Controllability of a string submitted to unilateral constraint

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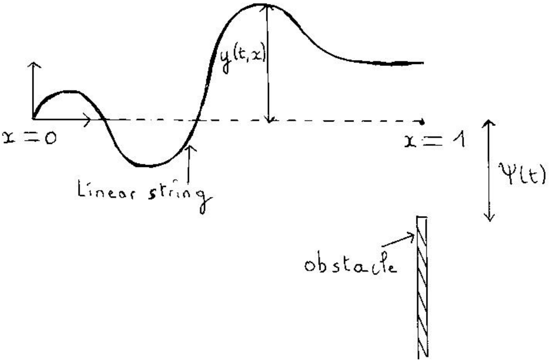
(with Farid Ammar-Khodja (Besançon) and Sorin Micu (Craiova))

Problem Statement

WE CONSIDER A LINEAR STRING OF LENGTH ONE AND DENSITY ONE, FIXED AT $x = 0$ AND SUBMITTED TO AN INITIAL EXCITATION (y^0, y^1) AT TIME $t = 0$. WE ASSUME THAT THE STRING IS SUBMITTED TO A UNILATERAL OBSTACLE ψ, TIME DEPENDENT, AT THE RIGHT EXTREMITY $x = 1$.



WE WANT TO ACT ON THE LEFT EXTREMITY, I.E. AT $x = 0$, IN ORDER TO STABILIZE THE STRING, AFTER A FINITE TIME T LARGE ENOUGH. INTUITIVELY, AT LEAST WHEN THE OBSTACLE IS TIME INDEPENDENT, THIS SHOULD BE POSSIBLE, BECAUSE THE OBSTACLE DOES NOT ADD ENERGY TO THE STRING !!!

Modelization of the Problem

Let $T > 0$ and $Q_T = (0, T) \times (0, 1)$. We consider the system

$$\begin{cases} y'' - y_{xx} = 0, & (t,x) \in Q_T, \\ y(t,0) = \mathbf{u(t)}, & t \in (0,T), \\ \mathbf{y(t,1) \geq \psi(t),\ \ y_x(t,1) \geq 0,\ \ (y(t,1) - \psi(t))y_x(t,1) = 0}, & t \in (0,T), \\ y(0,x) = y^0(x), \quad y'(0,x) = y^1(x), & x \in (0,1) \end{cases} \tag{1}$$

- u : Dirichlet control function at $x = 0$;
- ψ : time dependent obstacle function at $x = 1$ models by the classical Signorini's conditions.

Problem

For any T fixed large enough and any (y^0, y^1) in a given space, assuming that $\psi(T) \leq 0$, does there exist a Dirichlet control u which drives the corresponding solution of (1) to rest, i.e.

$$y(T) = y'(T) = 0, \quad in \quad (0,1) \quad ?$$

$\Longrightarrow$ NONLINEAR null control problem !

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Related litterature

ASYMPTOTIC BEHAVIOR - STABILIZATION

Munoz-Rivera and co-workers (Naso (JMMA,2008) , Oquendo (Funk. Ekvac, 1999,2001), Nakao (JMMA,2001)): Thermo-elastic beams with an internal dissipation, Visco-elastic material.

EXISTENCE AND/OR UNIQUENESS FOR HYPERBOLIC SYSTEM WITH UNILATERAL OBSTACLE

Schatzman (JMAA, 1980). In dimension $N = 1$ with a concave interior obstacle

Lebeau-Schatzman (JDE, 1984). In dimension $N \geq 1$ with $u = \psi = 0$. Existence and uniqueness result for sufficiently smooth initial data.

Kim (CPDE, 1989). Existence in dimension $N \geq 1$.

NUMERICAL ANALYSIS AN APPROXIMATIONS

Bercovier-Schatzmann (Math. Comp, 1989): Numerical analysis of the system.

Dumont-Paoli (M2AN, 2006). Approximation in the case of several obstacles.

