Liu, Levaggi, Orlov, Pisano, Utkin, Shu, Xu]

Considering **linear hyperbolic systems** with boundary input disturbances

- Construction of a sliding surface Σ and a sliding controller u

  $\rightarrow$ Backstepping method [Guo, Jin,Tang, Krstic]

  $\rightarrow$ Gradient of a Lyapunov function [T.L, Balogoun, Marx, Plestan]

- Existence and uniqueness of solutions

  $\rightarrow$ Semigroup theory $C^0([0, T))$ with T the time when the solution of PDEs reaches the sliding surface Σ [Guo, Jin,Tang, Krstic]

  $\rightarrow$ Coupled PDEs-ODE $C^0([0, +\infty))$ [T.L, Balogoun, Marx, Plestan]

Linear 2×2 hyperbolic system

Let $R_1^0, R_2^0 \in L^p(\mathbb{R})$ with $p \in [1, +\infty]$, $\lambda_1 > 0$, $\lambda_2 > 0$ and $k_2 \in \mathbb{R}$. We consider the linear hyperbolic system

$$\begin{cases} \partial_t R_1(t,x) + \lambda_1 \partial_x R_1(t,x) = 0 \\ \partial_t R_2(t,x) - \lambda_2 \partial_x R_2(t,x) = 0 \\ R_1(t,0) = u(t) + d(t) \\ R_2(t,L) = k_2 R_1(t,L) \\ R_1(0,x) = R_1^0(x), R_2(0,x) = R_2^0(x) \end{cases} \tag{PDE}$$

Objective : find a Sliding Surface Σ on which (PDE) becomes

$$\begin{cases} \partial_t R_1(t,x) + \lambda_1 \partial_x R_1(t,x) = 0 \\ \partial_t R_2(t,x) - \lambda_2 \partial_x R_2(t,x) = 0 \\ R_1(t,0) = k_1 R_2(t,0) \\ R_2(t,L) = k_2 R_1(t,L) \end{cases} \tag{PDE-SS}$$

in finite time $T > 0$ with k_1 chosen such that $|k_1 k_2| < 1$. From [Bastin, Coron (2016)] $(0,0)$ is exponentially stable for (PDE-SS)

From [Bastin, Coron (2016)], a candidate **Lyapunov function** for (PDE-SS) is

$$V \ : \ (f, g) \ \in \ L^2(\mathbb{R})^2 \ \to \ \int_0^L \frac{p_1}{\lambda_1} f^2(x) e^{\frac{-\nu x}{\lambda_1}} \ + \ \frac{p_2}{\lambda_2} g^2(x) e^{\frac{+\nu x}{\lambda_2}} dx$$

where p_1 and p_2 satisfies $p_2 - p_1 k_1^2 > 0$ and $p_1 > p_2 k_2^2 \exp(\nu(\frac{L}{\lambda_1} + \frac{L}{\lambda_2}))$

Sliding Mode Strategy

The sliding variable $\sigma : (f, g) \in L^p(\mathbb{R})^2 \to \int_0^L \frac{1}{\lambda_1} f(x) e^{\frac{-\nu x}{\lambda_1}} + \frac{k_1}{\lambda_2} g(x) e^{\frac{+\nu x}{\lambda_2}} dx$

The sliding surface $\Sigma = \{ (f, g) \in L^p(\mathbb{R})^2 \, / \, \sigma(f, g) = 0 \}$

The sliding controller $u : t \in \mathbb{R}_+ \to k_1 R_2(t, 0) - K \text{sign}(\sigma(R_1(t, \cdot), R_2(t, \cdot))$

with $k_1 = \frac{e^{-\nu(\frac{L}{\lambda_1} + \frac{L}{\lambda_2})}}{k_2}$. Note that, for any $\nu > 0$, $|k_1 k_2| < 1$.

Existence and stabilization

$$\begin{cases} \partial_t R_1(t,x) + \lambda_1 \partial_x R_1(t,x) = 0, \\ \partial_t R_2(t,x) - \lambda_2 \partial_x R_2(t,x) = 0, \\ R_1(t,0) = k_1 R_2(t,0) - K \mathrm{sign}(\sigma(R_1(t,\cdot), R_2(t,\cdot)) + d(t) \\ R_2(t,L) = k_2 R_1(t,L), \\ R_1(0,x) = R_1^0(x), R_2(0,x) = R_2^0(x), \end{cases} \tag{PDE}$$

Theorem (Existence)

Let $p \in [1,\infty]$ and $R_1^0, R_2^0 \in L^p(0,L)$. If $p \neq +\infty$, the system (PDE) admits a weak solution

$$(R_1, R_2) \in C^0([0,\infty); L^p((0,L); \mathbb{R}^2)).$$

If $p = \infty$, $(R_1, R_2) \in C^0([0,\infty); L^1((0,L); \mathbb{R}^2))$.

