Observability inequalities for elliptic equations in 2-d

Conférence : Contrôle, Problèmes inverses et Applications

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Introduction to controllability

$$\begin{cases} y'(t) = f(y(t), h(t)), \\ y(0) = y_0. \end{cases}$$

$y(t) \in \mathcal{Y}$: ***the state***, $h(t) \in \mathcal{H}$: ***the control***.

Controllability

$T > 0$, y_0, $y_f \in \mathcal{Y}$.

Does there exists $h : [0, T] \to \mathcal{H}$ such that $\begin{cases} y' = f(y, h), \\ y(0) = y_0, \end{cases} \implies y(T) = y_f$?

- ***small-time controllability*** : $T << 1$,
- ***large-time controllability*** : $T >> 1$,
- ***global controllability*** : $\forall y_0 \in \mathcal{Y}$,
- ***local controllability*** : $\forall y_0$ closed to y_f,
- ***null-controllability*** : $y_f = 0$.

Heat equation

$T > 0$, $\Omega \subset \mathbb{R}^N$, $\omega \subset \Omega$.

$$\begin{cases} \partial_t y - \Delta y = h 1_\omega & \text{in } (0,T) \times \Omega, \\ y = 0 & \text{on } (0,T) \times \partial\Omega, \\ y(0,\cdot) = y_0 & \text{in } \Omega. \end{cases} \tag{Heat}$$

In (Heat), $y(t,\cdot) : \Omega \to \mathbb{R}$ is the state and $h(t,\cdot) : \omega \to \mathbb{R}$ is the control.

Modelling:

- Ω is a room,
- $y(t,x)$: temperature at time $t \in (0,T)$, at point $x \in \Omega$,
- $h(t,x)$: action of a heater/cooler localized in ω.

Goal: Drive the temperature y to a prescribed target in time T, by using the heater/cooler h, localized in ω.

Small-time null-controllability

$$\begin{cases} \partial_t y - \Delta y = h 1_\omega & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = y_0 & \text{in } \Omega. \end{cases} \tag{Heat}$$

Theorem (Lebeau, Robbiano - Fursikov, Imanuvilov (1995-1996))

(Heat) *is **small-time (globally) null-controllable**, i.e.*

$$\forall T > 0,\ \forall y_0 \in L^2(\Omega),\ \exists h \in L^2(0,T;L^2(\omega)) \text{ such that } y(T,\cdot) = 0.$$

- heat equation $\Rightarrow$ regularizing effects $\Rightarrow$ exact controllability cannot hold.
- heat equation $\Rightarrow$ infinite speed of propagation $\Rightarrow$ small-time controllability.

Fattorini, Russell (1971): 1D.

Hilbert Uniqueness Method

$$\begin{cases} \partial_t y - \Delta y = h 1_\omega & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = y_0 & \text{in } \Omega. \end{cases} \quad \begin{cases} -\partial_t \varphi - \Delta \varphi = 0 & \text{in } (0,T)\times\Omega, \\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi(T,\cdot) = \varphi_T & \text{in } \Omega. \end{cases}$$

Proposition (H.U.M.)

The heat equation is null-controllable in time $T > 0$ iff there exists $C_T > 0$

$$\|\varphi(0,\cdot)\|_{L^2(\Omega)}^2 \leq C_T \left(\int_0^T \int_\omega \varphi^2 dxdt \right), \ \forall \varphi_T \in L^2(\Omega). \qquad \text{(Observability)}$$

Moreover, if such a $C_T > 0$ exists, then $\forall y_0 \in L^2(\Omega)$, there exists $h \in L^2(q_T)$

$$\|h\|_{L^2(q_T)} \leq C_T \|y_0\|_{L^2(\Omega)}, \qquad \text{(Cost)}$$

such that the solution y of (Heat) *satisfies $y(T,\cdot) = 0$.*

Carleman estimate

Let $\omega_0 \subset\subset \omega$ a nonempty open set.
$\exists \eta^0 \in C^2(\overline{\Omega})$ such that $\eta^0 > 0$ in Ω, $\eta^0 = 0$ on $\partial\Omega$, and $|\nabla \eta^0| > 0$ in $\overline{\Omega \setminus \omega_0}$.

$$\xi(t,x) := e^{\lambda \eta_0(x)} t^{-1}(T-t)^{-1}.$$

Theorem (Fursikov, Imanuvilov (1995-1996))

There exist $\lambda_1 = \lambda_1(\Omega,\omega) \geq 1$, $s_1 = C(\Omega,\omega)(T+T^2)$, $C_1 = C_1(\Omega,\omega)$ *such that for every* $\lambda \geq \lambda_1$, $s \geq s_1$,

$$\lambda^4 \int_{Q_T} e^{-2s\xi}(s\xi)^3 |\varphi|^2 dtdx$$

$$\leq C_1 \left(\int_{Q_T} e^{-2s\xi} |\partial_t \varphi + \Delta \varphi|^2 dtdx + \lambda^4 \int_{(0,T)\times\omega} e^{-2s\xi}(s\xi)^3 |\varphi|^2 dtdx \right),$$

where $\varphi \in C^2(\overline{Q_T})$ *with* $\varphi = 0$ *on* Σ_T.

The parameters λ and s play an important role:

- crucial in the proof of the Carleman estimate,
- useful when considering more general parabolic equations.

Proof of the observability estimate

Carleman estimate applied to $-\partial_t \varphi - \Delta \varphi = 0$:

$$\int_{Q_T} t^{-3}(T-t)^{-3} e^{-2s\xi} |\varphi|^2 dxdt \leq C_1 \int_{(0,T)\times\omega} t^{-3}(T-t)^{-3} e^{-2s\xi} |\varphi|^2 dxdt. \quad (1)$$

We have

$$t^{-3}(T-t)^{-3} e^{-2s\xi} \geq C T^{-6} e^{-C(\Omega,\omega)\left(1+\frac{1}{T}\right)} \text{ in } (T/4, 3T/4)\times\Omega, \quad (2)$$

$$t^{-3}(T-t)^{-3} e^{-2s\xi} \leq C(\Omega,\omega) T^{-6} \text{ in } (0,T)\times\omega. \quad (3)$$

By (1), (2) and (3), we get

$$\int_{(T/4,3T/4)\times\Omega} |\varphi|^2 dxdt \leq e^{C(\Omega,\omega)\left(1+\frac{1}{T}\right)} \int_{(0,T)\times\omega} |\varphi|^2 dxdt. \quad (4)$$