Our controllability result

- Constructive proof based on a fixed point argument and the characteristic method

Theorem (Ammar-Khodja/Micu/AM'10)

Let $T > 2$ and $\psi \in H^1(0,T)$ with the property that there exists $\tilde{T} \in (2, min(3,T))$ such that $\psi(t) \leq 0$ for any $t \in [\tilde{T}, T]$. For any $(y^0, y^1) \in H^1(0,1) \times L^2(0,1)$ with

$$\psi(0) \leq y^0(1)$$

there exists a control function $u \in H^1(0,T)$ such that (1) admits a unique solution $y \in C\left([0,T], H^1(0,1)\right) \cap C^1\left([0,T], L^2(0,1)\right)$ satisfying $y(T) = y'(T) = 0$ in $(0,1)$.

F. Ammar-Khodja, S. Micu, A. Münch, *Controllability of a string submitted to unilateral constraint*, Ann. I. H. Poincaré - AN 27 (2010) 1097-1119.

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- Step 1 Consider, for any $u, f \in H^1(0,T)$, and any $(\phi^0, \phi^1) \in H^1(0,1) \times L^2(0,1)$, the linear problem

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  Upon the compatibility condition

$$u(0) = \phi^0(0), \quad f(0) = \phi^0(1),$$

  system (2) is well-posed with $\phi \in C([0,T], H^1(0,1)) \cap C^1([0,T], L^2(0,1))$ and

$$\|(\phi(t), \phi'(t))\|_{H^1(0,1)\times L^2(0,1)} \leq C\Big(\|(\phi^0, \phi^1)\|_{H^1(0,1)\times L^2(0,1)} + \|(u,f)\|_{H^1(0,T)^2} \Big)$$

- Step 2 Compute explicitly for any $T > 2$, the set of controls

$$U = U(\phi^0, \phi^1, f) = \{u; \quad (\phi, \phi')(T) = (0,0)\}$$

- Step 3 Compute, for any $u \in U$ the Dirichlet-Neumann map A defined

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- Step 4 Observe then that the Signorini's conditions become equivalent to

$$\left\{\begin{array}{ll} f-\psi \geq 0, & t\in(0,T) \\ A(\phi^0,\phi^1,u,f)\geq 0, & t\in(0,T) \\ (f-\psi)A(\phi^0,\phi^1,u,f)=0, & t\in(0,T). \end{array}\right. \tag{3}$$

- Step 5 Find at least one $u\in U$ for ϕ and one f such that $\phi(\cdot,1)=f$ AND f solution of the inequation (3) posed at $x=1$.

In the sequel, we work with $T\in(2,3)$.

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Direct versus penalty approach

1. The direct approach consists in solving directly the inequation

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2. One may also use a penalty method which consists in relaxing the Signorini's inequations by the equation

$$y_{\epsilon,x}(1,t) = \epsilon^{-1}[y_\epsilon(\cdot,1)-\psi]^-, \quad t \in (0,T)$$

where $[s]^- = -min(0,s)$ and $0 < \epsilon << 1$. Then (4) becomes

$$A(\phi^0,\phi^1,u_\epsilon,f_\epsilon) = \epsilon^{-1}[f_\epsilon-\psi]^-, \quad t \in (0,T)$$

and one has to find a couple (u_ϵ, f_ϵ) uniformly bounded w.r.t. ϵ.

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Explicit computation of the admissible control set U

We define the space

$$\mathbb{H} = \left\{ \left(\left(\phi^0, \phi^1\right), (u, f) \right) \in H^1(0,1) \times L^2(0,1) \times H^1(0,T)^2,\ u(0) = \phi^0(0),\ f(0) = \phi^0(1) \right\}.$$

Setting

$$p = \phi' - \phi_x,\ q = \phi' + \phi_x, \tag{5}$$

it follows that the system in ϕ is equivalent to

$$\begin{cases} p' + p_x = q' - q_x = 0, & (t,x) \in Q_T, \\ (p+q)(\cdot,0) = 2u', & t \in (0,T), \\ (p+q)(\cdot,1) = 2f' & t \in (0,T), \\ p^0 = \phi^1 - \phi^0_x,\ q^0 = \phi^1 + \phi^0_x, & x \in (0,1). \end{cases} \tag{6}$$