Theorem (Stabilization)

For any $K > \|d\|_\infty$ and $(R_1^0(\cdot), R_2^0(\cdot))^T \in L^p((0,L); \mathbb{R}^2)$. Then the origin of (PDE) is globally asymptotically stable in the $L^p((0,L); \mathbb{R}^2)$-topology.

Existence of weak solutions : a coupled PDE-ODE

We consider the **weak** coupled **PDE**-**ODE**

$$\begin{cases} \partial_t S_1(t,x) + \lambda_1 \partial_x S_1(t,x) = 0 \\ \partial_t S_2(t,x) - \lambda_2 \partial_x S_2(t,x) = 0 \\ S_1(t,0) = k_1 S_2(t,0) + \dot{\gamma}(t) + \nu\gamma(t) \\ S_2(t,L) = k_2 S_1(t,L) \\ \dot{\gamma}(t) = -\nu\gamma(t) - K\text{sign}(\gamma(t)) + d(t) \\ \gamma(0) = \sigma(R_1^0, R_2^0) \\ S_1(0,x) = R_1^0(x), S_2(0,x) = R_2^0(x) \end{cases}$$

For any $K > \|d\|_{L^\infty(\mathbb{R}_+)}$, for any γ Filippov solution of ODE,

$$\dot{\gamma} + \nu\gamma \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \quad \Longrightarrow \text{ existence of weak solutions for PDE-ODE}$$

The function $\sigma : t \to \sigma(S_1(t,\cdot), S_2(t,\cdot))$ satisfies

$$\begin{cases} \dot{\sigma}(t) = -\nu\sigma(t) + \dot{\gamma}(t) + \nu\gamma(t) \\ \sigma(0) = \sigma(R_1^0, R_2^0) \end{cases} \Longrightarrow \begin{cases} \sigma(t) = \gamma(t) \\ \dot{\gamma}(t) + \nu\gamma(t) = -K\text{sign}(\sigma(t)) + d(t) \end{cases}$$

$$\Longrightarrow \text{ existence of weak solutions for PDE}$$

Let (R_1, R_2) a weak solution of (PDE). The sliding variable $\sigma : t \to \sigma(R_1(t,\cdot), R_2(t,\cdot))$ is a Filippov solution of

$$\begin{cases} \dot{\sigma}(t) \in -\nu\sigma(t) - K\mathrm{sign}(\sigma(t)) + d(t), \\ \sigma(0) = \sigma(R_1^0, R_2^0) \end{cases}$$

Since $d \in L^\infty(\mathbb{R}_+)$ and $K > \|d\|_{L^\infty(\mathbb{R})}$,

$$\exists T \ / \ \forall t > T, \ \sigma(t) = \dot{\sigma}(t) = 0.$$

The (PDE) becomes for any $t \geq T$

$$\begin{cases} \partial_t R_1(t,x) + \lambda_1 \partial_x R_1(t,x) = 0, \\ \partial_t R_2(t,x) - \lambda_2 \partial_x R_2(t,x) = 0, \\ R_1(t,0) = k_1 R_2(t,0) \\ R_2(t,L) = k_2 R_1(t,L), \end{cases}$$

with $|k_1 k_2| < 1$. From [Bastin, Coron (2016)] $(0,0)$ is exponentially stable

Non uniforme 2*2 hyperbolic system

$$\begin{cases} \partial_t R_1(t,x) + \lambda_1(x)\partial_x R_1(t,x) = 0, \\ \partial_t R_2(t,x) - \lambda_2(x)\partial_x R_2(t,x) = 0, \\ R_1(t,0) = k_1 R_2(t,0) - K\text{sign}(\sigma(R_1(t,\cdot),R_2(t,\cdot)) + d(t) \\ R_2(t,L) = k_2 R_1(t,L), \\ R_1(0,x) = R_1^0(x), R_2(0,x) = R_2^0(x), \end{cases} \tag{PDE}$$