Dissipativity in time of the L^2-norm:

$$\|\varphi(0,.)\|_{L^2(\Omega)} \leq \frac{2}{T} \int_{T/4}^{3T/4} \|\varphi(t,.)\|_{L^2(\Omega)} dt. \quad (5)$$

By (4) and (5), we get

$$\|\varphi(0,.)\|_{L^2(\Omega)} \leq e^{C(\Omega,\omega)\left(1+\frac{1}{T}\right)} \|\varphi\|_{L^2((0,T)\times\omega)}. \quad \text{(Observability)}$$

Parabolic equations

Let $a \in L^\infty(Q_T)$ and consider

$$\begin{cases} \partial_t y - \Delta y + a(t,x)y = h 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0,\cdot) = y_0 & \text{in } \Omega. \end{cases} \qquad \text{(Parabolic)}$$

Theorem (Fernandez-Cara, Zuazua (2000))

(Parabolic) *is **small-time globally null-controllable**, i.e.*

$$\forall T > 0,\ \forall y_0 \in L^2(\Omega),\ \exists h \in L^2(0,T;L^2(\omega)) \text{ such that } y(T,\cdot) = 0.$$

Moreover, $h \in L^2(0,T;L^2(\omega))$ *can be chosen such that*

$$\|h\|_{L^2(q_T)} \leq C_T \|y_0\|_{L^2(\Omega)}, \qquad \text{(Cost)}$$

with

$$C_T = \exp\left(C(\Omega,\omega)\left(1 + \frac{1}{T} + \|a\|_\infty^{2/3} + T\|a\|_\infty\right)\right).$$

Observability estimate for the parabolic equation

$$\begin{cases} -\partial_t \varphi - \Delta \varphi + a(t,x)\varphi = 0 & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(T,\cdot) = \varphi_T & \text{in } \Omega. \end{cases} \qquad \text{(Adjoint)}$$

Carleman estimate applied to $-\partial_t \varphi - \Delta \varphi + a\varphi = 0$:

$$\lambda^4 \int_{Q_T} e^{-2s\xi}(s\xi)^3|\varphi|^2 dtdx \leq C \left(\int_{Q_T} e^{-2s\xi}|a\varphi|^2 dtdx + \lambda^4 \int_{(0,T)\times\omega} e^{-2s\xi}(s\xi)^3|\varphi|^2 dtdx \right).$$

Take $\lambda = \lambda_1$ and $s \geq C(\Omega,\omega)T^2 \|a\|_\infty^{2/3}$, we get

$$\lambda^4 \int_{Q_T} e^{-2s\xi}(s\xi)^3|\varphi|^2 dtdx \leq C\lambda^4 \int_{(0,T)\times\omega} e^{-2s\xi}(s\xi)^3|\varphi|^2 dtdx.$$

$$\Rightarrow \int_{(T/4,3T/4)\times\Omega} |\varphi|^2 dxdt \leq e^{C(\Omega,\omega)\left(1+\frac{1}{T}+\|a\|_\infty^{2/3}\right)} \int_{(0,T)\times\omega} |\varphi|^2 dxdt.$$

Dissipativity in time of the L^2-norm:

$$\|\varphi(0,.)\|_{L^2(\Omega)} \leq \exp\left(CT\left(\|a\|_\infty\right)\right) \frac{2}{T} \int_{T/4}^{3T/4} \|\varphi(t,.)\|_{L^2(\Omega)} dt.$$

$$\Rightarrow \|\varphi(0,.)\|_{L^2(\Omega)} \leq e^{C(\Omega,\omega)\left(1+\frac{1}{T}+\|a\|_\infty^{2/3}+T\|a\|_\infty\right)} \|\varphi\|_{L^2((0,T)\times\omega)}. \qquad \text{(Observability)}$$

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Semilinear parabolic equations

Take $f \in C^1(\mathbb{R};\mathbb{R})$ such that $f(0) = 0$ and consider

$$\left\{ \begin{array}{ll} \partial_t y - \Delta y + f(y) = h 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0,.) = y_0 & \text{in } \Omega. \end{array} \right. \tag{HeatSL}$$

$f(0) = 0 \Rightarrow 0$ is a stationary state.
In particular if $y(T,\cdot) = 0$, then by setting $h \equiv 0$ for $t \geq T$ then $y \equiv 0$ for $t \geq T$.

Goal/Question: Null-controllability of the semilinear equation (HeatSL)?

Small-time local null-controllability

$$\begin{cases} \partial_t y - \Delta y + f(y) = h 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0,.) = y_0 & \text{in } \Omega. \end{cases} \tag{HeatSL}$$

Theorem

(HeatSL) *is **small-time locally null-controllable**, i.e.*

$\forall T > 0$, $\exists \delta_T > 0$ $\forall \|y_0\|_{L^\infty} \leq \delta_T$, $\exists h \in L^\infty(0,T;L^\infty(\omega))$ *such that* $y(T,\cdot) = 0$.

Linear test:

$$\begin{cases} \partial_t y - \Delta y + f'(0) y = h 1_\omega & \text{in } (0,T) \times \Omega, \\ y = 0 & \text{on } (0,T) \times \partial\Omega, \\ y(0,\cdot) = y_0 & \text{in } \Omega, \end{cases}$$

globally null-controllable, then (HeatSL) is locally null-controllable.

What about global null-controllability?

$$\begin{cases} \partial_t y - \Delta y + f(y) = h 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0,.) = y_0 & \text{in } \Omega. \end{cases} \tag{HeatSL}$$

We will also assume that f satisfies the restrictive growth condition ($\alpha > 0$)

$$\frac{f(s)}{|s| \log^\alpha(1+|s|)} \to 0 \text{ as } |s| \to +\infty.$$

Under this assumption, *blow-up* may occur if $h = 0$ in (HeatSL).
Take for example $f(s) = -|s| \log^p(1+|s|)$ with $p > 1$ (Osgood's condition).

Goal/Question: ***Global null-controllability*** of (HeatSL)?