If $\left(\left(p^0, q^0\right), (u,f)\right) \in L^2(0,1)^2 \times H^1(0,T)^2$ system (6) admits a unique generalized solution $(p,q) \in C\left([0,T], L^2(0,1)^2\right)$. In view of (5), this solution corresponds to a solution ϕ of (2) satisfying

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U via the characteristic method

Proposition

Let $T \in (2,3)$ *and assume that* $\left((\phi^0,\phi^1),(u,f)\right) \in \mathbb{H}$*. Then the solution* (p,q) *of (6) satisfies* $(p,q)(T) = 0$ *in* $(0,1)$ *if and only if* $\left((\phi^0,\phi^1),(u,f)\right)$ *satisfies*

$$\begin{cases} u'(t) = f'(t+1) + \frac{1}{2}q^0(t) & \text{if } T-2 < t < 1 \\ u'(t) = f'(t+1) + f'(t-1) - \frac{1}{2}p^0(2-t) & \text{if } 1 < t < T-1 \\ u'(t) = f'(t-1) - \frac{1}{2}p^0(2-t) & \text{if } T-1 < t < 2 \\ u'(t) + u'(t-2) = f'(t-1) + \frac{1}{2}q^0(t-2) & \text{if } 2 < t < T. \end{cases} \quad (7)$$

Remark

u is "free" on $(0, T-2)$.

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The Control Dirichlet-to-Neumann map

$$\mathbb{H}_c = \left\{ \left(\phi^0, \phi^1, u, f\right) \in \mathbb{H} \mid \text{(7) is verified and } u(T) = f(T) = 0 \right\}.$$

Lemma

Let $T \in (2,3)$ and $(\phi^0, \phi^1, u, f) \in \mathbb{H}_c$. The Control Dirichlet-to-Neumann map $A_c : \mathbb{H}_c \to L^2(0,T)$ defined by

$$A_c(\phi^0, \phi^1, u, f) = \phi_x(\cdot, 1) \quad \left(= \frac{q(t,1) - p(t,1)}{2} \right)$$

is given by

$$A_c(\phi^0, \phi^1, u, f)(t) = \begin{cases} f'(t) - p^0(1-t) & 0 < t < 1 \\ f'(t) - 2u'(t-1) - q^0(t-1) & 1 < t < T-1 \\ -f'(t) & T-1 < t < T \end{cases} \tag{8}$$

where $p^0 = \phi^1 - \phi^0_x$ and $q^0 = \phi^1 + \phi^0_x$.

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Note that the expression of $A_c(\phi^0, \phi^1, u, f)$ in (8) involves only the part of u defined on $(0, T-2)$, i.e. the "free" part of u.

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Lemma

Let $T \in (2,3)$ and $(\phi^0, \phi^1, u, f) \in \mathbb{H}_c$. The Control Dirichlet-to-Neumann map $A_c : \mathbb{H}_c \to L^2(0,T)$ defined by

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$$A_c(\phi^0, \phi^1, u, f)(t) = \begin{cases} f'(t) - p^0(1-t) & 0 < t < 1 \\ f'(t) - 2u'(t-1) - q^0(t-1) & 1 < t < T-1 \\ -f'(t) & T-1 < t < T \end{cases} \tag{8}$$

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Proposition

Problem (1) is null controllable in time $T \in (2,3)$ *if and only if, for any* $(y^0, y^1) \in H^1(0,1) \times L^2(0,1)$, $\psi \in H^1(0,T)$ *with the condtions*

$$\psi(0) \leq y^0(1), \quad \psi(T) \leq 0,$$

there exist a control $u \in H^1(0,T)$ *and a function* $f \in H^1(0,T)$ *such that*