The sliding variable $\sigma : (f,g) \in L^p(\mathbb{R})^2 \to \int_0^L \frac{1}{\lambda_1(x)} e^{-\int_0^x \frac{\nu}{\lambda_1(s)} ds} f(x) + \frac{k_1}{\lambda_2(x)} e^{-\int_0^x \frac{\nu}{\lambda_2(s)} ds} g(x) dx.$

The sliding controller $u : t \in \mathbb{R}_+ \to k_1 R_2(t,0) - K\text{sign}(\sigma(R_1(t,\cdot),R_2(t,\cdot))$

with $k_1 k_2 \ = \ \exp(-\nu \int_0^L \frac{1}{\lambda_1(s)} + \frac{1}{\lambda_2(s)} ds)$

Theorem (Stabilization)

For any $K > \|d\|_\infty$ and $(R_1^0(\cdot), R_2^0(\cdot))^T \in L^p((0,L);\mathbb{R}^2)$. Then the origin of (PDE) *is globally asymptotically stable in the $L^p((0,L);\mathbb{R}^2)$-topology.*

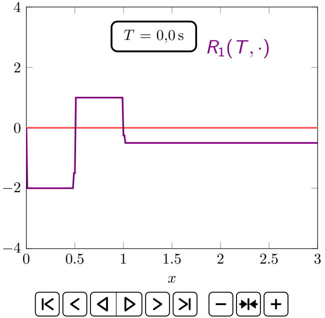
IV. Simulations

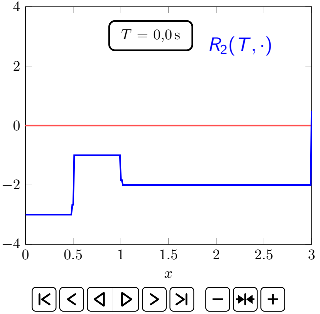
Simulation without controller

$$\begin{cases} \partial_t R_1(t,x) + \lambda_1 \partial_x R_1(t,x) = 0, \\ \partial_t R_2(t,x) - \lambda_2 \partial_x R_2(t,x) = 0, \\ R_1(t,0) = d(t) \\ R_2(t,L) = k_2 R_1(t,L), \\ R_1(0,x) = R_1^0(x), R_2(0,x) = R_2^0(x), \end{cases}$$

The parameters are

- $L = 3$
- $\lambda_1 = 2,\ \lambda_2 = 1$
- $k_2 = -1$
- $d(t) = \sin(t)$





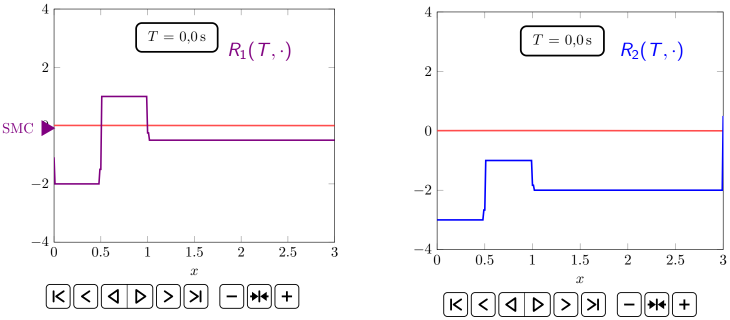
Plotting of an approximate solution $(R_1(T,\cdot), R_2(T,\cdot))$ using an **upwind scheme**.

Simulation with Sliding Mode Controller (SMC)

$$\begin{cases} \partial_t R_1(t,x) + \lambda_1 \partial_x R_1(t,x) = 0, \\ \partial_t R_2(t,x) - \lambda_2 \partial_x R_2(t,x) = 0, \\ R_1(t,0) = \textbf{SMC} + d(t) \\ R_2(t,L) = k_2 R_1(t,L), \\ R_1(0,x) = R_1^0(x), R_2(0,x) = R_2^0(x), \end{cases}$$

The parameters are

- $L = 3$
- $\lambda_1 = 2, \ \lambda_2 = 1$
- $k_1 = -0.6376, \ k_2 = -1$
- $d(t) = \sin(t)$



Plotting of an approximate solution $(R_1(T,\cdot), R_2(T,\cdot))$ using an **upwind scheme**.