Fernandez-Cara, Zuazua's results

$$\begin{cases} \partial_t y - \Delta y + f(y) = h 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0,\cdot) = y_0 & \text{in } \Omega. \end{cases} \qquad \text{(HeatSL)}$$

Theorem (Fernandez-Cara, Zuazua (2000))

• *(Positive result) Assume that* $f(s) = o_{+\infty}(|s| \log^{3/2}(1+|s|))$, *then* (HeatSL) *is* ***small-time globally null-controllable***, *i.e.*

$$\forall T > 0,\ \forall y_0 \in L^\infty(\Omega),\ \exists h \in L^\infty(0,T;L^\infty(\omega)) \text{ such that } y(T,\cdot) = 0.$$

• *(Negative result) Set* $f(s) := |s| \log^p(1+|s|)$ *with* $p > 2$ *and assume that* $\Omega \setminus \overline{\omega} \neq \emptyset$, *then one* ***cannot prevent blow-up*** *in* (HeatSL), *i.e.*

$$\forall T > 0,\ \exists y_0 \in L^\infty(\Omega),\ \forall h \in L^\infty(0,T;L^\infty(\omega)),\ y \text{ blows-up in time } T^* < T.$$

Proof of the positive result

Linearization: $g(s) := f(s)/s$, take $z \in L^\infty(Q_T)$ and consider

$$\left\{\begin{array}{ll} \partial_t y - \Delta y + g(z)y = h1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0,\cdot) = y_0 & \text{in } \Omega. \end{array}\right. \quad \text{(Parabolic)}$$

Null-controllability with cost estimate: $\forall \tau > 0$, $\exists \|h\|_{L^\infty(q_\tau)} \leq C_\tau \|y_0\|_{L^\infty(\Omega)}$ with $C_\tau := \exp\left(C\left(1 + \frac{1}{\tau} + \|g(z)\|_\infty^{2/3} + \tau\|g(z)\|_\infty\right)\right)$ such that $y(\tau,\cdot) = 0$.

Act in very small-time: $\tau := \min\left(T, \|g(z)\|_{L^\infty(Q_T)}^{-1/3}\right) \Rightarrow C_\tau := \exp\left(C\left(\|g(z)\|_\infty^{2/3}\right)\right)$.

Fixed-point argument:

$$\Phi : z \in L^\infty(Q_T) \mapsto \{y \in L^\infty(Q_T)\ ;\ \exists \|h\|_{L^\infty(q_\tau)} \leq C_\tau \|y_0\|_{L^\infty(\Omega)},\ y(\tau,\cdot) = 0\}.$$

If we prove that $\exists y \in \Phi(y)$ then $\partial_t y - \Delta y + f(y) = h1_\omega$ and $y(T,\cdot) = 0$.
Invariant ball: Using $g(s) = o_{+\infty}(\log^{3/2}(|s|))$, we get

$$\begin{aligned} \forall z \in B_R,\ \forall y \in \Phi(z),\ \|y\|_{L^\infty(Q_T)} &\leq \exp\left(\|g(z)\|_{L^\infty(Q_T)}^{2/3}\right)\|y_0\|_{L^\infty(\Omega)} \\ &= o_{+\infty}(\exp(\log(R)))\|y_0\|_{L^\infty(\Omega)} \leq R. \end{aligned}$$

Rmk: Constructive proof in Ervedoza, Lemoine, Münch (2021).

Proof of the negative result

Localized eigenfunction method: Take $\rho \in C_c^\infty(\Omega \setminus \overline{\omega})$ such that $\int_\Omega \rho(x)dx = 1$ and multiply $\partial_t y - \Delta y + f(y) = h1_\omega$ by ρ and integrate in Ω,

$$\frac{d}{dt}\left(\int_\Omega y(t,x)\rho(x)dx\right) = \int_\Omega \Delta y \rho - \int_\Omega f(y)\rho.$$

Setting $u(t) = -\int_\Omega y(t,x)\rho(x)dx$, and integrating by parts

$$\frac{du}{dt} = -\int_\Omega y\Delta\rho + \int_\Omega f(|y|)\rho.$$

By Young's inequality, we have

$$\left|\int_\Omega y\Delta\rho dx\right| \leq \int_\Omega |y| \left|\frac{\Delta\rho}{\rho}\right| \rho dx \leq \frac{1}{2}\int_\Omega f(|y|)\rho dx + \frac{1}{2}\int_\Omega f^*\left(\frac{2\Delta\rho}{\rho}\right)\rho dx$$

So,

$$\frac{du}{dt} \geq -\frac{C}{2} + \frac{1}{2}\int_\Omega f(|y|)\rho dx,\ \ C := \int_\Omega f^*\left(\frac{2|\Delta\rho|}{\rho}\right)\rho dx < +\infty\ (p > 2).$$

Therefore, Jensen's inequality and parity of f lead to

$$\frac{du}{dt} \geq -\frac{C}{2} + \frac{f(u)}{2} \Rightarrow \text{ Blow-up.}$$

Open questions

$$\begin{cases} \partial_t y - \Delta y + f(y) = h 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0,\cdot) = y_0 & \text{in } \Omega. \end{cases} \tag{HeatSL}$$

Open questions: What happen for $f(s) \approx_{|s| \to +\infty} |s| \log^p(1+|s|)$, $p \in [3/2, 2]$?

1. Can one prevent the blow-up from happening?
2. (HeatSL) is large-time globally null-controllable?
3. (HeatSL) is small-time globally null-controllable?

Le Balc'h (2020): 2. is true for f semi-dissipative.

Can we improve the cost of null-controllability of $\partial_t y - \Delta y + a(t,x)y = h1_\omega$:

$$\|h\|_{L^2(q_T)} \leq \exp\left(C(\Omega,\omega)\left(1 + \frac{1}{T} + \|a\|_\infty^{2/3} + T\|a\|_\infty\right)\right)\|y_0\|_{L^2(\Omega)}? \tag{Cost}$$

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Optimal observability inequality for parabolic equations

Let $a \in L^\infty((0,T)\times\Omega)$, $\begin{cases} -\partial_t \varphi - \Delta\varphi + a(t,x)\varphi = 0 & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(T,\cdot) = \varphi_T & \text{in } \Omega. \end{cases}$

Fernández-Cara, Zuazua (2000) proved:

$$\|\varphi(0,\cdot)\|^2_{L^2(\Omega)} \leq C \left(\int_0^T \int_\omega \varphi^2 dxdt \right) \ \forall \varphi_T \in L^2(\Omega),$$

where $\boxed{C = C(\Omega,\omega,T,a) = \exp\left(C(\Omega,\omega)\left(1 + \frac{1}{T} + T\|a\|_{L^\infty} + \|a\|_{L^\infty}^{2/3}\right)\right).}$

Theorem (Duyckaerts, Zhang, Zuazua (2008))

Optimality of $\|a\|_\infty^{2/3}$ for $\varphi_T \in L^2(\Omega;\mathbb{C})$, $a \in L^\infty(Q_T;\mathbb{C})$.

Le Balc'h (2020): $\|a\|_\infty^{1/2}$ for $\varphi_T \in L^2(\Omega;\mathbb{R})$, $a \in L^\infty(Q_T;\mathbb{R}^+)$.