1. $(y^0, y^1, u, f) \in \mathbb{H}_c$.
2. y *is the solution of (2) with nonhomogeneous terms* $(u,f) \in (H^1(0,T))^2$ *and initial data* (y^0, y^1).
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Characterisation of the controls for (y, y')

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Resolution of the Signorini's conditions

The Signorini's conditions are :

$$\begin{cases} f-\psi \geq 0, & t\in(0,T) \\ A_c(y^0,y^1,u,f)\geq 0, & t\in(0,T) \\ (f-\psi)A_c(y^0,y^1,u,f)=0, & t\in(0,T). \end{cases} \tag{9}$$

- On $(0, T-1)$, problem (9) writes:

$$\begin{cases} f-\psi \geq 0, \\ f'-v \geq 0, \\ (f-\psi)\,(f'-v)=0 \\ f(0)=y^0(1) \end{cases}, \quad (0,T-1), \tag{10}$$

  where

$$v(t)=\begin{cases} p^0(1-t) & 0<t<1 \\ 2u'(t-1)-q^0(t-1) & 1<t<T-1 \end{cases},$$

- On $(T-1, T)$:

$$\begin{cases} f-\psi \geq 0, \\ f' \leq 0, \\ (f-\psi)\,f'=0, \\ f(T)=0 \end{cases}, \quad (T-1,T), \tag{11}$$

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Lemma (Benilan-Pierre, 1979)

Let $h \in H^1(0,T)$ *and* $\theta_0 \geq h(0)$. *Then the function*

$$\theta(t) = \max\left(\theta_0, \sup_{0\leq s\leq t} h(s)\right), \; t \in [0,T[$$

belongs to $H^1(0,T)$ *and is the unique solution of the problem*

$$\left\{\begin{array}{ll} \theta \geq h & \text{in } (0,T) \\ \theta' \geq 0 & \text{in } (0,T) \\ \theta'\,(\theta - h) = 0 & \text{in } (0,T) \\ \theta(0) = \theta_0. & \end{array}\right. \tag{12}$$

Ph. Benilan, M. Pierre, *Inéquations différentielles ordinaires avec obstacles irréguliers*, Ann. Fac. Sci. Toulouse (1979)-1-8.

$$v(t) = \begin{cases} p^0(1-t) & 0 < t < 1 \\ 2u'(t-1) - q^0(t-1) & 1 < t < T-1 \end{cases}, \tag{13}$$

Proposition

Let $v \in L^2(0, T-1)$ defined by (13) and $V(t) = \int_0^t v(s)ds$. Then the unique solution of (10) is given by

$$f(t) = V(t) + \max\left(y^0(1), \sup_{0\le s\le t} (\psi(s) - V(s))\right), \quad t \in (0, T-1)\,. \tag{14}$$

The unique solution of (11) is given by

$$f(t) = \left[\sup_{t\le s\le T} \psi(s)\right]^+, \quad t \in (T-1, T). \qquad [s]^+ = \max(0, s) \tag{15}$$

$$v(t) = \begin{cases} p^0(1-t) & 0 < t < 1 \\ 2u'(t-1) - q^0(t-1) & 1 < t < T-1 \end{cases}, \qquad V_u(t) = \int_0^t v(s)ds$$

Proposition

There exists u such that the function f given by (14) and (15) belongs to $H^1(0,T)$, i.e.

$$\lim_{t\to(T-1)^-} f(t) = \lim_{t\to(T-1)^+} f(t).$$

This amounts to find $u \in H^1(0, T-2)$ such that

$$V_u(T-1) + \max\left(y^0(1), \sup_{0\le s\le T-1}(\psi(s) - V_u(s))\right) = \left[\sup_{T-1\le s\le T}\psi(s)\right]^+.$$

The answer is positive (u is free in $(0, T-2)$). Once u is fixed in $(0, T-2)$ and f in $(0,T)$, we compute u in $(T-2, T)$ using that $(\phi^0, \phi^1, u, f) \in \mathbb{H}_c$. This shows the controllability of the problem.