V. Sliding Mode Control to scalar conservation laws

We consider the one-dimensional Burgers equation :

$$\begin{cases} \partial_t u(t.x) + \partial_x f(u(t,x)) = 0, \\ u(t,0) = u_L(t) + d(t), \\ u(t,L) = u_R(t), \\ u(0,x) = u_0(x), \end{cases} \qquad \text{with the flux } f \,:\, u \to \frac{u^2}{2} \tag{PDE}$$

Objective : construct a boundary feedback law $(u_L(\cdot), u_R(\cdot))$ which globally stabilizes the stationary entropy solution

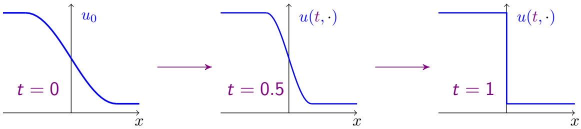
$$y_l = \begin{cases} k & \text{if } 0 \leq x < p \\ -k & \text{if } p < x \leq L \end{cases} \qquad k \geq 0,\ p \in (0,L)$$

The sliding variable $\sigma : f \in L^1(0,L) \to \int_0^L (f(x) - y_l(x))dx$

The left controller $u_L : t \in \mathbb{R}_+ \to y_l(0) - K\mathbf{sign}(\sigma(u(t,\cdot)))$

The right controller $u_R : t \in \mathbb{R}_+ \to y_l(L)$

Barrier 1 : non-regularity of solutions

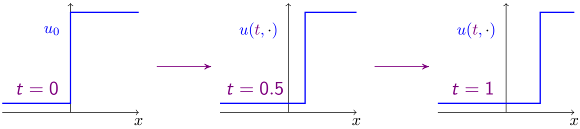


$\Longrightarrow$ creation of a discontinuity (*called shock*) in finite time

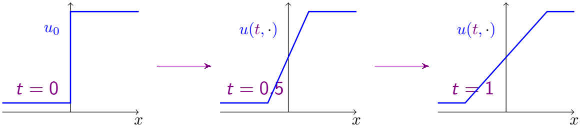
The function u is a **weak solution** to (PDE), for $(t,x) \in (0,+\infty) \times \mathbb{R}$, i.e for all $\varphi \in C_c^1(\mathbb{R}^2, \mathbb{R})$,

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} (u\partial_t\varphi + f(u)\partial_x\varphi) dxdt + \int_{\mathbb{R}} u_0(x)\varphi(0,x)dx = 0.$$

Barrier 2 : infinity of weak solutions



Solution 1 : moving of a discontinuity



Solution 2 : dissipation of a discontinuity

The function u is **a weak-entropy solution** to (PDE) if $\forall k \in \mathbb{R}$, $\forall \varphi \in C_c^1(\mathbb{R}^2, \mathbb{R}_+)$,

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} (|u-k|\partial_t \varphi + \operatorname{sgn}(u-k)(f(u)-f(k))\partial_x \varphi)dxdt + \int_{\mathbb{R}} |u_0 - k|\varphi(0,x)dx \geq 0.$$

[S.N. Kruzkov, P.D. Lax, T.P. Liu, O.A. Oleinik]

Barrier 3 : entropy boundary conditions

For $u \in \mathbb{R}$ we define $Adm_l(u)$ and $Adm_r(u)$ as follows

$$Adm_L(u) := \left\{ \begin{array}{lll} \{z \in \mathbb{R} & / & f'(z) \leq 0\} \\ \{z \in \mathbb{R} & / & f'(z) < 0 \text{ and } f(z) \geq f(u)\} \cap \{u\} \end{array} \begin{array}{l} \text{if } f'(u) \leq 0, \\ \text{if } f'(u) > 0. \end{array} \right\}$$

$$Adm_R(u) := \left\{ \begin{array}{lll} \{z \in \mathbb{R} & / & f'(z) \geq 0\} \\ \{z \in \mathbb{R} & / & f'(z) > 0 \text{ and } f(z) \geq f(u)\} \cap \{u\} \end{array} \begin{array}{l} \text{if } f'(u) \geq 0, \\ \text{if } f'(u) < 0. \end{array} \right\}$$

For almost all time $t > 0$

$$u(t, 0+) = u_L(t) + d(t) \quad \text{becomes} \quad u(t, 0+) \in Adm_L(u_L(t) + d(t))$$

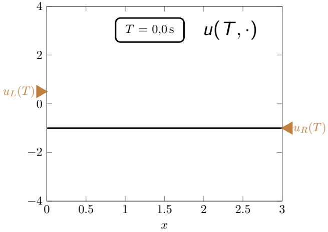
$$u(t, L-) = u_R(t) \quad \text{becomes} \quad u(t, L-) \in Adm_R(u_R(t))$$