Proof of the optimality by Meshkov's function

Meshkov's result: There exist $V \in L^\infty(\mathbb{C};\mathbb{C})$ and $u \neq 0$ such that

$$-\Delta u + V(x)u = 0 \text{ and } u(x) \leq \exp(-|x|^{4/3}).$$

Scaling argument: We set $u_R(x) = u(Rx)$ and $a_R(x) = R^2 V(Rx)$, we have

$$-\Delta u_R + a_R(x)u = 0 \text{ and } u_R(x) \leq \exp(-R^{4/3}|x|^{4/3}).$$

Test the observability inequality with $\varphi_R = u_R$: Assume that $d(0,\overline{\omega}) > 0$ then

$$\|\varphi_R(0,\cdot)\|^2_{L^2(\Omega)} \sim \|u_R\|^2_{L^2(\mathbb{R}^2)} \sim \frac{1}{R^2},$$
$$\|\varphi_R\|_{L^2((0,T)\times\omega)} \leq \exp(-R^{4/3}),$$
$$\|a_R\|_{L^\infty(Q_T)} \sim R^2.$$

So for $T \leq \|a_R\|^{-1/3}_{L^\infty(Q_T)}$, we get for $c > 0$ sufficiently small

$$\lim_{R\to+\infty} \left\{ \frac{\|\varphi_R(0,\cdot)\|_{L^2(\Omega)}}{\exp\left(c\,\|a_R\|^{2/3}_{L^\infty(\Omega)}\right)\int_0^T\int_\omega |\varphi_R(t,x)|^2 dtdx} \right\} = +\infty.$$

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The Landis conjecture on exponential decay

Conjecture (Landis, 1960's)

$$V \in L^\infty(\mathbb{R}^N), \left\{ \begin{array}{l} -\Delta u + V(x)u = 0 \textit{ in } \mathbb{R}^N, \\ |u(x)| \leq \exp(-|x|^{1+\varepsilon}), \ \varepsilon > 0, \end{array} \right. \qquad \Longrightarrow u \equiv 0.$$

Example: $u(x) = \exp(-|x|)$ in $\{|x| > 1\}$, smoothly extended to $\mathbb{R}^N$.

Proof in 1D (M. Pierre): $-u'' + V(x)u = 0$ in $\mathbb{R}$, $|u(x)| \leq \exp(-|x|^{1+\varepsilon})$.
By integrating, we easily get $|u'(x)| \leq C\exp(-|x|^{1+\varepsilon})$.
Duality argument: Let ϕ s.t. $-\phi'' + V\phi = \mathrm{sign}(u)$, $\phi(0) = \phi'(0) = 0$.
Gronwall's argument: $|\phi(x)| + |\phi'(x)| \leq C\exp(C|x|)$.
$\int_{-R}^{R} |u| = \int_{-R}^{R} u \cdot \mathrm{sign}(u) = \int_{-R}^{R} u(-\phi'' + V\phi) = [-\phi' u + \phi u]_{-R}^{R} \leq e^R e^{-R^{1+\varepsilon}} \to 0.$

Meshkov's counterexample (1991): $\exists V \in L^\infty(\mathbb{R}^2; \mathbb{C})$ and $u \in L^\infty(\mathbb{R}^2; \mathbb{C}) \neq 0$ such that $-\Delta u + V(x)u = 0$ in $\mathbb{R}^2$ and $|u(x)| \leq \exp(-|x|^{4/3})$.

Optimality (Meshkov):
$-\Delta u + V(x)u = 0$ in $\mathbb{R}^N$ and $|u(x)| \leq \exp(-|x|^{4/3+\varepsilon})$, $\varepsilon > 0 \Rightarrow u \equiv 0$.

Landis conjecture for real-valued potentials

Open questions (Kenig, Bourgain, 2005):

- Is the (**qualitative**) Landis conjecture true for real-valued potentials?

$$V \in L^\infty(\mathbb{R}^N; \mathbb{R}), \; \begin{cases} -\Delta u + V(x) u = 0 \text{ in } \mathbb{R}^N, \\ |u(x)| \leq \exp(-|x|^{1+\varepsilon}), \; \varepsilon > 0, \end{cases} \Longrightarrow u \equiv 0.$$

- **Quantitative** Landis conjecture: for $|V| \leq 1$ real-valued and $|u| \leq 1$ such that $-\Delta u + Vu = 0$, $|u(0)| = 1$, do we have: $\forall R >> 1$, $\forall |x_0| = R$,

$$\sup_{|x-x_0|<1} |u(x)| \geq \exp(-R \log^\alpha(R))?$$

Logunov, Malinnikova, Nadirashvili, Nazarov (2020) in the plane $\mathbb{R}^2$.

Conclusion of the first part

In brief, recall the story

- Null-controllability of $\partial_t y - \Delta y + a(t,x)y = h1_\omega$ Cost $= \exp\left(C\left(\|a\|_\infty^{2/3}\right)\right)$.
- Observability of $-\partial_t \varphi - \Delta\varphi + a\varphi = 0$, $|\varphi(0)|_{L^2} \leq \exp\left(C\left(\|a\|_\infty^{2/3}\right)\right)|\varphi|_{L^2(q_T)}$.
- Global null-controllability of $\partial_t y - \Delta y + |y|\log^p(1+|y|) = h1_\omega$, $p < 3/2$.
- Blow-up of $\partial_t y - \Delta y + |y|\log^p(1+|y|) = h1_\omega$, $p > 2$.
- Optimality of $\|a\|_\infty^{2/3}$ by Meshkov's counterexample for $a \in L^\infty(Q_T;\mathbb{C})$.
- Landis conjecture: $-\Delta u + V(x)u = 0$, $|u(x)| \leq \exp(-|x|^{1+\varepsilon}) \Rightarrow u \equiv 0$.
- False in 2-d for $V \in L^\infty(\mathbb{R}^2;\mathbb{C})$ by Meshkov's counterexample.
- True in 2-d for $V \in L^\infty(\mathbb{R}^2;\mathbb{R})$ by Logunov and al.

Goal: improve observability estimates for elliptic equations in 2-d and obtain new elliptic control results.