$$v(t) = \begin{cases} p^0(1-t) & 0 < t < 1 \\ 2\mathbf{u'}(t-1) - q^0(t-1) & 1 < t < T-1 \end{cases}, \qquad V_{\mathbf{u}}(t) = \int_0^t v(s)ds$$

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Penalty approach

For any $\epsilon > 0$, find y_ϵ solution of the penalized problem

$$\begin{cases} y_\epsilon'' - y_{\epsilon,xx} = 0 & (t,x) \in Q_T, \\ y_\epsilon(t,0) = \boldsymbol{u_\epsilon}(t) & t \in (0,T), \\ y_{\epsilon,x}(t,1) = \epsilon^{-1}[y_\epsilon(t,1) - \psi(t)]^- & t \in (0,T), \\ y_\epsilon(0,x) = y^0(x), \;\; y_\epsilon'(0,x) = y^1(x) & x \in (0,1) \end{cases} \tag{16}$$

where $[y_\epsilon(t,1) - \psi(t)]^- = -\min\{0, y_\epsilon(t,1) - \psi(t)\}$.

Problem

For any T fixed large enough and any (y^0, y^1) in a given space, assuming that $\psi(T) \leq 0$, does there exist a Dirichlet control $\boldsymbol{u_\epsilon}$, uniformly bounded w.r.t. ϵ, which drives the corresponding solution of (16) to rest, i.e.

$$y_\epsilon(T) = y_\epsilon'(T) = 0, \quad \textit{in} \quad (0,1) \quad ?$$

Let $(y^0, y^1) \in H_0^1(0,1) \times L^2(0,1)$, $u_\epsilon, f_\epsilon \in H^1(0,T)$, $u_\epsilon(0) = y^0(0)$, $f_\epsilon(0) = y^0(1)$ and y_ϵ solution of

$$\begin{cases} y_\epsilon'' - y_{\epsilon,xx} = 0 & (t,x) \in Q_T, \\ y_\epsilon(t,0) = \mathbf{u_\epsilon(t)}, \quad y_\epsilon(t,1) = f_\epsilon(t) & t \in (0,T), \\ y_\epsilon(0,x) = y^0(x), \; y_\epsilon'(0,x) = y^1(x) & x \in (0,1) \end{cases} \tag{17}$$

Proposition

System (17) is well-posed and $y_\epsilon \in C([0,T]; H^1(0,1)) \cap C^1([0,T], L^2(0,1))$. Moreover, for any $u_\epsilon \in H^1(0,1)$, there exists a unique $f_\epsilon \in H^1(0,T)$ such that

$$f_\epsilon(0) = y^0(1), \quad A(y^0, y^1, u_\epsilon, f_\epsilon) = \epsilon^{-1}[f_\epsilon - \psi]^- \tag{18}$$

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Let $T \in (2,3)$ and $\psi \in H^1(0,T)$ with $\psi(T) \leq 0$. Problem (16) is null controllable in time T if and only if, for any $(y^0, y^1) \in H^1(0,1) \times L^2(0,1)$ with $y^0(1) \geq \psi(0)$, there exist a control $u_\epsilon \in H^1(0,T)$ and a function $f_\epsilon \in H^1(0,T)$ such that

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Proposition

Let $T \in (2,3)$ and $\psi \in H^1(0,T)$ with $\psi(T) \leq 0$. Problem (16) is null controllable in time T if and only if, for any $(y^0, y^1) \in H^1(0,1) \times L^2(0,1)$ with $y^0(1) \geq \psi(0)$, there exist a control $u_\epsilon \in H^1(0,T)$ and a function $f_\epsilon \in H^1(0,T)$ such that

1. $(y^0, y^1, u_\epsilon, f_\epsilon) \in \mathbb{H}_C$.
2. y_ϵ *is the solution of (2) with nonhomogeneous terms* (u_ϵ, f_ϵ) *and initial data* (y^0, y^1).
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Corollary