We consider the one-dimensional Burgers equation :

$$\begin{cases} \partial_t u(t.x) + \partial_x f(u(t,x)) = 0, \\ u(t,0) = u_L(t) + d(t), \\ u(t,L) = u_R(t), \\ u(0,x) = u_0(x), \end{cases}$$

Example : $L = 3$, $u_0(\cdot) = -1$, $d(t) = 0$, $u_L(t) = 0.5$, $u_R(t) = -1$

We have $u_L(t) \neq u(t, 0+)$ for any $t \geq 0$



$\longrightarrow$ Switching phenomena (inactive control)

$\longrightarrow$ Delay phenomena

Let the stationary entropy solution y_l defined by $y_l = \begin{cases} k & \text{if } 0 \leq x < p \\ -k & \text{if } p < x \leq L \end{cases}$

and $u_L(t) = y_l(0) - K\mathbf{sign}(\sigma(u(t,\cdot)))$ and $u_R(t) = y_l(L)$

Conjecture : For any $u_0 \in L^\infty(0,L)$, there exists a finite time T such that for any $t \geq T$, $u(t,\cdot) = y_l(\cdot)$

Idea

Step 1. Using conservation of mass, we have

$$\dot{\sigma}(t) = f(u(t,0+)) - f(u(t,L-))$$

Step 2. Assuming that $\|d\|_\infty$ is small enough

$$\exists T_1 > 0 \text{ such that } u(t,0+) = u_L(t) + d(t) \text{ and } u(t,L-) = u_R(t)$$

Step 3. For every $t \geq T_1$ σ is a Filippov solution of

$$\begin{cases} \dot{\sigma}(t) = f(y_l(L) - K\textbf{sign}(\sigma(t)) + d(t)) - f(y_l(L)) \\ \sigma(T_1) = \int_0^T u(T,x)dx \end{cases}$$

Step 4. For every $t \geq T_1$, the (PDE) is rewritten as a coupled (PDE-ODE)

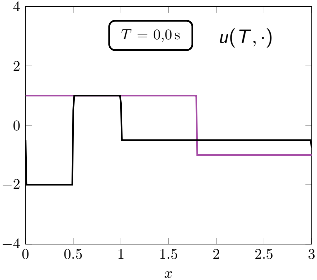
$$\exists T_2 > T_1 \text{ such that } \begin{cases} \sigma(t) = \dot{\sigma}(t) = 0 \\ u_L(t) = y_l(0) \text{ and } u_R(t) = y_l(L) \end{cases} \quad \forall t \geq T_2$$

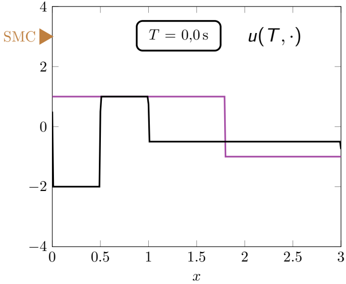
$$\implies u(t,\cdot) = y_l(\cdot) \quad \forall t \geq T_2$$

Simulation

$$\begin{cases} \partial_t u(t.x) + \partial_x f(u(t,x)) = 0, \\ u(t,0) = y_l(0) + \sin(t), \\ u(t,L) = y_l(L), \\ u(0,x) = u_0(x), \end{cases}$$

$$\begin{cases} \partial_t u(t.x) + \partial_x f(u(t,x)) = 0, \\ u(t,0) = \textbf{SMC} + \sin(t), \\ u(t,L) = y_l(L), \\ u(0,x) = u_0(x), \end{cases}$$





Plotting of an approximate solution $u(T,\cdot)$ using an **Godunov scheme**.

Conclusion

Summary : three PDEs with boundary disturbances

- Linear 2*2 hyperbolic systems
- Non uniforme 2*2 hyperbolic systems
- Scalar conservation laws (*work in progress*)

Construction of a boundary feedback law which globally stabilizes a stationary solution

The sliding controller is designed to attain a sliding surface in finite time on which

- the boundary disturbance $d(\cdot)$ is rejected
- the global asymptotic stability is ensured

The sliding surface may be derived from the gradient of a Lyapunov function

Further works :

- Generalization to semi-group
- Systems of conservation laws

Thank you for your attention