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Optimal observability inequality in 2-d

Theorem (Ervedoza, Le Balc'h (2021))

Let $\Omega \subset \mathbb{R}^2$ and $\omega \subset \Omega$.
*For every **real-valued potential** $V \in L^\infty(\Omega;\mathbb{R})$ and function $u \in H^2(\Omega) \cap H_0^1(\Omega)$,*

$$\|u\|_{H^2(\Omega)} \leq C \left(\|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)} \right), \qquad \text{(Observability)}$$

where $C > 0$ is given by $\boxed{C = \exp\left(C(\Omega,\omega) \left(1 + \|V\|_\infty^{1/2} \log^{1/2} (\|V\|_\infty) \right) \right).}$

- V has to be real-valued (Meshkov's counterexample).
- $V \in L^\infty(\Omega;\mathbb{R}) \Rightarrow$ one can assume that $\underline{u \text{ is real-valued}}$.
- Today, Ω has to be smooth and simply connected.
- (Observability) proved by Logunov and al for Ω a 2-d manifold without boundary and $-\Delta u + Vu = 0$.

Applications to control theory

Take $f \in C^1(\mathbb{R};\mathbb{R})$ such that $f(0)=0$ and consider the elliptic control problem

$$\begin{cases} -\Delta y + f(y) = F + h1_\omega & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{(LaplaceNL)}$$

where $F \in L^\infty(\Omega)$.

Goal: Find a pair $(y,h) \in [H_0^1(\Omega) \cap L^\infty(\Omega)] \times L^\infty(\omega)$ satisfying (LaplaceNL).

Theorem (Ervedoza, Le Balc'h (2021))

- *(Positive result) Assume that* $f(s) = o_{+\infty}(|s| \log^p(1+|s|))$, $p < 2$, *then*

  $\forall F \in L^\infty(\Omega)$, $\exists (y,h) \in [H_0^1(\Omega) \cap L^\infty(\Omega)] \times L^\infty(\omega)$ *satisfying* (LaplaceNL).

- *(Negative result) Take* $f(s) = |s| \log^p(1+|s|)$, $p > 2$. *Then,*

  $\exists F \in L^\infty(\Omega)$, $\forall h \in L^\infty(\omega)$, (LaplaceNL) *has no solution* $y \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

- Negative result is based on the localized eigenfunction method (OK in N-d).
- Positive result is true in 1-d, with $p = 2$.
- Positive result is true in N-d, with $p = 3/2$.

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Strategy of the proof of the main result

Theorem (Ervedoza, Le Balc'h (2021))

Let $\Omega \subset \mathbb{R}^2$ and $\omega \subset \Omega$.
*For every **real-valued potential** $V \in L^\infty(\Omega; \mathbb{R})$ and function $u \in H^2(\Omega) \cap H_0^1(\Omega)$,*

$$\|u\|_{H^2(\Omega)} \leq C_V \left(\|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)} \right), \qquad \text{(Observability)}$$

where $C_V > 0$ is given by $\boxed{C_V = \exp\left(C(\Omega, \omega) \left(1 + \|V\|_\infty^{1/2} \log^{1/2}(\|V\|_\infty) \right) \right).}$

The proof is divided into five main steps:

1. Reduction to concentric balls.
2. Reduction to a weak observability inequality for smooth functions.
3. Construction of a multiplier in a perforated domain.
4. A quasiconformal change of variable to transform the divergence equation.
5. Carleman estimate conjugated with Harnack inequalities.

3, 4, 5 are crucially inspired by Logunov and al (2020).

Reduction to concentric balls (Step 1)

Up to a translation: $0 \in \omega$.
Smooth Riemann mapping theorem: $\exists \varphi : \overline{\Omega} \to \overline{B(0,1)}$, one-to-one, $\varphi(0) = 0$,

$$\varphi \in \mathcal{O}(\Omega),\ \varphi \in C^\infty(\overline{\Omega}),\ 0 < c \leq |\varphi'| \leq C \text{ in } \overline{\Omega}.$$

Open mapping theorem: φ maps ω to a neighborhood of 0.
Cauchy-Riemann's equation: Set $\hat{u} := u \circ \varphi^{-1}$, we have

$$\Delta \hat{u}(x) = |\nabla \Re(\varphi^{-1})|^2 \Delta u(\varphi^{-1}(x))\ \forall x \in B(0,1).$$

So setting $\hat{V} = |\nabla \Re(\varphi^{-1})|^2 V$, we obtain

$$-\Delta \hat{u} + \hat{V}\hat{u} = |\nabla \Re(\varphi^{-1})|^2(-\Delta u(\varphi^{-1}) + Vu(\varphi^{-1})) \in L^2(B(0,1)).$$

$$\text{WLOG, } \boxed{\omega = B(0,r) \subset \Omega = B(0,R), \quad 0 < r < R.}$$

A weak inequality for smooth functions (Step 2)

For every $V \in L^\infty(\Omega;\mathbb{R})$ and $u \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\text{LHS} = \|u\|_{H^2(\Omega)} \leq C_V \left(\|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)} \right) = \text{RHS}. \quad \text{(Observability)}$$

Sobolev embeddings and local elliptic regularity: Take $\omega_0 \subset\subset \omega$, we have

$$\|u\|_{L^\infty(\omega_0)} \leq C \|u\|_{H^2(\omega_0)} \leq C \left(\|-\Delta u\|_{L^2(\omega)} + \|u\|_{L^2(\omega)} \right) \leq \text{RHS}.$$

Global elliptic regularity:

$$\text{LHS} \leq C \|\Delta u\|_{L^2(\Omega)} \leq C_V \left(\|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^\infty(\Omega)} \right).$$

Density argument: The set

$\mathcal{U} = \{u \in C^\infty(\overline{\Omega};\mathbb{R})\ ;\ 0$ is a regular value of u and u is a non-zero constant on $\partial\Omega\}$

is dense in $H^2(\Omega) \cap H_0^1(\Omega)$ for the $H^2(\Omega)$-topology (Sard's lemma).