Let $T \in (2,3)$ and $(y^0, y^1) \in H^1(0,1) \times L^2(0,1)$, $\psi \in H^1(0,T)$ with $\psi(T) \leq 0$ and $y^0(1) - \psi(0) \geq 0$. Then, the previous problem admits a sequence (u_ϵ, f_ϵ) of solutions such that

$$\begin{aligned} f_\epsilon^2(t) &\leq C, \quad t \in [0,T] \\ \|f_\epsilon\|_{H^1(0,T)} &\leq C \\ \|u_\epsilon\|_{H^1(0,T)} &\leq C \\ \int_0^T \left([f_\epsilon(t) - \psi(t)]^-\right)^2 dt &\leq C\epsilon^2. \end{aligned}$$

$\Longrightarrow$ This allows to pass to the limit w.r.t. to ϵ and get a controllability result.

Numerical experiments : Constant obstacle $\psi(t) = L = -1/10$

$$T = 2.2, \quad (y^0(x), y^1(x)) = \left(x(1-\frac{x}{2}), -3x\right), \quad x \in (0,1)$$

$$u(t) = -\frac{t}{2}\left(2t - 1 + \frac{L}{T-2}\right), \quad t \in (0, T-2)$$

$$f(t) = \begin{cases} t(-3+t) + \frac{1}{2} & 0 \leq t \leq t_L \\ L & t_L \leq t \leq 1 \\ \frac{L(-t+T-1)}{T-2} & 1 \leq t \leq T-1 \\ 0 & T-1 \leq t \leq T \end{cases}$$

with $t_L = (3 - \sqrt{7+4L})/2 \in (0,1)$. The function f then provides u in $(T-2, T)$

$$\begin{cases} u(t) = -\frac{L}{2} + \frac{t}{2} - t^2 & T-2 < t < 1 \\ u(t) = \frac{3}{2} - \frac{L}{2} + \frac{t^2}{2} - \frac{5t}{2} & 1 < t < t_L + 1 \\ u(t) = -3 + \frac{L}{2} + \frac{5t}{2} - \frac{t^2}{2} & t_L + 1 < t < 2 \\ u(t) = -\frac{1}{2}\frac{L(t-T)}{T-2} & 2 < t < T \end{cases}$$

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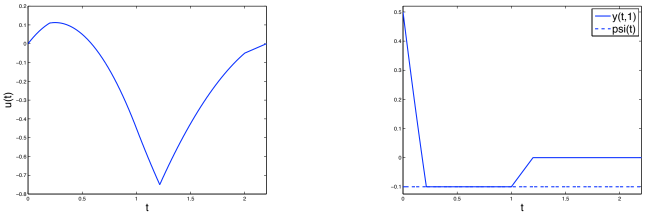


Figure: Control u and corresponding displacement $y(\cdot, 1)$ vs. $t \in [0, T]$.

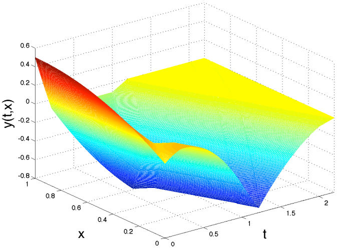
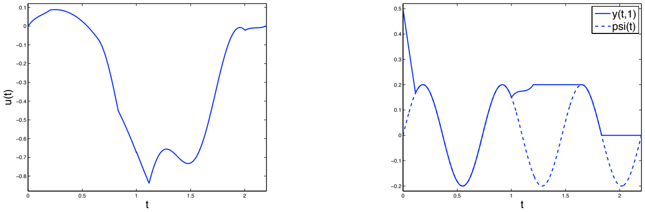


Figure: Controlled solution y in Q_T.