WLOG, one has to prove that for every $V \in L^\infty(\Omega;\mathbb{R})$ and $u \in \mathcal{U}$,

$$\boxed{\|u\|_{L^\infty(\Omega)} \leq C_V \left(\|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^\infty(\omega)} \right).}$$

Decomposition of the nodal set (Step 3)

We introduce the *nodal set* of $u \in \mathcal{U}$:

$$Z := \{x \in \Omega \; ; \; u(x) = 0\}.$$

Recall that 0 is a regular value of u and $u \neq 0$ on $\partial\Omega$, so

$$Z = \cup_{i \in I} \mathcal{C}_i, \; \mathcal{C}_i \text{ are disjoint smooth Jordan curves that do not intersect } \partial\Omega.$$

Take $\varepsilon > 0$, a small parameter that will be fixed later.

$$\forall x_0 \in \mathcal{C}_i, \; \forall r \in (0, \varepsilon], \; \partial B(x_0, r) \cap \mathcal{C}_i \neq \emptyset. \tag{P-ε}$$

We then decompose

$$Z = Z_\varepsilon^1 \cup Z_\varepsilon^2, \; \forall \mathcal{C}_i \subset Z_\varepsilon^1, \; \mathcal{C}_i \text{ satisfies (P-}\varepsilon\text{)}, \; \forall \mathcal{C}_i \subset Z_\varepsilon^2, \; \mathcal{C}_i \text{ does not satisfy (P-}\varepsilon\text{)}.$$

Picture of the nodal set (Step 3)

$$\forall x_0 \in \mathcal{C}_i,\ \forall r \in (0,\varepsilon],\ \partial B(x_0,r) \cap \mathcal{C}_i \neq \emptyset. \tag{P-ε}$$

$Z = Z_\varepsilon^1 \cup Z_\varepsilon^2$, $\forall \mathcal{C}_i \subset Z_\varepsilon^1$, $\mathcal{C}_i$ satisfies (P-ε), $\forall \mathcal{C}_i \subset Z_\varepsilon^2$, $\mathcal{C}_i$ does not satisfy (P-ε).

Pointwise estimates (Step 3)

$$\forall x_0 \in \mathcal{C}_i,\ \forall r \in (0,\varepsilon],\ \partial B(x_0,r) \cap \mathcal{C}_i \neq \emptyset. \tag{P-ε}$$

$Z = Z_\varepsilon^1 \cup Z_\varepsilon^2$, $\forall \mathcal{C}_i \subset Z_\varepsilon^1$, $\mathcal{C}_i$ satisfies (P-ε), $\forall \mathcal{C}_i \subset Z_\varepsilon^2$, $\mathcal{C}_i$ does not satisfy (P-ε).

$$-\Delta u + Vu = f \in L^2(\Omega).$$

Proposition

There exist $C > 0$ and $c > 0$ such that for every $\boxed{\varepsilon^2 \left\|V\right\|_{L^\infty(\Omega)} \leq c,}$

- $\forall \mathcal{C} \subset Z_\varepsilon^2$, $\|u\|_{H_0^1(\mathcal{O}_\mathcal{C})} + \|u\|_{L^\infty(\mathcal{O}_\mathcal{C})} \leq C\|f\|_{L^2(\mathcal{O}_\mathcal{C})}$,
- $\forall \mathcal{O} \subset \Omega \setminus Z_\varepsilon^1$, $\left(\forall x \in \mathcal{O},\ u(x) \geq -C\left\|f\right\|_{L^2(\Omega)}\right)$ *or* $\left(\forall x \in \mathcal{O},\ u(x) \leq C\left\|f\right\|_{L^2(\Omega)}\right)$

Construction of the perforated domain (Step 3)

Lemma

There exists $C_0 \geq 2^{14}$ s.t. for every $\varepsilon > 0$, there exist finitely many closed disks of radius ε, whose union is denoted by F_ε satisfying the following properties:

- *these disks are $C_0\varepsilon$-separated from each other, from Z_ε^1, from $\partial\Omega$, from $x_{\max}$ and from 0,*
- *the set $Z_\varepsilon^1 \cup F_\varepsilon \cup \partial\Omega$ is a $C_0\varepsilon$-net in Ω,*
- *the Poincaré constant $\boxed{C_P(\Omega_\varepsilon) \leq C\varepsilon}$ with $\Omega_\varepsilon = \Omega \setminus (Z_\varepsilon^1 \cup F_\varepsilon)$.*

A positive multiplier in the perforated domain (Step 3)

Recall that $C_P(\Omega_\varepsilon) \leq C\varepsilon$.

Lemma

There exist $C > 0$ *and* $c > 0$ *such that for every* $\varepsilon > 0$*, with* $\boxed{\varepsilon^2 \|V\|_{L^\infty(\Omega)} \leq c,}$ *there exists* $\varphi \in H^1(\Omega_\varepsilon)$ *such that*

- $-\Delta\varphi + V\varphi = 0 \quad$ *in* Ω_ε,
- $\tilde{\varphi} := \varphi - 1 \in H_0^1(\Omega_\varepsilon)$ *and* $\|\tilde{\varphi}\|_\infty \leq C\varepsilon^2 \|V\|_{L^\infty(\Omega)}$.

Summary of Step 3

The main steps are the following

- Decomposition of the nodal set: $Z := \{x \in \Omega \ ; \ u(x) = 0\} = Z_1^\varepsilon \cup Z_2^\varepsilon$,
- Punctual estimate: $\forall \mathcal{O} \subset \Omega \setminus Z_\varepsilon^1$,
  $\left(\forall x \in \mathcal{O},\ u(x) \geq -C \|f\|_{L^2(\Omega)}\right)$ or $\left(\forall x \in \mathcal{O},\ u(x) \leq C \|f\|_{L^2(\Omega)}\right)$
- Perforation of the domain: $\Omega_\varepsilon = \Omega \setminus (Z_\varepsilon^1 \cup F_\varepsilon) \Rightarrow C_P(\Omega_\varepsilon) \leq C\varepsilon$,
- First choice of ε: $\varepsilon^2 \|V\|_{L^\infty(\Omega)} \leq c$,
- Construction of the multiplier:
  $-\Delta\varphi + V\varphi = 0$ in Ω_ε $\tilde{\varphi} := \varphi - 1 \in H_0^1(\Omega_\varepsilon)$ and $\|\tilde{\varphi}\|_\infty \leq C\varepsilon^2 \|V\|_{L^\infty(\Omega)}$.

Reduction to a homogeneous divergence equation (Step 4)

Recall that $-\Delta u + Vu = f$ in Ω and $-\Delta \varphi + V\varphi = 0$ in $\Omega_\varepsilon = \Omega \setminus (Z_\varepsilon^1 \cup F_\varepsilon)$.

The function $v = u/\varphi$ satisfies - $\nabla \cdot (\varphi^2 \nabla v) = f\varphi$ in $\Omega_\varepsilon' = \Omega \setminus F_\varepsilon$.

Lax-Milgram: $\exists ! \psi \in H_0^1(\Omega_\varepsilon')$, $-\nabla \cdot (\varphi^2 \nabla \psi) = f\varphi$ in Ω_ε', $\|\psi\|_{L^\infty(\Omega_\varepsilon')} \leq C \|f\|_{L^2(\Omega)}$.