Numerical experiments : Non constant obstacle $\psi(t) = \sin(6\pi t/T)/5$



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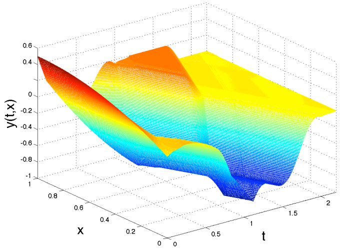

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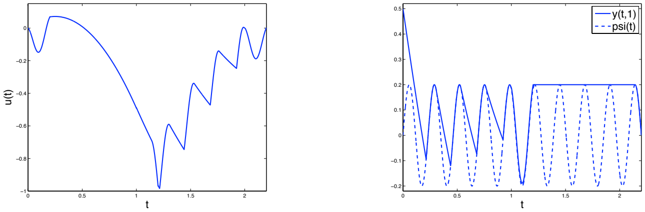

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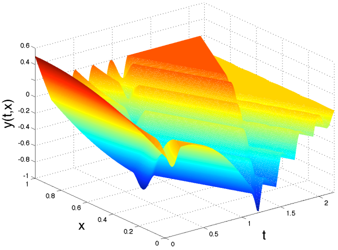

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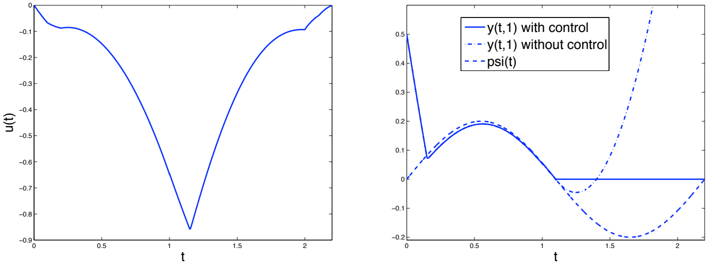

Figure: Penalty method - $\epsilon = 1/200$ - Control u_ϵ (**Left**) and corresponding displacement $y_\epsilon(\cdot, 1)$ (**Right**) vs $t \in [0, T]$.

ϵ	$1/100$	$1/200$	$1/400$	$1/800$
$\|u_\epsilon\|_{L^2(0,T)}$	5.58×10^{-1}	5.53×10^{-1}	5.50×10^{-1}	5.49×10^{-1}
$\|\epsilon^{-1}[y_\epsilon(\cdot,1) - \psi]^-\|_{L^2(0,T)}$	1.837	1.844	1.848	1.850
$\min_{t\in[0,T]}(y_\epsilon(t,1) - \psi(t))$	-3.09×10^{-2}	-1.57×10^{-2}	-7.97×10^{-3}	-4.01×10^{-3}

Table: Penalty approach - $\psi(t) = \sin(2\pi t/T)/5$.

- Same technique for $(y(T), y'(T)) = (z^0, z^1)$ with any $(z^0, z^1) \in H^1(0,1) \times L^2(0,1)$ with $z^0(1) \geq \psi(T)$.
- Same technique for a case of a lower and an upper obstacle :

$$\psi_l(t) \leq y(t,1) \leq \psi_u(t), t \in (0,T), \qquad \psi_l, \psi_u \in H^1(0,T)$$

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$$\begin{cases} y'' - y_{xx} = 0 & (t,x) \in (0,T) \times (0,1), \\ y(t,0) = u(t), \quad y_x(t,1) = f(t,y) & t \in (0,T), \\ y(0,x) = y^0(x), \ y'(0,x) = y^1(x) & x \in (0,1) \end{cases}$$

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- How to obtain the control of minimal $H^1(0,T)$ norm ?
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- Extension to the parabolic situation

$$\begin{cases} \theta_t - \theta_{xx} = 0, & (t,x) \in Q_T, \\ \theta(t,0) = \boldsymbol{u(t)}, & t \in (0,T), \\ \boldsymbol{\theta(t,1) \geq \psi(t),\ \theta_x(t,1) \geq 0,\ (\theta(t,1) - \psi(t))\theta_x(t,1) = 0}, & t \in (0,T), \\ \theta(0,x) = \theta^0(x) & x \in (0,1) \end{cases}$$

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More details on

F. Ammar-Khodja, S. Micu, A.Münch,
Exact controllability of a string submitted to a boundary unilateral constraint,
Annales de l'Institut Henri Poincaré (C). 27(4) (2010)

THANK YOU FOR YOUR ATTENTION