Lemma

The function $\hat{v} = v - \psi$ satisfies $\nabla \cdot (\varphi^2 \nabla \hat{v}) = 0$ in Ω_ε'.
There exists $C > 0$ such that for every disk $D \subset F_\varepsilon$,

$$\left(\forall x \in \Omega_\varepsilon',\ d(x, D) \leq C_0 \varepsilon,\ \hat{v}(x) \geq -C \|f\|_{L^2(\Omega)}\right)$$

or

$$\left(\forall x \in \Omega_\varepsilon',\ d(x, D) \leq C_0 \varepsilon,\ \hat{v}(x) \leq C \|f\|_{L^2(\Omega)}\right).$$

Quasiconformal change of variable (Step 4)

Recall that $\nabla \cdot (\varphi^2 \nabla \hat{v}) = 0$ in Ω'_ε.

Lemma

There exists a K-quasiconformal mapping $L : \Omega \to \Omega$, $L(0) = 0$, with K s.t.

$$\boxed{1 \leq K \leq 1 + C\varepsilon^2 \|V\|_{L^\infty(\Omega)},}$$

such that $h := \hat{v} \circ L^{-1}$ satisfies $\Delta h = 0$ in $L(\Omega'_\varepsilon)$.

Poincaré's lemma for divergence free vector: $\exists \tilde{v}$ s.t. $\varphi^2 \hat{v}_x = \tilde{v}_y$ and $\varphi^2 \hat{v}_y = -\tilde{v}_x$.
Beltrami's equation: $w := \hat{v} + i\tilde{v}$ satisfies $\frac{\partial w}{\partial \bar{z}} = \mu \frac{\partial w}{\partial z}$ with $\mu = \frac{1-\varphi^2}{1+\varphi^2} \frac{\hat{v}_x + i\hat{v}_y}{\hat{v}_x - i\hat{v}_y}$.
Estimate on φ: $|\mu| \leq \frac{1-\varphi^2}{1+\varphi^2} \leq C\varepsilon^2 \|V\|_{L^\infty(\Omega)}$.
Beltrami: $\exists \psi$ K-quasiconformal, $\psi(0) = 0$, $\frac{\partial \psi}{\partial \bar{z}} = \mu \frac{\partial \psi}{\partial z}$ in $\mathbb{C}$, $1 \leq K \leq 1 + C\varepsilon^2 \|V\|_\infty$.
Stoilow factorization theorem: $\exists W$ hol. s.t. $w = W \circ \psi$ so $\hat{v} \circ \psi^{-1} = \Re(W)$ is harmonic.
Riemann mapping theorem: $\psi(\Omega)$ is simply connected so $\exists \alpha : \psi(\Omega) \to \Omega$ and $\alpha(0) = 0$.
Set $L := \alpha \circ \psi$, K-quasiconformal, $L(0) = 0$ maps Ω to Ω and $h = \hat{v} \circ L^{-1}$ is harmonic.

Image of the perforated domain by L (Step 4)

Recall that $L : \Omega \to \Omega$, $L(0) = 0$, K-quasiconformal, $1 \leq K \leq 1 + C\varepsilon^2 \|V\|_{L^\infty}$.

Lemma

There exist a positive constant $c > 0$ such that for every $\varepsilon > 0$ satisfying

$$\boxed{\varepsilon \leq c \|V\|_{L^\infty(\Omega)}^{-1/2} \log^{-1/2}\left(\|V\|_{L^\infty(\Omega)}\right),}$$

- *$L(\omega)$ contains $B(0, r/32)$,*
- *$\forall D \subset F_\varepsilon$, $L(D) \subset D'$, a disk of size $\varepsilon' = 32\varepsilon$,*
- *these disks are $C_0\varepsilon/32$-separated from each other, from $L(Z_\varepsilon^1)$, from $\partial\Omega$ $(= L(\partial\Omega))$, from $L(x_{\max})$ and from 0.*

The main ingredient is Mori's theorem: $\frac{1}{16}\left|\frac{z_1 - z_2}{R}\right|^K \leq \frac{|L(z_1) - L(z_2)|}{R} \leq 16\left|\frac{z_1 - z_2}{R}\right|^{1/K}$.

Summary of Step 4

By the change of variable L, the equation $\nabla \cdot (\varphi^2 \nabla \hat{v}) = 0$ in Ω'_ε becomes

$$\boxed{\Delta h = 0 \text{ in } L(\Omega'_\varepsilon).}$$

Moreover, we have for $\varepsilon \leq c \left\| V \right\|_{L^\infty(\Omega)}^{-1/2} \log^{-1/2} \left(\left\| V \right\|_{L^\infty(\Omega)} \right)$,

- $L(\omega) \sim B(0, r/32)$,
- $L(\Omega_\varepsilon) \sim \Omega \setminus \cup_{i \in I} D(x'_i, \varepsilon')$,
- $D(x'_i, \varepsilon')$ are $16\varepsilon'$-separated from each other, from $L(Z^1_\varepsilon)$, from $\partial\Omega$, from $L(x_{\max})$ and from 0,
- h is constant on $\partial\Omega$,
- punctual estimate of h near the disks $D' = D(x'_i, \varepsilon')$,

$$\left(\forall x \in \Omega'_\varepsilon,\ \mathrm{d}(x, D') \leq 16\varepsilon,\ h(x) \geq -C \left\| f \right\|_{L^2(\Omega)} \right)$$

or

$$\left(\forall x \in \Omega'_\varepsilon,\ \mathrm{d}(x, D) \leq 16\varepsilon,\ h(x) \leq C \left\| f \right\|_{L^2(\Omega)} \right).$$

A Carleman estimate (Step 5)

Set $\eta(x) = \sqrt{R^2+1} - \sqrt{|x|^2+1}$ and for $\lambda \geq 1$, $\xi(x) = e^{\lambda \eta(x)}$.

Theorem

There exist $\lambda_1 \geq 1$, $s_1 > 0$, $C_1 > 0$ such that for every $\lambda \geq \lambda_1$, $s \geq s_1$,

$$\lambda^4 \int_\Omega e^{2s\xi}(s\xi)^3|\varphi|^2 dx \leq C_1 \left(\int_\Omega e^{2s\xi}|\Delta\varphi|^2 dx + \lambda^4 \int_\omega e^{2s\xi}(s\xi)^3|\varphi|^2 dx \right),$$

where $\varphi \in H^2(B(0,R))$, constant on $\partial B(0,R)$.

Carleman estimate in the perforated domain (Step 5)

Cut-off near the disks: $\chi \equiv 0$ on $B(0,3)$ $\chi \equiv 1$ on $\mathbb{R}^2 \setminus B(0,4)$, and set

$$\varphi(x) = \begin{cases} h(x) \prod_{i \in I} \chi\left(\frac{x - x_i'}{\varepsilon'}\right) \text{ for } x \in L(\Omega_\varepsilon'), \\ 0 \text{ for } x \in \Omega \setminus L(\Omega_\varepsilon'). \end{cases}$$

Carleman estimate: $s^3 \int_\Omega e^{2s\xi} |\varphi|^2 dx \leq C \left(\int_\Omega e^{2s\xi} |\Delta \varphi|^2 dx + s^3 \int_\omega e^{2s\xi} |\varphi|^2 dx \right)$.

$\Delta h = 0$: $\int_\Omega e^{2s\xi} |\Delta\varphi|^2 \leq C \sum \int_{B(x_i', 4\varepsilon') \setminus B(x_i', 3\varepsilon')} \left(\frac{1}{|\varepsilon'|^4} |h|^2 + \frac{1}{|\varepsilon'|^2} |\nabla h|^2 \right) e^{2s\xi}$

Harnack's inequality: $\int_{B(x_i', 4\varepsilon') \setminus B(x_i', 3\varepsilon'))} \frac{1}{|\varepsilon'|^2} |\nabla h|^2 e^{2s\varphi} \leq C \int_{B(x_i', 4\varepsilon') \setminus B(x_i', 3\varepsilon'))} \frac{1}{|\varepsilon'|^4} |h|^2 e^{2s\xi}$.

$$\boxed{s^3 \sum \int_{B(x_i', 8\varepsilon') \setminus B(x_i', 4\varepsilon')} |h|^2 e^{2s\xi} \leq C \left(\sum \int_{B(x_i', 4\varepsilon') \setminus B(x_i', 3\varepsilon')} \frac{1}{|\varepsilon'|^4} |h|^2 e^{2s\xi} + s^3 \int_\omega e^{2s\xi} |h|^2 dx \right).}$$

Weight property: $\int_{B(x_i', 4\varepsilon') \setminus B(x_i', 3\varepsilon')} e^{2s\xi} \leq e^{-2s\varepsilon'} \int_{B(x_i', 8\varepsilon') \setminus B(x_i', 4\varepsilon')} e^{2s\xi}$.

Harnack's inequality: $\int_{B(x_i', 4\varepsilon') \setminus B(x_i', 3\varepsilon')} \frac{1}{|\varepsilon'|^4} |h|^2 e^{2s\xi} \leq e^{-2s\varepsilon'} \int_{B(x_i', 8\varepsilon') \setminus B(x_i', 4\varepsilon')} |h|^2 e^{2s\xi}$.

For $s\varepsilon'^4 e^{2s\varepsilon'} \geq C$, i.e. $s \geq C\varepsilon^{-1} \log(\varepsilon^{-1})$ we obtain $\boxed{s^3 \int_\Omega e^{2s\xi} |\varphi|^2 \leq Cs^3 \int_\omega e^{2s\xi} |h|^2}$.

Summary of Step 5

For $\varphi = h$, far from the disks, we obtain for $s \geq C\varepsilon^{-1}\log(\varepsilon^{-1})$,

$$\boxed{s^3 \int_\Omega e^{2s\xi}|\varphi|^2 \leq Cs^3 \int_\omega e^{2s\xi}|h|^2.}$$

How can we finish the proof?

- $\varepsilon \leq c\,\|V\|_\infty^{-1/2}\log^{-1/2}(\|V\|_\infty) \Rightarrow \boxed{s \geq \|V\|_\infty^{1/2}\log^{3/2}(\|V\|_\infty).}$
- $|\varphi|_{L^2(\Omega)} \leq \exp(C\|V\|_\infty^{1/2}\log^{3/2}(\|V\|_\infty))(|f|_{L^2(\Omega)} + |h|_{L^2(\omega)})$.
- Mean-value: $|h(L(x_{max})| \leq \exp(C\|V\|_\infty^{1/2}\log^{3/2}(\|V\|_\infty))(|f|_{L^2(\Omega)} + |h|_{L^2(\omega)})$.
- Coming back: $|u(x_{max})| \leq \exp(C\|V\|_\infty^{1/2}\log^{3/2}(\|V\|_\infty))(|f|_{L^2(\Omega)} + |u|_{L^\infty(\omega)})$.

Rmk: To obtain $\exp(C\|V\|_\infty^{1/2}\log^{1/2}(\|V\|_\infty))$, we have to use the antisymmetric term in the Carleman estimate.

Conclusion

In brief, recall the story

- Null-controllability of $\partial_t y - \Delta y + a(t,x)y = h1_\omega$ Cost $= \exp\left(C\left(\|a\|_\infty^{2/3}\right)\right)$.
- Observability of $-\partial_t \varphi - \Delta \varphi + a\varphi = 0$, $|\varphi(0)|_{L^2} \leq \exp\left(C\left(\|a\|_\infty^{2/3}\right)\right)|\varphi|_{L^2(q_T)}$.
- Global null-controllability of $\partial_t y - \Delta y + |y|\log^p(1+|y|) = h1_\omega$, $p < 3/2$.
- Blow-up of $\partial_t y - \Delta y + |y|\log^p(1+|y|) = h1_\omega$, $p > 2$.
- Optimality of $\|a\|_\infty^{2/3}$ by Meshkov's counterexample for $a \in L^\infty(Q_T;\mathbb{C})$.
- Landis conjecture: $-\Delta u + V(x)u = 0$, $|u(x)| \leq \exp(-|x|^{1+\varepsilon}) \Rightarrow u \equiv 0$.
- False in 2-d for $V \in L^\infty(\mathbb{R}^2;\mathbb{C})$ by Meshkov's counterexample.
- True in 2-d for $V \in L^\infty(\mathbb{R}^2;\mathbb{R})$ by Logunov and al.

What's new? In 2-d,

- $\boxed{\|u\|_{H^2(\Omega)} \leq \exp\left(C\left(\|V\|_\infty^{1/2}\log^{1/2}\left(\|V\|_\infty\right)\right)\right)\left(\|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)}\right),}$
- Existence of a pair (y,h) s.t. $-\Delta y + |y|\log^p(1+|y|) = F + h1_\omega$, $p < 2$,
- No existence of a pair $-\Delta y + |y|\log^p(1+|y|) = F + h1_\omega$, $p > 2$.

Perspectives: Ω connected, optimality of $\|V\|_\infty^{1/2}\log^{1/2}\left(\|V\|_\infty\right)$, parabolic equations with spatial potential in 2-d, multi-d case...